A Framework for the Analysis and Construction of Domain Decomposition Preconditioners*

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Abstract: Domain Decomposition is a class of techniques for the solution of partial differential equations on a domain by solving smaller problems on subdomains. They are particularly useful for solving problems on irregular domains and on parallel computers. The key ingredient is the system of equations governing the variables on the interfaces between the subdomains, which is often solved by preconditioned iterative methods. Since each iteration involves solving problems on each subdomain, it is essential to keep the number of iterations low by using a good preconditioner. In this paper, we present a framework for analyzing and constructing such efficient preconditioners. We use two approaches. The first is based on spectral analysis and can be used to invert exactly the interface operator for general piecewise constant coefficient elliptic operators on rectangular regions in any dimension. Methods for adapting these techniques to nonconstant coefficient problems and irregular domains will be discussed. The second approach is based on treating the interface operator as a localized pseudo-differential operator on the interface unknowns and is applicable to more general operators than the spectral approach. One of our objectives is to illuminate the relationships among the most common preconditioners in the literature.

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1. Introduction

The term Domain Decomposition generally refers to a class of techniques for solving partial differential equations on a given domain by first decomposing the domain into smaller ones and then obtaining the overall solution by solving smaller problems on these subdomains. In this sense the idea is rather old and can be traced to Schwarz's alternating procedure, in which existence of solutions to boundary value problems are proved by an iteration involving solutions on overlapping subdomains. This idea is also widely used in many fields of scientific computing. In structural mechanics, these techniques are known as substructuring or frontal methods and are especially useful when the size of the complete problem is too large for the main memory of the computing machine. In computational fluid dynamics, it is common to decompose the physical domain into different regions and use slightly different forms of the governing equations in each (e.g. the boundary layer equations near a body and potential flow in the far field.)

In the past several years there has been an explosion of activities this research area. The primarily reason is no doubt the advent of parallel computing and the obvious opportunity for parallelism in these methods. Another development has been in the improvement in the efficiency of these methods, primarily through improved handling of the coupling between the subdomain solutions. For example, while Schwarz's procedure is known to converge slowly, acceleration of this method can lead to computationally efficient algorithms [15]. However, we shall not address this class of methods in this paper.

Instead we shall restrict our attention to the class of domain decomposition techniques which use non-overlapping subdomains. The basic idea is to reduce the differential operator on the whole domain to an operator (not necessarily a differential one) on the interfaces between the subdomains. The equations for the interfaces are then solved by iterative methods, such as preconditioned conjugate gradient methods. Typically, each iteration involves the solution of a problem on each of the subdomains and therefore for efficiency reasons, it is essential to keep the number of iterations small by using a good preconditioner. Several such preconditioners have been proposed in the recent literature [3,5,6,10,13,17]. In most aspects, their derivations are mostly unrelated. Our main purpose in this paper is to give a uniform framework in which efficient preconditioners can be derived and their properties analyzed. Moreover, within this framework most of the preconditioners in the literature can be related, compared and generalized.

We use two approaches. Our main approach is based on spectral analysis and can be used to invert exactly the interface operator for general piecewise constant coefficient elliptic operators on rectangular regions in any dimensions. For these operators, our technique leads to domain decomposed fast direct solvers. For more general operators on irregular domains where the exact inverses cannot be derived explicitly, these techniques can easily be adapted to construct efficient preconditioners for the interface operator. Our second approach is based on approximating the interface operator by treating it as a localized pseudo-differential operator. Since this approach does not depend on the special form of the differential operator, it is applicable to more general operators than the spectral approach.

The outline of the paper is as follows. In section 2, we introduce our formulation of the interface system. In section 3, we consider the spectral approach. In particular, for the case of a rectangle decomposed into two smaller ones, we give the exact inverse of the interface operator for a variety of second order elliptic operators and discretizations in 2D and 3D: the Laplace operator (5 point and 9 point discretization), the Helmholtz operator, operators with first order terms with central and upwind differencing, and operators with piecewise constant coefficients in each subdomain. Moreover, we extend these to the multiple subdomain case. In section 5, we consider the use of these exact inverses as preconditioners in the case of irregular domains. The exact inverses allow a comparison of the various preconditioners in the literature to be made. Both numerical and

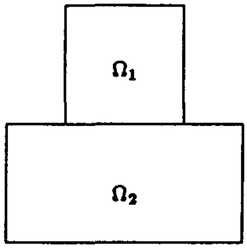

Figure 1: The domain Ω and its partition.

theoretical results will be given. Section 5 considers extensions to non-separable problems. Finally, section 6 considers briefly the operator approach.

2. Formulation

We will first formulate our approach in the simplest case of a domain split into two subdomains with one interface. Consider the problem:

$$Lu = f \qquad \text{on} \quad \Omega \tag{2.1}$$

with boundary condition

$$u = u_b \qquad \text{on} \quad \partial\Omega$$

where L is a linear elliptic operator and the domain Ω is as illustrated in Fig. 1 . We will call the interface between Ω_1 and Ω_2, Γ.

If we order the unknowns for the internal points of the subdomains first and those in the interface Γ last, then the discrete solution vector $u = (u_1, u_2, u_3)$ satisfies the linear system

$$Au = b \quad , \tag{2.2}$$

which can be expressed in block form as:

$$\begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} . \tag{2.3}$$

The system (2.3) can be solved by Block-Gaussian Elimination as follows:

Step 1: Compute

$$C = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}, \tag{2.4}$$

$$w_1 = A_{11}^{-1} b_1 \tag{2.5}$$

$$w_2 = A_{22}^{-1} b_2 \tag{2.6}$$

and solve

$$C u_3 = b_3 - A_{13}^T w_1 - A_{23}^T w_2. \tag{2.7}$$

Step 2: Compute

$$u_1 = w_1 - A_{11}^{-1} A_{13} u_3 \tag{2.8}$$

and

$$u_2 = w_2 - A_{22}^{-1} A_{23} u_3. \tag{2.9}$$

Note that, except for (2.7), the algorithm only requires the solution of problems with A_{11} and A_{22}, which corresponds to solving independent problems on the subdomains. The matrix C (2.4) is the Schur complement of A_{33} in A and it is sometimes called the *capacitance matrix* in this context. It corresponds to the reduction of the operator L on Ω to an operator on the boundary Γ.

3. The Spectral Approach for Separable Operators

The basic idea of the spectral approach is to diagonalize the matrix C by appropriately chosen eigenvectors [6]. Because of the form of C in (2.4), it is clear that a vector w would be an eigenvector of C if it is an eigenvector for each of the three terms in (2.4). It can be verified that the product of the last two terms with w corresponds imposing local averages of w (namely $A_{13}w$ and $A_{23}w$) as Dirichlet boundary conditions on Γ, solving for the solutions (say v_1 and v_2) on each subdomain and evaluating local averages of these solutions near Γ (namely $A_{13}^T v_1$ and $A_{23}^T v_2$.) Thus the issue of finding the eigenvectors of C is closely related to the separability of the operators A_{11}^{-1} and A_{22}^{-1} along the direction of Γ. In particular, if Ω is rectangular with Γ parallel to one of its edges and the operator L is also separable in the directions of the two edges of Ω, then the eigenvectors of C can often be easily found in terms of the separating eigenfunctions of L. For constant coefficient operators such as the Laplacian with Diriclet boundary conditions, the eigenvectors are simply the discrete Fourier functions defined on Γ. For more complicated operators, such as ones with first order terms, these eigenfunctions are slightly more complicated. For variable coefficient operators, these eigenfunctions may have to be computed numerically. Generally, this technique works for separable operators on rectangular domains, similar to the situation for conventional fast elliptic solvers. Analogously, the spectral approach leads to domain decomposed fast elliptic solvers.

3.1. Laplace Operator

We first consider the case where L is the Laplacian operator, discretized by the standard second order centered differencing, and Ω is a rectangle divided into two or more strips like is shown in Fig. 2. Using the spectral technique explained earlier, the exact eigenvectors and eigenvalues of C can be derived [2, 6, 7]. The eigenvectors are discrete sine functions.

For the case of two strips, C has the following eigenvalue decomposition: [6]

$$W \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} W^T \quad , \tag{3.1}$$

where W is the matrix whose columns are

$$w_j = \sqrt{\frac{2}{n+1}} (\sin j\pi h, \sin 2j\pi h, \cdots, \sin nj\pi h)^T \tag{3.2}$$

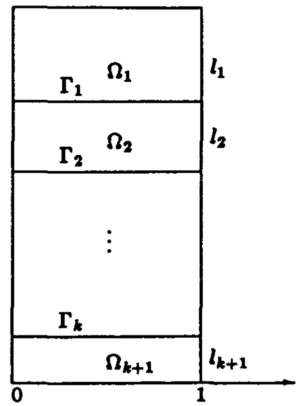

Figure 2: Rectangular domain divided into strips.

and

$$\lambda_j = -\left(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}} + \frac{1+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}}\right)\sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \tag{3.3}$$

$$\sigma_j = 4\sin^2\left(\frac{j\pi h}{2}\right) \quad , \tag{3.4}$$

$$\gamma_j = \left(1 + \frac{\sigma_j}{2} + \sqrt{\sigma_j + \frac{\sigma_j^2}{4}}\right)^2 \tag{3.5}$$

for $j = 1, \ldots, n$, where h is the grid size, and m_1 and m_2 are the number of rows of grid points in the y-direction in Ω_1 and Ω_2 respectively. By using the decomposition (3.1), the capacitance system (2.7) can be solved by fast Fourier transforms. Once the solution u_3 on the interface is computed, we can compute u_1 and u_2 by (2.8) and (2.9), which correspond to solving two independent problems on the subdomains with boundary condition u_3 on Γ.

In the multistrip case, the matrix C has the block-tridiagonal structure:

$$C = \begin{pmatrix} C_1 & B_2 & & \\ B_2 & C_2 & \ddots & \\ & \ddots & \ddots & B_k \\ & & B_k & C_k \end{pmatrix} \quad . \tag{3.6}$$

The C_i's correspond to the reduce operator on Γ_i and the B_i's correspond to the coupling between the interfaces. All blocks C_i and B_i have the same matrix of eigenvectors W, i.e. for $i = 1, \ldots, k$, we have

$$W^T C_i W = \Lambda_i = \operatorname{diag}(\lambda_{i1}, \ldots, \lambda_{in}) \tag{3.7}$$

and for $i = 2, \ldots, k$, we have

$$W^T B_i W = D_i = \mathrm{diag}(\delta_{i1}, \ldots, \delta_{in}) \tag{3.8}$$

where

$$\lambda_{ij} = -\left(\frac{1+\gamma_j^{m_i+1}}{1-\gamma_j^{m_i+1}} + \frac{1+\gamma_j^{m_{i+1}+1}}{1-\gamma_j^{m_{i+1}+1}}\right)\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}, \tag{3.9}$$

and

$$\delta_{ij} = \sqrt{\gamma_j^{m_i}}\left(\frac{1-\gamma_j}{1-\gamma_j^{m_i+1}}\right). \tag{3.10}$$

By first diagonalizing C with a block diagonal matrix with W as the diagonal blocks and then rearranging the equations, C can be reduced to a set of n decoupled tridiagonal systems of dimension k, where $k+1$ is the number of subdomains [7].

Although it first appears that the algorithm requires the solution of two problems on each subdomain, one for computing the right hand side and one for computing the solution on each subdomain, the extra work can be saved if care if taken to save some intermediate results from the first solves. We refer the interested reader to [8], [16] where the parallel implementations of these algorithms are also discussed.

3.2. More General Operators and Discretizations

The spectral technique can be extended to more general operators and discretizations and in higher dimensions. We give a few examples here.

The capacitance matrix for the second order centered finite difference discretization of the operator

$$u_{xx} + \beta u_{yy} \quad , \tag{3.11}$$

where the coefficient β takes constant values β_i on each subdomain Ω_i, has the same form as (3.6) except that the eigenvalues of C_i and B_i are given by [17]

$$\lambda_{ij} = -\left(\frac{1+\gamma_{ij}^{m_i+1}}{1-\gamma_{ij}^{m_i+1}}\right)\sqrt{\frac{\sigma_j^2}{4} + \beta_i\sigma_j} \quad - \quad \left(\frac{1+\gamma_{i+1,j}^{m_{i+1}+1}}{1-\gamma_{i+1,j}^{m_{i+1}+1}}\right)\sqrt{\frac{\sigma_j^2}{4} + \beta_{i+1}\sigma_j} \tag{3.12}$$

and

$$\delta_{ij} = \beta_i\sqrt{\gamma_{ij}^{m_i}}\left(\frac{1-\gamma_{ij}}{1-\gamma_{ij}^{m_i+1}}\right) \quad ,$$

where

$$\gamma_{ij} = \frac{1}{\beta_i^2} \quad \left(\frac{\sigma_j}{2} + \beta_i - \sqrt{\frac{\sigma_j^2}{4} + \beta_i\sigma_j}\right)^2 \quad . \tag{3.13}$$

The capacitance matrix for the second order centered finite difference discretization of the Helmholtz operator

$$\Delta u + \alpha u \tag{3.14}$$

also has the form (3.6), with the eigenvalues of C_i and B_i given by [17]:

$$\lambda_{ij} = -\left(\frac{1+\gamma_j^{m_i+1}}{1-\gamma_j^{m_i+1}} + \frac{1+\gamma_j^{m_{i+1}+1}}{1-\gamma_j^{m_{i+1}+1}}\right)\sqrt{\frac{\mu_j^2}{4} - 1} \tag{3.15}$$

and

$$\delta_{ij} = \sqrt{\gamma_j^{m_i}} \left(\frac{1-\gamma_j}{1-\gamma_j^{m_i+1}} \right) \quad , \tag{3.16}$$

where

$$\mu = -\sigma_j - 2 + \alpha h^2 \tag{3.17}$$

and

$$\gamma_j = \left(-\frac{\mu_j}{2} - \sqrt{\frac{\mu_j^2}{4} - 1} \right)^2 \quad . \tag{3.18}$$

The capacitance matrix for the 3D Laplacian has the same form as the 2D version with σ_j replaced by the eigenvalues of the 2D Laplace operator [16].

Different discretizations can also be treated within this framework. For example, consider the following nine point fourth order discretization of the Laplacian [21]:

$$\Delta_9 = 1/6h^2 \text{tridiagonal}(S_i, T_i, S_i)$$

where S_i = tridiagonal(1, 4, 1) and T_i = tridiagonal(4, −20, 4). The capacitance matrix C_9 associated with Δ_9 has the same form as (3.1), with σ_j replaced by

$$\sigma_j = \frac{4 \sin^2\left(\frac{i\pi h}{2}\right)}{1 - \frac{2}{3}\sin^2\left(\frac{i\pi h}{2}\right)} \quad . \tag{3.19}$$

It can also be easily shown that

$$\lim_{\substack{h\to 0 \\ m\to\infty}} K(C_5^{-1}C_9) = \sqrt{\frac{3}{2}},$$

where K denotes the spectral condition number and C_5 denotes the capacitance matrix corresponding to the 5-point discrete Laplacian. The above result shows that C_5 is spectrally equivalent to C_9 and is a reasonably good preconditioner for it.

3.3. Non-Self-Adjoint Operators With First Order Terms

Our framework can also be extended to second order elliptic problems with first order derivative terms, such as the operators $L_x \equiv \Delta + \alpha u_x$ and $L_y \equiv \Delta + \alpha u_y$. Since the discretizations of these non-self-adjoint operators lead to nonsymmetric matrices, the spectral approach becomes more complicated. Consider a rectangle split by an interface Γ along the x-direction, as in Figure 2. For the operator L_y, it is easy to see that the Fourier matrix W can still be used to diagonalize C because the $y-$derivative does not affect the separating eigen-modes in the $x-$direction. The eigenvalues λ_j's of course depends on the value of α. On the other hand, for the operator L_x, W cannot be used to diagonalized C because the Fourier modes are no longer eigenfunctions of the operator L_x in the $x-$direction. It turns out, however, that the eigenfunctions of C can still be found analytically - they are simply given by DW, where D is a suitably chosen diagonal matrix that depends on α. The eigenvalues depend on α as well. These formulas are too complicated to be presented here and the interested reader is referred to a recent report by Chan and Hou [19], where results for both centered and upwind discretizations for the first order terms are presented.

What we would like to show here is the effectiveness of these exact preconditioners when applied to problems where the first derivative terms are not negligible. Consider the situation where one needs a preconditioner M for the boundary operator $C(\alpha)$ corresponding to L_y. Without the knowledge of the exact preconditioners, a natural approach is to use $M = C(0)$, for which we know the exact diagonalization. Moreover, spectral equivalence results [10] guarantee that for fixed α the condition number of the preconditioned system remains bounded independent of h. It turns out, however, that for fixed h and large values of α the preconditioned system can have a large condition number. For example, for the case where $h = 0.02, m_1 = 50, m_2 = 100$, and upwind differencing is used, the values of $K(C^{-1}(0)C(\alpha))$ are approximately 15 and 40 for values of the "cell-Reynolds-number" $\alpha h/2$ equal to 0.4 and 0.8 respectively.

4. Irregular Domains and Preconditioners

For general irregular domains, the eigenvalues and eigenvectors of the capacitance matrix cannot be computed analytically via the spectral techniques, and hence one must find alternative methods for solving the capacitance system (2.7).

Note that the computation of the capacitance matrix C is expensive, since it requires the solution of $m + 1$ systems with A_{11} and A_{22}, and it is also expensive to invert for m large, because it is dense in general.

Instead of solving the system (2.7) directly, iterative methods such as preconditioned conjugate gradient methods (PCG) can be applied, in which only matrix vector products Cy for arbitrary $y \in R^m$ are required. As explained earlier, this product can be computed by one solve on each subdomain with boundary condition on Γ determined by y. Since each iteration involves the solution of problems on the subdomains, keeping the number of iterations small is very important for the efficiency of the method. This can be achieved by choosing a good preconditioner for C. In this section, we shall survey some preconditioners in the literature, highlighting the relationships among them. We shall also analyze their performance, with special emphasis on the dependence on the mesh size h *and* the departure from regularity of the domain.

4.1. Survey of Preconditioners

We summarize several of the preconditioners which have been proposed in the literature [3,5,6,10,13]. We summarise them here in our notations to make it easier to compare them.

1. In [10], Dryja proposed the following preconditioner for (2.4):

$$M_D = W diag(\lambda_1^D, \lambda_2^D, \cdots, \lambda_n^D) W^T \quad , \tag{4.1}$$

where the columns of W are given by (3.2) and

$$\lambda_j^D = -2\sqrt{\sigma_j} \tag{4.2}$$

with σ_j given by (3.4). This preconditioner is based on the Sobolov trace theorem [22]. He proved that $K(M_D^{-1}C)$ is bounded independently of the mesh size h.

2. Golub and Mayers [13] proposed the preconditioner:

$$M_G = W diag(\lambda_1^G, \lambda_2^G, \cdots, \lambda_n^G) W^T \quad , \tag{4.3}$$

where

$$\lambda_j^G \equiv -2\sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \quad . \tag{4.4}$$

The derivation is motivated by considering the generating function for the solution for the case where the boundaries of the two domains move away to infinity. Empirical results in [13] show that M_G performs better than M_D.

3. Another interesting preconditioner was given by Björstad and Widlund [3] (based on a suggestion of Dryja's) and has the following form:

$$M_B = A_{33} - 2A_{13}^T A_{11}^{-1} A_{13} \quad .$$

It is easy to show that the eigenvalue decomposition of M_B is

$$M_B = W diag(\lambda_1^B, \lambda_2^B, \cdots, \lambda_n^B) W^T \quad , \tag{4.5}$$

where

$$\lambda_j^B = -2\left(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}}\right)\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}.$$

The underlying motivation for this preconditioner is exploiting symmetries in the operator and the domain about the interface. When Ω_1 and Ω_2 are identical (and hence $A_{11} = A_{22}$), it is easy to see that M_B is an exact preconditioner. Björstad and Widlund showed that the product $M_B^{-1}Cv$ can be computed by solving a mixed Neumann-Dirichlet problem in one of the subdomains and a Dirichlet problem in the other one. The basic idea is that if both the operator L and the domain are symmetric about Γ then the solution can be found by solving on only one of the subdomains with a Neumann boundary condition on Γ. They also proved that $K(CM_B^{-1})$ is uniformed bounded for certain finite element approximation of Dirichlet problems for self-adjoint second order elliptic problems in plane regions. Their method has the advantage that it can be applied to more general operators and domain shapes. However, in the particular case of the Laplacian operator on a union of rectangles, it is less efficient than applying a single FFT computation on the interface grid points, as the factorization (4.5) suggests. They also proved that M_B is spectrally equivalent to C [3].

4. Finally, Chan [6] suggested a procedure for extending the exact preconditioner (3.1) for rectangular regions to construct preconditioners for irregular regions. The idea is to use as preconditioner the exact capacitance matrix corresponding to a best *rectangular* approximation to the irregular domain sharing the same interface. The motivation is to improve Dryja's and Golub/Mayer's preconditioners by taking into account the aspect ratio of the subdomains. We will call this preconditioner M_C.

Although M_D, M_G, M_B and M_C were derived independently, we have expressed them in the same matrix factorization formats. Since the eigenvectors are the same, to compare them we only need to look at their eigenvalues λ_j's. On a rectangle, for which M_C is exact, M_D, M_G and M_B can be viewed as progressively better approximations to M_C. The λ_j^G's are exact for rectangles with infinitely large aspect ratios because the coefficient in front of λ_j in (3.9) tends to -2 in the limit of m_1 and m_2 tending to ∞. It can also be easily observed that λ_j^D is a first order approximation to λ_j^G for the small λ_j's but underestimates the larger λ_j's. Finally, it it easy to see that the λ_j^B's are exact only for the case when $m_1 = m_2$. For more detailed analysis, the reader is referred to [6].

4.2. Performance of Preconditioners: Numerical Results

In Fig. 3 we compare the preconditioners M_D, M_G and M_C for the Poisson equation on a T-shaped region Ω as given in Fig. 1, where we vary the aspect ratio of the subdomain Ω_1. We consider a uniform grid on Ω with $n = 15$ grid points on the interface Γ. By varying m_1, the number of interior grid points in the y direction on the subdomain Ω_1, we computed the condition number of the preconditioned capacitance system for different aspect ratios defined as $\frac{m_1+1}{n+1}$. As we can see from the plots, M_C performs very well, even when Ω_1 becomes very narrow, while the performance of the others deteriorate as the aspect ratio becomes small. The curves for M_C, M_G

and M_B are indistinguishable for aspect ratios larger than one and they are all better than M_D. See [14] for a careful numerical comparison of these and other preconditioners for constant and variable coefficients operators.

4.3. Dependence on Irregularity of Domain: Some Theoretical Results

Besides the empirical evidence of the performance of the various preconditioners, there are also some theoretical results available. The most common results of this kind are *spectral equivalence* results which asserts that a particular preconditioner is spectrally equivalent to the exact boundary operator as the grid size h tends to zero *for a fixed domain* [3,10]. This essentially guarantees that the number of iterations needed to solve the preconditioned system to a given accuracy is independent of h. For domains partitioned with interfaces and cross-points, these spectral results must be relaxed to allow for a slight increase (of the form $log(h^{-1})$ in the conditioning of the preconditioned operator [3,5].

In a somewhat orthogonal direction, we have recently obtained some theoretical results concerning the performance of preconditioners as the *shape* of the domain varies. This issue is of obvious practical importance in applying the preconditioners to domains of varying shapes. We prove that [18] on any L-shaped domain, the preconditioned capacitance matrix for the preconditioner M_C is bounded by 2.16, independent of h and the *aspect ratios* of the subdomains. Moreover, the convergence rate is essentially the same irrespective of how the domain is partitioned (there are two ways of partitioning an L-shape domain into two rectangular domains). Similar results are also obtained for C-shaped regions. This independence of the aspect ratios is a special property of the preconditioner M_C not shared by the other preconditioners in general (see Fig. 3 for example), and can be traced directly to the fact that the aspect ratios of the subdomains are incorporated into the exact preconditioner for the approximating rectangle from which M_C is derived.

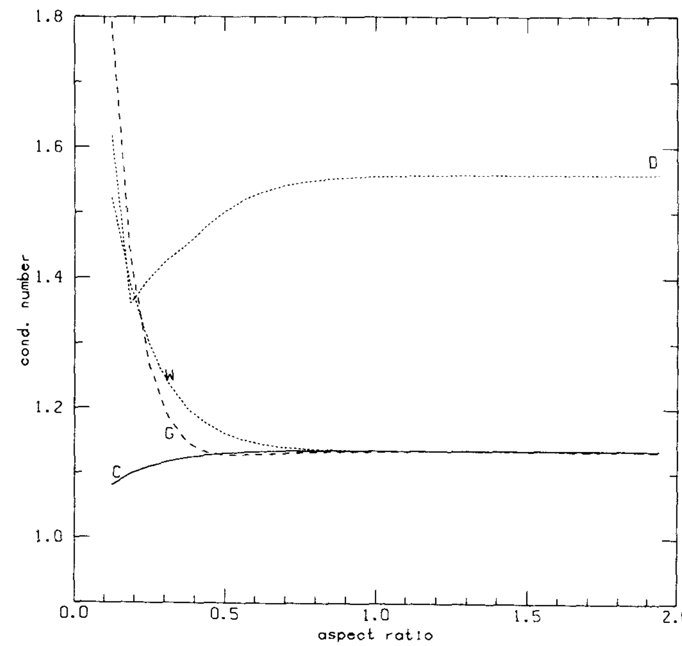

Figure 3: T-shaped region. Condition number of the preconditioned capacitance matrix with Chan's (C), Dryja's(D), Björstad and Widlund's (W) and Golub and Mayers' (G) preconditoners.

5. Non-Separable Problems

If the operator A is non-separable, there usually are no fast solvers available for A_{11} and A_{22}. Therefore, in each iteration of an iterative method for solving the capacitance system, the matrix-vector product Cy cannot be evaluated inexpensively, making methods which work only with the interface system ineffective.

An alternative is to solve the system (2.3) on the whole domain instead of just the capacitance system on the interface. We will show that preconditioners for (2.3) can be derived from preconditioners for the capacitance matrix. Let B_{11} and B_{22} be approximations to A_{11} and A_{22}. The former could be separable approximations to the latter or they could represent some truncated inner iteration for solving systems with the latter [20]. Based on the following decomposition of the matrix A in (2.3):

$$A = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ A_{31} & A_{32} & C \end{pmatrix} \begin{pmatrix} I & & A_{11}^{-1}A_{13} \\ & I & A_{22}^{-1}A_{23} \\ & & I \end{pmatrix} , \tag{5.1}$$

where C is the Schur complement (2.4), we can derive a preconditoner for A given by:

$$\tilde{M} = \begin{pmatrix} B_{11} & & \\ & B_{22} & \\ A_{31} & A_{32} & M \end{pmatrix} \begin{pmatrix} I & & B_{11}^{-1}A_{13} \\ & I & B_{22}^{-1}A_{23} \\ & & I \end{pmatrix} , \tag{5.2}$$

where M is a good preconditioner for the matrix C. We can see that $\tilde{M}$ is easily invertible by block-elimination, since fast solvers can be applied to solve systems with B_{11} and B_{22}.

Preconditioners of the form (5.2) were first used by Bramble, Pasciak and Schatz [4,5,5a]. They used both M_D and M_B as the preconditioner M for C. As a generalization of their idea, any of the preconditioners given for the constant coefficients case can be applied here as M. In fact, a theorem by Eisenstat in [14] shows that, when $B_{ii} = A_{ii}$, the PCG algorithm applied to (2.7) with preconditioner M and initial guess u_3^0 is equivalent to the PCG algorithm applied to (2.3) with preconditioner given by (5.2) and initial guess $(A_{11}^{-1}(b_1 - A_{13}u_3^0), A_{22}^{-1}(b_2 - A_{23}u_3^0), u_3^0)$. In [14], numerical experiments were performed with these and other preconditioners.

6. The Operator Approach

So far, our approach for deriving preconditioners for C depends on special differential properties of the operator A (except for M_B which only uses symmetry arguments). This raises the question of how effective they will be when applied to other more different and complicated operators, (e.g. the steady state Navier-Stokes operator), without first somehow reducing the problem to one of a second order elliptic problem that we have already treated here. For example, Dryja's preconditioner M_D is intimately tied to the Sobolov Trace Theorem for second order elliptic problems and it cannot be expected to perform well for other types of operators. It is therefore desirable to derive preconditioners in a more general way that depends less critically on the particular form of the differential operator, but more on the other computable quantitites of the given operator. An example is the exploitation of symmetry in the style of M_B, which can be expected to be applicable for more general class of problems.

Here we investigate another approach which depends on efficiently "probing" the operator C to gain information on its structure. This information can then be used to construct an effective preconditioner. Our main motivation is the empirical observation that , in the case of the Laplace operator, the elements of the matrix C decay rapidly away from the main diagonal [13]. It is therefore reasonable to consider k-diagonal approximations to C. It would not, however, be efficient to compute the elements of C in order to do this. *We now present a method for computing* a k-diagonal approximation to C without requiring the computation of C explicitly. The idea is motivated by sparse Jacobian evaluation techniques [9]. For example, for the case $k = 3$, the approximant M to C can be computed in compact form by evaluating the three products $Cu_i, i = 1,2,3$, where $u_1 = (1,0,0,1,0,\ldots)^T$, $u_2 = (0,1,0,0,1,\ldots)^T$ and $u_3 = (0,0,1,0,0,\ldots)^T$. The motivation is clear, for if C were indeed tridiagonal, ($k = 3$), then all of its nonzero elements can be found in the three vectors $Cu_i, i = 1,2,3$. Note that the computation of each product Cu_i involves solving one problem on each subdomain with u_i as boundary condition on the interface.

Note that in principle this approach can be applied to any operator A and requires only a solver for the subdomains. However, it can only be expected to be effective for those operators for which the reduced boundary operator C is predominantly local (corresponding to the rapidly decaying elements away from the diagonal.)

In Figure 4, we plot the eigenvalues of the tridiagonal preconditioner computed by the above method (denoted by M_3) together with the eigenvalues of M_D, M_G and C for the problem of a Laplacian on a square divided into two strips, with $n = 15$ and $m_1 = m_2 = 7$. For this problem, the plots for C and M_G are indistinguishable. The preconditioner M_D underestimates the large eigenvalues of C whereas M_3 seems to follow the exact eigenvalues more closely.

The generalization to other values of k is obvious. Moreover, it can be easily verified that the matrix M computed this way preserves the row-sums of C. The case $k = 1$, however, deserves special mention. The method described above would compute a diagonal approximation to C, with diagonal entries given by Ce, where $e = (1,1,\ldots,1)^T$. However, since the first term A_{33} in the definition of C in (2.4) is already known explicitly (and it is tridiagonal), it is only necessary to apply the above approximation procedure to the last two terms in (2.4). The resulting matrix M is thus tridiagonal, namely, A_{33} with the diagonal entries modified in such a way that the row sums

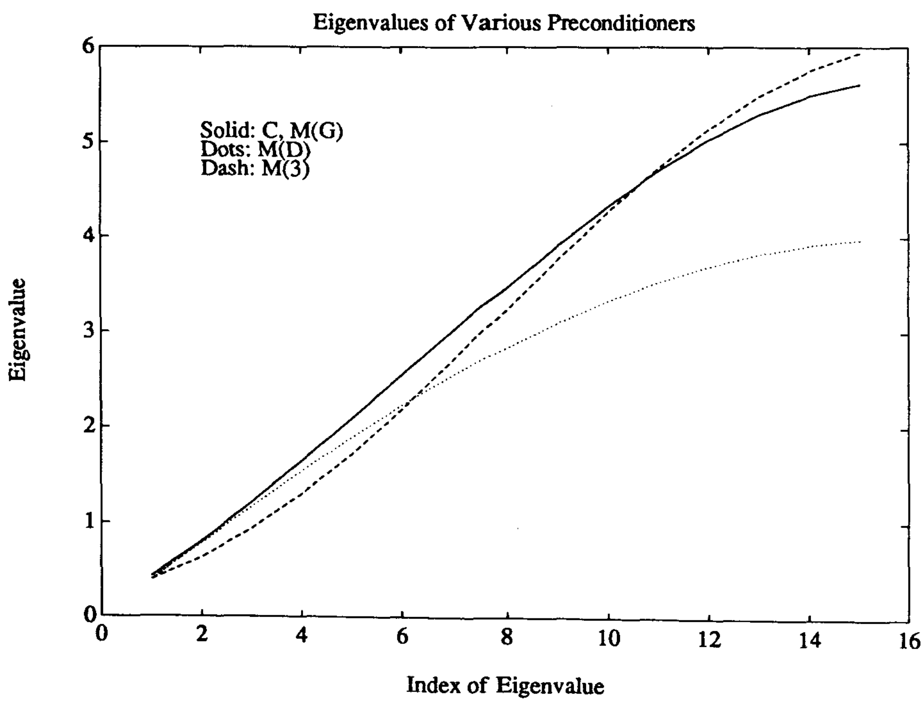

Figure 4: Comparison of Eigenvalues of Preconditioners for Poisson's Equation with $n = 15, m_1 = m_2 = 7$.

of C are preserved. Viewed this way, the case $k = 1$ is similar in spirit to the Dupont-Kendall-Rachford procedure [11] for obtaining an easily invertible banded approximant for C. This special procedure for the case $k = 1$ was sugested independently by Eisenstat [12]. See [14] for numerical experiments with this class of preconditioners.

In general, for a k-diagonal approximation to C, k problems on each subdomain must be solved, which may seem prohibitively expensive except for small values of k. However, the main advantage of this family of preconditioners is that they are less dependent on special properties (e.g. eigenstructures) of the differential operator underlying A. Moreover, for nonlinear problems where a Newton type outer iteration may be involved, one preconditioner can be reused several times and the cost of computing it can be amortized over the overall iteration.

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