An Iterative Procedure for Domain Decomposition Methods: A Finite Element Approach

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Abstract. For conforming finite element approximations of elliptic boundary value problems, a new domain decomposition technique is proposed. It is based on a block iterative procedure among subdomains in which the transmission conditions at interfaces are attributed partly to one subdomain and partly to its adjacent. No preconditioning is needed, but one should simply solve a sequence of discretized mixed boundary-value-problems on each subdomain. An optimal strategy for the determination of a relaxation parameter to be used at the subdomain interfaces is indicated.

1. Introduction - In recent years a considerable attention has been devoted to the use of domain decomposition (or substructing) techniques for the numerical solution of partial differential equations. Among others, the following reasons underly the development of these techniques. The equations in the different subdomains may be of different type, or, more simply, they might contain different parameters. Besides, when dealing with complicated geometries, a subdivision of the entire domain by simply shaped subdomains on which special solution techniques can be applied may increase the overall efficiency of the numerical scheme. This is, e.g., the case of the numerical approximations based on spectral methods (see, e.g., Canuto, Hussaini, Quarteroni and Zang [1; Ch. 13]). A further important reason is that very often domain decomposition techniques are well suited for computations in parallel environments.

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Apart from the Schwarz method, the leading idea of the most part of domain decomposition methods currently in use is the following. After its discretization, the given differential problem is partitioned into subproblems corresponding to non overlapping subregions of the entire physical domain. Then, by a block elimination procedure, independent systems are derived for each subdomain. The remaining unknowns pertaining to the interface boundaries are coupled by a global system. The interactions between the two sets of unknowns are then handled by a suitable iterative method (e.g., the conjugate gradient method, or the Chebyshev method, or else the Richardson method). At this step, the use of a properly designed preconditioner may remarkably reduce the number of iterations.

In this paper we propose a different approach with the aim of simplifying at most the computational complexity of the problem, bypassing the solution of a global system and then the construction of proper preconditioners. We focus our attention on discretizations by finite elements of second-order elliptic boundary value problems. We state first, for the continuous problem, an equivalence principle between the original single-domain problem and the multi-domain problem in which the transmission conditions at subdomain interfaces are properly taken into account. Then we take inspiration from this principle to build up an iterative procedure to compute the finite element solution of the single-domain problem by means of a sequence of finite element problems on each subdomain. We simply iterate between two adjacent subdomains by imposing in one of them the condition of continuity of the solution. On the other one the continuity of the normal derivative of the solution is imposed in the weak sense. The original finite element problem is reduced to a sequence of finite element approximations of mixed boundary value problems on each subdomain, which may be faced by standard single-domain finite element solvers. The effectiveness of the previous iterative procedure can be achieved by a proper choice of a relaxation parameter to be used at subdomain interfaces. To this end, an optimal strategy for its automatic selection is indicated.

The above iterative method is inspired by a similar one that was formerly proposed by Funaro, Quarteroni and Zanolli [3] for the differential problem itself, and consequently applied to numerical discretizations using spectral methods.

The convergence analysis, which is concerned with a partition of the domain into two subdomains only, exhibits an interval in which the relaxation parameter should be taken at each iteration in order to achieve convergence. Besides, for conforming finite elements with arbitrary degree, the error reduction factor per iteration is independent of the finite element mesh size.

We report at the end of this paper some numerical experiences that show the effectiveness of the method here proposed. The reader can find in [5] further numerical results as well as the mathematical proofs which are not reported here.

2. The differential problem and its multidomain formulation - Let Ω be an open bounded polygonal domain of $\mathbf{R}^2$ whose boundary will be denoted by $\partial\Omega$. We consider the boundary value problem:

(2.1) $Lu=f$ in Ω ; $u=0$ on $\partial\Omega$, where f is a given function and

$$Lu := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u$$

with a_{ij} symmetric, uniformly positive definite, bounded, and piecewise smooth on Ω, and $a_0(x)\geq 0$. In (2.1) homogeneous Dirichlet conditions are used in order to simplify the exposition.
Setting

$$a(u,v) := \sum_{ij=1}^{2}\int_\Omega \left(a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\right)dx + \int_\Omega a_0 uv\,dx,$$

it is well known that if $f\in H^{-1}(\Omega)$, then (2.1) has a unique solution that satisfies

(2.2) $u\in H_0^1(\Omega)$: $a(u,v)=\langle f,v\rangle \quad \forall v\in H_0^1(\Omega)$.

We remind that

$$H_0^1(\Omega)=\{u\in L^2(\Omega)\,|\,\nabla u\in L^2(\Omega),\ u=0 \text{ on } \partial\Omega\}$$

and that $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ (see, e.g., Lions and Magenes [4]). In the sequel, for the sake of simplicity, we shall assume that $f\in L^2(\Omega)$.
We assume that Ω is partitioned into two non intersecting subdomains Ω_1 and Ω_2, i.e., $\bar\Omega=\bar\Omega_1\cup\bar\Omega_2$, and we denote by Γ the common boundary of Ω_1 and Ω_2. Then we define:

$$V_i := \{v\in H^1(\Omega_i),\ v_{|\partial\Omega\cap\partial\Omega_i}=0\},\quad \mathring{V}_i := H_0^1(\Omega_i),\ \text{for } i=1,2,$$

$$\text{and } \Phi := \{v_{|\Gamma} : v\in H_0^1(\Omega)\}.$$

It is known that $\Phi=H_{00}^{\frac{1}{2}}(\Gamma)$(see, e.g., Lions and Magenes [4]). Then, for any $\phi\in\Phi$ we denote by $R_1\phi$ and $R_2\phi$ the "harmonic" extensions of ϕ to Ω_1 and Ω_2 respectively. Precisely:

(2.3) $R_1\phi\in V_1$: $a_1(R_1\phi,v)=0 \quad \forall v\in\mathring{V}_1$; $R_1\phi=\phi$ on Γ,

(2.4) $R_2\phi\in V_2$: $a_2(R_2\phi,v)=0 \quad \forall v\in\mathring{V}_2$; $R_2\phi=\phi$ on Γ.

where we have set

$$(2.5)\qquad a_k(u,v):=\sum_{i,j=1}^{2}\int_{\Omega_k}\left(a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\right)dx+\int_{\Omega_k}a_0uvdx.$$

The following equivalence statement introduces the multidomain formulation of problem (2.2).

<u>Lemma 2.1</u> - <u>The function u is the solution of</u> (2.2) <u>if the functions</u> $u_1:=u_{|\Omega_1}\varepsilon V_1$ <u>and</u> $u_2:=u_{|\Omega_2}\ V_2$ <u>satisfy the following</u> <u>split problem</u>:

$$(2.6)\qquad a_1(u_1,v)=\langle f,v\rangle_1\quad \forall v\varepsilon \mathring{V}_1\ ;\ u_1=u_2 \text{ on } \Gamma\ ,$$

$$(2.7)\qquad a_2(u_2,v)=\langle f,v\rangle_2\quad \forall v\varepsilon \mathring{V}_2,$$

$$(2.8)\qquad a_2(u_2,R_2\phi)=-a_1(u_1,R_1\phi)+\langle f,R_1\phi\rangle_1+\langle f,R_2\phi\rangle_2.$$

<u>Here</u> $\langle .,.\rangle_k$ <u>denotes the scalar product of</u> $L^2(\Omega_k)$, k=1,2.

<u>Proof</u> - We have the following characterization for $H_0^1(\Omega)$:

$$(2.9)\ H_0^1(\Omega)=V_1^*\oplus\Phi^*\oplus V_2^*$$

where we have set:

$$(2.10)\qquad \Phi^*=\{v\varepsilon H_0^1(\Omega):\exists\phi\varepsilon\Phi,\ v_{|\Omega_1}=R_1\phi,\ v_{|\Omega_2}=R_2\phi\},$$

$$V_1^*=\{v\varepsilon H_0^1(\Omega):v_{|\Omega_1}\varepsilon\mathring{V}_1,\ v_{|\Omega_2}\equiv 0\},$$

and $$V_2^*=\{v\varepsilon H_0^1(\Omega:\ v_{|\Omega_2}\varepsilon\mathring{V}_2,\ v_{|\Omega_1}\ 0\}.$$

For any $v\varepsilon H_0^1(\Omega)$, let now denote by v_k its restriction to Ω_k, for k=1,2. Then (2.2) is equivalent to

$$(2.11)\qquad a_1(u_1,v_1)+a_2(u_2,v_2)=\langle f,v_1\rangle_1+\langle f,v_2\rangle_2\quad \forall v\varepsilon H_0^1(\Omega).$$

Therefore, (2.6), (2.7) and (2.8) are obtainable from (2.2) by taking respectively $v\varepsilon V_1^*$, $v\varepsilon V_2^*$ and $v\varepsilon\Phi^*$ as test functions. On the other hand, in view of (2.9) any equation of (2.2) can be obtained by summation of equations of the form (2.6), (2.7) and (2.8). This concludes the proof.■

<u>Remark 2.1</u> - The solution of (2.6)-(2.8) satisfies

$$(2.12)\qquad \partial_1u_1+\partial_2u_2=0\quad \text{on}\quad \Gamma,$$

where ∂_j is the conormal derivative operator associated with the bilinear form a_j. Indeed, we note first that (2.6) and (2.7) yield:

(2.13) $\quad Lu_k = f \quad \text{in } \mathcal{D}'(\Omega_k), \quad k=1,2.$

Let now ϕ be any function of $\mathcal{D}(\Gamma)$. Integrating by parts within each subdomain and using (2.13) yields:

$$a_1(u_1,R_1\phi)+a_2(u_2,R_2\phi)=\langle Lu_1,R_1\phi\rangle_1+\langle Lu_2,R_2\phi\rangle_2$$
$$+\langle\partial_1u_1,\gamma_0R_1\phi\rangle+\langle\partial_2u_2,\gamma_0R_2\phi\rangle =$$

$$\langle f,R_1\phi\rangle_1+\langle f,R_2\phi\rangle_2+\langle\partial_1u_1+\partial_2u_2,\phi\rangle \quad \forall\phi\in\Phi.$$

In the above relations γ_0 is the trace operator from $H^1(\Omega)$ to Φ, and the symbol $\langle.,.\rangle$ indicates the duality between Φ and its dual space Φ' (see again Lions and Magenes [4]).

Now the property (2.12) can be established using (2.8).■

3. <u>The finite element approximation</u> - We shall keep in this section the notations of section 2 concerning the multidomain partition of Ω. Let $\mathcal{T}_h$ be a regular decomposition [2] of Ω into triangles T not crossing the interface Γ. (Thus, each element T is either contained in $\bar{\Omega}_1$ or in $\bar{\Omega}_2$). Define the conforming finite element space:

(3.1) $\quad \mathring{V}_h := \{ v\in C^\circ(\bar{\Omega}) : v_{|T}\in P_r(T) \ \forall T\in\mathcal{T}_h, \quad v=0 \text{ on } \partial\Omega \}$

As usual, we have denoted by $P_r(T)$ the space of polynomials of degree $\le r$ on T ($r\ge1$).

The finite element approximation of problem (2.2) is then:

(3.2) $\quad u_h\in\mathring{V}_h \ : \ a(u_h,v) = \langle f,v\rangle \quad \forall v\in\mathring{V}_h,$

and the following error estimate holds (see, e.g., Ciarlet [2]):

(3.3) $\quad \|u-u_h\|_{H^1(\Omega)} \le Ch^r\|u\|_{H^{r+1}(\Omega)} .$

We define, for i=1,2

(3.4) $\quad V_{i,h} := \{ v\in C^\circ(\bar{\Omega}_i) : v_{|T}\in P_r(T) \ \forall T\in\mathcal{T}_h, \ T\subset\Omega_i, \ v=0 \text{ on } \partial\Omega_i\backslash\Gamma \},$

(3.5) $\quad \mathring{V}_{i,h} := \{ v\in V_{i,h} : v=0 \text{ on } \partial\Omega_i \},$

(3.6) $\quad \|v\|_i^2 = a_i(v,v) \quad v\in V_{i,h}.$

Let us denote by Σ_h the decomposition of Γ induced by the triangulation $\mathcal{T}_h$ of Ω, and let I be the current interval of Σ_h. Then we define the space:

$$(3.7) \quad \Phi_h := \{\phi \varepsilon C^\circ(\bar{\Gamma}) : \phi_{|I} \varepsilon P_r(I) \ \forall I \varepsilon \Sigma_h, \ \phi_{|\partial\Gamma} = 0\},$$

and, for i=1,2, the following extension operators:

$$(3.8) \quad \rho_{i,h} : \Phi_h \to V_{i,h}, \ \rho_{i,h}\phi_{|\Gamma} = \phi, \ \rho_{i,h}\phi_{|T} = 0 \text{ if } \partial T \cap \Gamma = \emptyset.$$

The actual computation of the finite element solution u_h of (3.2) can be carried out by means of the following iterative procedure, suggested by the split problem (2.6)-(2.8):

Let $g^0 \varepsilon \Phi_h$ be given; then for n≥1 let $u^n_{1,h} \varepsilon V_{1,h}$ and $u^n_{2,h} \varepsilon V_{2,h}$ be the solutions

$$(3.9) \quad a_1(u^n_{1,h}, v) = \langle f, v\rangle_1 \quad \forall v \varepsilon \mathring{V}_{1,h},$$

$$(3.10) \quad u^n_{1,h} = g^{n-1} \quad \text{on } \Gamma,$$

$$(3.11) \quad a_2(u^n_{2,h}, v) = \langle f, v\rangle_2 \quad \forall v \varepsilon \mathring{V}_{2,h},$$

$$(3.12) \quad a_2(u^n_{2,h}, \rho_{2,h}\phi) = -a_1(u^n_{1,h}, \rho_{1,h}\phi) + \langle f, \rho_{1,h}\phi\rangle_1 + \langle f, \rho_{2,h}\phi\rangle_2, \quad \forall \phi \varepsilon \Phi_h,$$

and

$$(3.13) \quad g^n := \theta_n u^n_{2,h|\Gamma} + (1-\theta_n) g^{n-1}.$$

<u>Remark 3.1</u> - In (3.13) $\{\theta_n\}$ is a sequence of positive relaxation parameters that will be determined in order to ensure and accelerate convergence of the iterative scheme (3.9)-(3.13). As we shall see in next section, these parameters can be automatically evaluated within the iterative procedure and do not require any initial guess. ■

<u>Remark 3.2</u> - The previous iterative method is inspired to a similar method that was formerly proposed in [3] and [6] for the differential problem (2.1) as well as for its numerical approximation based on the Chebyshev collocation method. A convergence analysis for both the differential and the numerical problem has been carried out in [3] for the case of a rectangular domain Ω partitioned by two rectangles.■

Before studying the convergence as $n\to\infty$ of the scheme (3.9)-(3.13), let us first note that if the sequence $\{u^n_{1,h}, u^n_{2,h}\}$ converges as $n\to\infty$, then its limit is precisely the finite element solution of (3.2). In fact, we can prove the following result.

Theorem 3.1 - Assume that there exists $\theta_{min}>0$ such that $\theta_n \geq \theta_{min}$ $\forall n\geq 1$. If the sequence $\{u^n_{1,h|\Gamma}\}$ converges as $n\to\infty$, then the whole sequence $\{u^n_{1,h}, u^n_{2,h}\}$ converges, and its limit is the finite element solution of problem (3.2), i.e.,

$$(3.14)\qquad \lim_n u^n_{1,h}=u_{h|\Omega_1}\ ,\ \lim_n u^n_{2,h}=u_{h|\Omega_2}\ .$$

■

In order to study the convergence of the sequence $\{u^n_{1,h|\Gamma}\}$ as $n\to\infty$, let us introduce the discrete-harmonic extension to Ω_i (i=1,2) of functions in Φ_h. For i=1,2 define:

$$(3.15)\qquad R_{i,h}:\Phi_h\to V_{i,h},\ a_i(R_{i,h}\phi,v)=0\ \forall v\in \overset{\circ}{V}_{i,h},\ R_{i,h}\phi=\phi \text{ on } \Gamma.$$

Define a norm in Φ_h and its associated scalar product by

$$(3.16)\qquad |||\phi|||^2=\|R_{1,h}\phi\|^2_1,\ ((\phi,\psi))=a_1(R_{1,h}\phi,R_{1,h}\psi).$$

Finally introduce the operator S from Φ_h in itself by:

$$(3.17)\qquad \psi\in\Phi_h \to S\psi=w_{2,h|\Gamma},$$

where $w_{2,h}$ is the finite element solution of the mixed (Dirichlet-Neumann) boundary value problem in Ω_2:

$$(3.18)\qquad a_2(w_{2,h},v)=0\quad \forall v\in \overset{\circ}{V}_{2,h},$$

$$(3.19)\qquad a_2(w_{2,h},\rho_{2,h}\phi)=-a_1(R_{1,h}\psi,\rho_{1,h}\phi)\quad \forall\phi\in\Phi_h.$$

(According to (3.15) we can also write $w_{2,h}=R_{2,h}S\psi$). Then, for any positive θ, we set

$$(3.20)\qquad S_\theta:\Phi_h\to\Phi_h,\ S_\theta\phi:=\theta S\phi+(1-\theta)\phi\quad \forall\phi\in\Phi_h.$$

After all these definitions, some explanation is in order. We shall prove that S_θ is a contraction (for some positive θ), that is, S_θ is a norm reducing operator:

(3.21) $\exists\, k(\theta)<1 : |||S_\theta\phi||| \le k(\theta)|||\phi||| \quad \forall\phi\in\Phi_h$.

Note that the finite element solution $\{u_{h|\Omega_1}, u_{h|\Omega_2}\}$ of problem (3.2) verifies a problem of the type (3.9)-(3.13). Of course, (3.10) has now to be intended as $u_{h|\Omega_1}=u_{h|\Omega_2}$.

Then, the iterative scheme (3.9)-(3.13), applied to the sequences $\{u^n_{1,h}-u_{h|\Omega_1}\}$, $\{u^n_{2,h}-u_{h|\Omega_2}\}$, can be interpreted in terms of S_θ to give

(3.22) $(u^{n+1}_{1,h}-u_h)_{|\Gamma}=S_\theta(u^n_{1,h|\Gamma}-u_{h|\Gamma})$.

Convergence will then follow from (3.21).
To prove (3.21) the following Lemma will be useful:

<u>Lemma 3.1</u> - <u>If</u> Σ_h <u>is a quasi-uniform</u> [2] <u>decomposition of</u> Γ, <u>then there exist two positive constants</u> C_o <u>and</u> C_1 <u>independent of</u> h <u>such that for any</u> $\phi\in\Phi_h$

(3.23) $\|R_{2,h}\phi\|_2 \le C_0\|R_{1,h}\phi\|_1 \;,\; \|R_{1,h}\phi\|_1 \le C_1\|R_{2,h}\phi\|_2$

<u>with</u> $R_{i,h}$ <u>defined in</u> (3.15) (i=1,2). ■

<u>Remark 3.3</u> - Introducing the quantities:

(3.24) $\sigma=\sup\left\{\dfrac{\|R_{1,h}\phi\|_1^2}{\|R_{2,h}\phi\|_2^2},\ \phi\in\Phi_h\right\}, \quad \tau=\mathrm{Sup}\left\{\dfrac{\|R_{2,h}\phi\|_2^2}{\|R_{1,h}\phi\|_1^2},\phi\in\Phi_h\right\}$,

it follows from (3.23) that σ and τ are bounded independently of h, since

(3.25) $C_0^{-2}\le\sigma\le C_1^2 \quad , \quad C_1^{-2}\le\tau\le C_0^2$.

As we shall see, this property will ensure that the convergence interval for the iterative scheme is independent of h. Also, the constants σ and τ can be used in numerical computations to evaluate automatically the relaxation parameters θ_n. ■

We can now prove the following theorem.

<u>Theorem 3.2</u> - <u>Under the hypotheses of Lemma</u> 3.1, <u>there exists</u> $\theta^*>0$ <u>such that</u>, $\forall$ h>0 <u>the following holds</u>:

(3.26) $\forall\theta\in(0,\theta^*)\,\exists\, k(\theta)\le 1$ s.t. $|||S_\theta\psi|||\le k(\theta)\,|||\psi||| \quad \forall\psi\in\Phi$.

Moreover, there exist θ',θ'' and k with $0<\theta'<\theta''<\theta^*$ and $k<1$ such that, for all $h>0$:

(3.27) $\forall\theta\in[\theta',\theta'']\quad k(\theta)\le k<1.$

Proof - From definitions (3.16) and (3.20) we have

(3.28) $$|||S_\theta\psi|||^2=\theta^2|||S\psi|||^2+2\theta(1-\theta)((\psi,S\psi))+(1-\theta)^2|||\psi|||^2=$$
$$=\theta^2\|R_{1,h}S\psi\|_1^2+(1-\theta)^2\|R_{1,h}\psi\|_1^2+$$
$$2\theta(1-\theta)a_1(R_{1,h}\psi,R_{1,h}S\psi)$$

Moreover we have

(3.29) $$a_1(R_{1,h}\psi,R_{1,h}S\psi)=a_1(R_{1,h}\psi,\rho_{1,h}S\psi)\qquad\text{(from (3.15))}$$
$$=-a_2(w_{2,h},\rho_{2,h}S\psi)\qquad\text{(from (3.19)}$$
$$=-a_2(R_{2,h}S\psi,R_{2,h}S\psi)\qquad\text{(from (3.18)).}$$

Using (3.29) and (3.24) in (3.28) we can write

(3.30) $$|||S_\theta\psi|||^2\le\theta^2\sigma\|R_{2,h}S\psi\|_2^2+(1-\theta)^2\|R_{1,h}\psi\|_1^2-2\theta(1-\theta)\|R_{2,h}S\psi\|_2^2.$$

With the same arguments as for (3.29) we can derive the following bounds:

(3.31) $$\frac{1}{\sqrt{\tau}}\|R_{1,h}\psi\|_1\le\|R_{2,h}S\psi\|_2\le\sqrt{\sigma}\,\|R_{1,h}\psi\|_1.$$

Hence (3.30) (for $0<\theta<1$) gives that

(3.32) $$|||S_\theta\psi|||^2\le[\theta^2\sigma^2+(1-\theta)^2-\frac{2\theta(1-\theta)}{\tau}]\|R_{1,h}\psi\|_1^2.$$

If we define

(3.33) $$k(\theta)=[\frac{\theta^2(\sigma^2\tau+\tau+2)-2\theta(\tau+1)+\tau}{\tau}]^{\frac{1}{2}},$$

we can readily see that (3.26) holds and that

(3.34) $$k(\theta)<1\quad\text{iff}\quad 0<\theta<\theta_h^*=\min\,(1,\frac{2(\tau+1)}{\sigma^2\tau+\tau+2}).$$

A consequence of (3.25) is that θ_h^* can be bounded from above and from below independently of h, so that

$\theta^{*}=\inf \theta^{*}_{h}$ is positive.

Then (3.27) easily follows from the continuity of $k(\theta)$.■

We can now conclude this section with the following convergence theorem.

Theorem 3.3 - *If the sequence* $\{\theta_n\}$ *is such that* $\theta' \leq \theta_n \leq \theta''$ $\forall n \geq 0$, *then, for each* $h>0$, *the solution* $\{u^n_{1,h}, u^n_{2,h}\}$ *of problem* (3.9)-(3.13) *converges, as* $n \to \infty$, *to* $\{u_{h|\Omega_1}, u_{h|\Omega_2}\}$, *where* u_h *is the solution of problem* (3.2). *Moreover we have*

$$(3.35) \qquad |||(u^{n+1}_{1,h}-u_h)_{|\Gamma}||| \leq k(\theta_n)\ldots k(\theta_0)\, |||(u^0_{1,h}-u_h)_{|\Gamma}|||$$

$$\leq k^{n+1}\, |||(u^0_{1,h}-u_h)_{|\Gamma}||| \ .$$

The constants θ', θ'' and k are defined in theorem 3.2.■

4. Numerical examples - We present in this section some examples of application of the iterative scheme (3.9)-(3.13) using continuous linear finite elements. The algorithm we use is based on the idea of determining a sequence of θ_n converging as quikly as possible to the optimal value of the relaxation parameter θ. By (3.32) this value is given by

$$(4.1) \qquad \theta_{opt} = \frac{\tau+1}{\sigma^2\tau+\tau+2} \ ,$$

where the constants σ and τ (given by (3.24)) are not known. However, we can build up a procedure which generates a sequence of discrete-harmonic functions on Ω_1 and Ω_2 with the same trace on Γ. This allows us to compute, at each iteration, two constants τ_n and σ_n as suggested by (3.24). Using these constants in (4.1) gives a value of θ_n to be used in our numerical scheme. We point out that the evaluation of θ_n does not require the solution of any additional problem in our algorithm, as it will be clear from the description below. With this choice of θ_n our numerical experiences show an impressive reduction of the initial error after very few iterations. We now describe in detail the algorithm.

Initialization

Let $g^0 \in \Phi_h$ be given, and let $\sigma_0=0$, $\tau_0=0$. Compute the solutions

$u^0_{1,h}$, $\tilde{u}^0_{1,h}$, $u^0_{2,h}$, $\tilde{u}^0_{2,h}$ of the following problems.

$$(4.2)\quad u^0_{1,h}\in V_{1,h}: a_1(u^0_{1,h},v)=\langle f,v\rangle_1 \quad \forall v\in \mathring{V}_{1,h},\ u^0_{1,h}=g^0 \text{ on } \Gamma$$

$$(4.3)\quad \tilde{u}^0_{1,h}\in V_{1,h}: a_1(\tilde{u}^0_{1,h},v)=0 \quad \forall v\in \mathring{V}_{1,h},\ \tilde{u}^0_{1,h}=u^0_{1,h} \text{ on } \Gamma,$$

$$(4.4)\begin{cases} u^0_{2,h}\in V_{2,h}: \quad a_2(u^0_{2,h},v)=\langle f,v\rangle_2 \quad \forall v\in \mathring{V}_{2,h}, \\ a_2(u^0_{2,h},\rho_{2,h}\phi)=-a_1(u^0_{1,h},\rho_{1,h}\phi)+\langle f,\rho_{1,h}\phi\rangle_1+\langle f,\rho_{2,h}\phi\rangle_2 \forall\phi\in\Phi_h, \end{cases}$$

$$(4.5)\quad \tilde{u}^0_{2,h}\in V_{2,h}: a_2(\tilde{u}^0_{2,h},v)=0 \quad \forall v\in \mathring{V}_{2,h},\ \tilde{u}^0_{2,h}=u^0_{2,h} \text{ on } \Gamma.$$

Note that, with the notation of section 2 we have

$$\tilde{u}^0_{1,h}=R_{1,h}(u^0_{1,h|\Gamma}) \ ,\ \tilde{u}^0_{2,h}=R_{2,h}(u^0_{2,h|\Gamma}).$$

<u>Step n (n≥1)</u>

Compute the solution $z^n_{1,h}$ of:

$$(4.6)\quad z^n_{1,h}\in V_{1,h} : a_1(z^n_{1,h},v)=0 \quad \forall v\in \mathring{V}_{1,h},\ z^n_{1,h}=\tilde{u}^{n-1}_{2,h} \text{ on } \Gamma$$

(i.e., $z^n_{1,h}=R_{1,h}(\tilde{u}^{n-1}_{2,h|\Gamma})$).

Then evaluate α_n, σ_n, τ_n and θ_n by:

$$(4.7)\quad \alpha_n=\|z^n_{1,h}\|^2_1/\|\tilde{u}^{n-1}_{2,h}\|^2_2 \ ,$$

$$(4.8)\quad \sigma_n=\max(\sigma_{n-1},\alpha_n) \ ,\ \tau_n=\max(\tau_{n-1},1/\alpha_n),$$

$$(4.9)\quad \theta_n=(\tau_n+1)/(\sigma_n^2\tau_n+\tau_n+2),$$

and simply take

$$(4.10)\quad u^n_{1,h}=u^{n-1}_{1,h}+\theta_n(z^n_{1,h}-\tilde{u}^{n-1}_{1,h}),$$

$$(4.11)\quad \tilde{u}^n_{1,h}=\tilde{u}^{n-1}_{1,h}+\theta_n(z^n_{1,h}-\tilde{u}^{n-1}_{1,h}).$$

Compute then the solution $\tilde{u}^n_{2,h}$ of:

$$(4.12)\quad \begin{cases} \tilde{u}^n_{2,h}\in V_{2,h}:\ a_2(\tilde{u}^n_{2,h},v)=0 \quad \forall v\in \overset{\circ}{V}_{2,h} \\ a_2(\tilde{u}^n_{2,h},\rho_{2,h}\phi)=-a_1(u^n_{1,h},\rho_{1,h}\phi)+\langle f,\rho_{1,h}\phi\rangle_1 \quad \forall\phi\in\Phi_h \end{cases}$$

(Note that again $\tilde{u}^n_{2,h}=R_{2,h}(\tilde{u}^n_{2,h|\Gamma})$). Then take

$$(4.13)\quad u^n_{2,h}=u^{n-1}_{2,h}+\tilde{u}^n_{2,h}-\tilde{u}^{n-1}_{2,h}\ .$$

- <u>Stop if</u>

$$(4.14)\quad \|u^n_{1,h}-u^{n-1}_{1,h}\|_1+\|u^n_{2,h}-u^{n-1}_{2,h}\|_2 \leq \varepsilon$$

(or any other reasonable stopping criterion one may prefer), otherwise go back to (4.6).

To sum up, the initialization of the algorithm requires the solution of the 4 problems (4.2)-(4.5), while at the generic iteration only the 2 problems (4.6) and (4.12) need to be solved . The solutions $z^n_{1,h}$ and $u^n_{2,h}$ permit to derive, without additional cost, the relaxation parameter θ_n and the actual sequences $u^n_{1,h}$,$u^n_{2,h}$.

We give now some examples of application of algorithm (4.2)-(4.14) to the model problem:

$$(4.15)\quad \begin{cases} -\Delta u+\lambda u=\lambda & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

whose solution, as well as the one of its finite element approximation, is given by $u\equiv u_h\equiv 1$. Starting from a vector g^0 randomly chosen, the stopping criterion for the algorithm was:

$$(4.16)\quad \|e^n_1\|_{L^\infty(\Omega_1)}+\|e^n_2\|_{L^\infty(\Omega_2)} \leq 10^{-5}\{\|e^0_1\|_{L^\infty(\Omega_1)}+\|e^0_2\|_{L^\infty(\Omega_2)}\}$$

where we have defined:

$$(4.17)\quad e^n_1:=u^n_{1,h}-u_{h|\Omega_1}\ ,\ e^n_2:=u^n_{2,h}-u_{h|\Omega_2}\quad n\geq 0.$$

Tables (4.1) and (4.2) report the results obtained for the two domains of figures (4.1) and (4.2) respectively and two different values of $\lambda(\lambda=0,100)$. For each case, we report the total number of unknowns (D.O.F.), the total number (NIT) of iterations to satisfy (4.16), and finally the average reduction factor (ERF) defined by:

$$(4.18)\quad \text{ERF}:=\max_{i=1,2}\ \{\|e^n_i\|_{L^\infty(\Omega_i)}/\|e^0_i\|_{L^\infty(\Omega_i)}\}^{1/n}\quad \text{for } n=\text{NIT}.$$

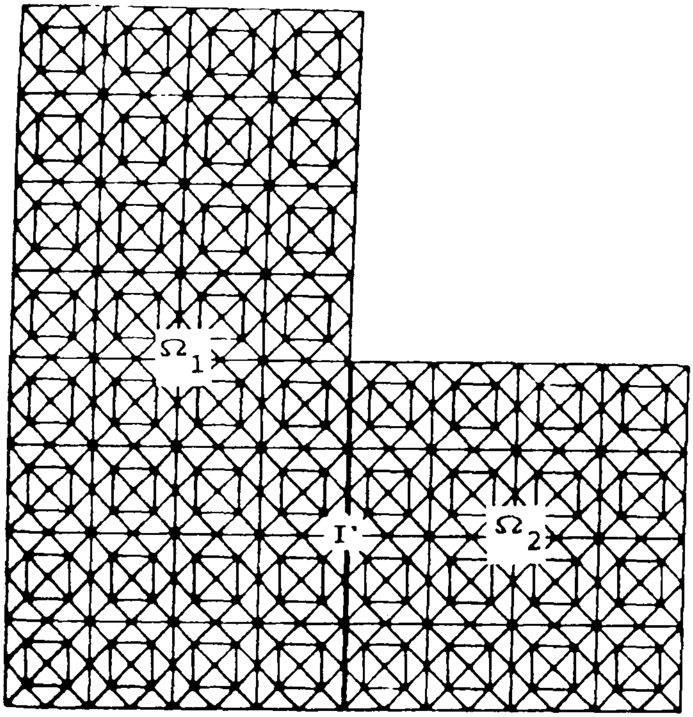

Fig. 4.1: $\Omega=\{(x,y)\in\mathbf{R}^2: 0<y<2 \text{ if } 0<x<1, \text{ and } 0<y<1 \text{ if } 1\leq x<2\}$.

Example of a finite element triangulation: 768 triangles and 355 nodes.

D.O.F. / λ	81		355		1475	
	NIT	E.R.F.	NIT	E.R.F.	NIT	E.R.F.
0	3	0.042	4	0.035	4	0.048
100	2	0.0002	2	0.009	3	0.006

Table 4.1: Numerical results for problem (4.15) on the domain of Fig. 4.1 and for three different finite element triangulations.

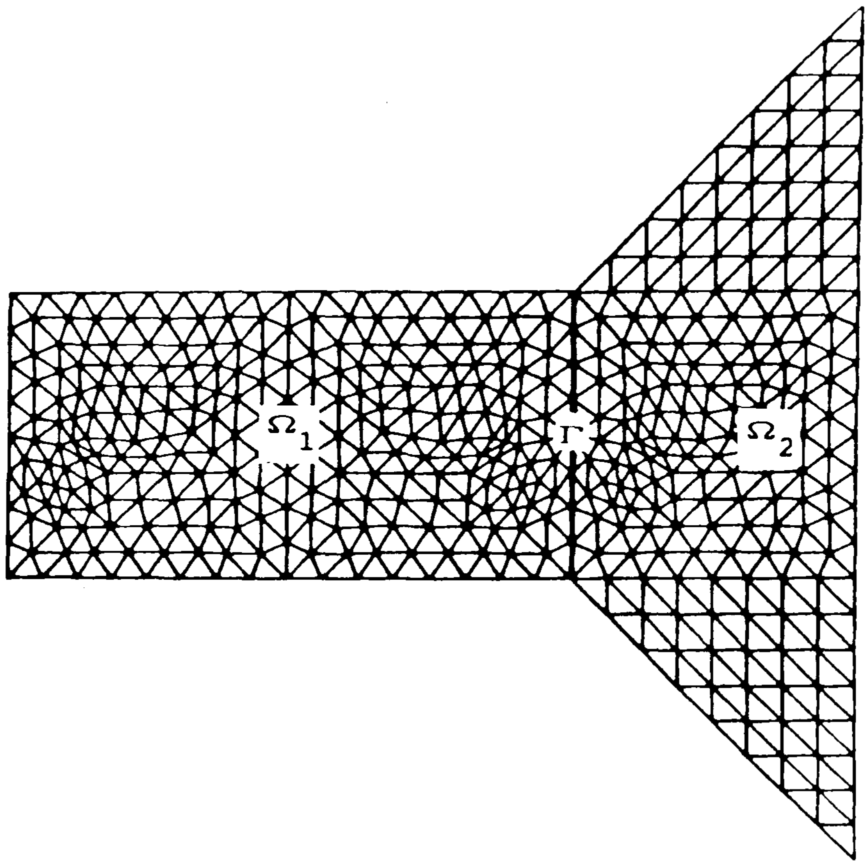

Fig. 4.2: $\Omega\{(x,y)\in\mathbb{R}^2$: $1<y<2$ if $0<x<2$, $3-x<y<x$ if $2\leq x<3\}$.
Example of a finite element triangulation:
824 triangles and 455 nodes.

D.O.F. / λ	128		455		1731	
	NIT	E.R.F.	NIT	E.R.F.	NIT.	E.R.F.
0	4	0.0338	5	0.0617	5	0.0793
100	2	0.0025	2	0.0014	3	0.0074

Table 4.2: Numerical results for the problem(4.15)on the domain of Fig. 4.2 and for three different finite element tringulations.

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