

Domain Decomposition and Mixed Finite Element Methods for Elliptic Problems

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Abstract. In this paper we describe the numerical solution of elliptic problems with nonconstant coefficients by domain decomposition methods based on a mixed formulation and mixed finite element approximations. Two families of conjugate gradient algorithms taking advantage of domain decomposition will be discussed and their performance will be evaluated through numerical experiments, some of them concerning practical situations arising from flow in porous media.

0. Introduction. These last years have seen the strong emergence of solution methods for partial differential equations based on the concept of domain decomposition. Indeed this approach is not new since the Schwarz alternating method to solve some class of elliptic problems goes back to the last century. However this technique, almost forgotten for a long time, is enjoying revival very likely due to the development of parallel computers and multiprocessor supercomputers. Giving reference to all the publications dealing with domain decomposition for partial differential equations has become an impossible task due to the active research presently done in the United States, Western Europe, Japan and U.S.S.R. For this reason we advise the interested reader to consult the other papers in the *Proceedings* and the references therein.

From a technical point of view domain decomposition techniques considered so far have been dealing with finite difference, conforming finite elements and spectral methods. To the knowledge of the two authors they have been the first to consider the combination of domain decomposition with mixed finite element [1, 2]. This approach, compared to more traditional ones founded on conforming finite difference or finite element method, has no problem in treating the vertex difficulties associated with the box decompositions [3, 4]. Another advantage is that it seems ideally suited to handling problems with highly discontinuous coefficients, since it contains built in harmonic averaging of

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coefficients which is very close to the one usually associated to homogenization techniques [5]. In this paper we would like to discuss domain decomposition for solving elliptic problems with nonconstant coefficients based on mixed formulation. We shall consider two classes of methods which can be viewed as duals of each other. To each we shall associate conjugate gradient algorithms, first for the continuous problem and then for the finite dimensional one obtained through mixed finite element discretizations. The numerical implementation of these algorithms will be discussed in detail and the possibilities of these methods will be illustrated by numerical experiments, some of them related to the numerical solution of the pressure equation originating from the mathematical models described by flow in porous media.

1. The Model Problem. Let Ω be a bounded domain of R^N . We consider on Ω the following *Neumann problem*:

$$\begin{cases} -\nabla \cdot (A \nabla u) = f & \text{in } \Omega, \\ (A \nabla u) \cdot \mathbf{n} = g & \text{on } \partial\Omega (= \Gamma); \end{cases} \tag{1.1}$$

\mathbf{n} : unit outward normal vector. For the above problem, in order to have a solution (defined within an arbitrary constant) we need to have the *compatibility condition*

$$\int_{\Omega} f \, dx + \int_{\Gamma} g \, d\Gamma = 0. \tag{1.2}$$

We have been considering directly the pure Neumann problem since it is the one that is the most difficult and physical.

2. A Mixed Variational Formulation of Problem (1.1). Define now \mathbf{p} by

$$\mathbf{p} = A \nabla u; \tag{2.1}$$

we have then

$$\nabla \cdot \mathbf{p} + f = 0, \tag{2.2}$$

$$\nabla u = A^{-1} \mathbf{p}. \tag{2.3}$$

Multiplying (2.2) and (2.3) by v and \mathbf{q} , respectively, we obtain

$$\int_{\Omega} (\nabla \cdot \mathbf{p} + f)v \, dx = 0, \quad \forall v \in L^2(\Omega), \tag{2.4}$$

$$\int_{\Omega} (A^{-1} \mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_0, \tag{2.5}$$

where

$$P_0 = \{\mathbf{q} \mid \mathbf{q} \in H(\Omega; \text{div}), \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}. \tag{2.6}$$

Take $f \in L^2(\Omega)$, $g \in H^{-1/2}(\Gamma)$, and A symmetric such that

$$A \in (L^\infty(\Omega))^{N \times N}, \tag{2.7}$$

$$A(x)\mathbf{q} \cdot \mathbf{q} \geq \alpha |\mathbf{q}|^2, \quad \forall \mathbf{q} \in R^N, \text{ a.e. on } \Omega, \tag{2.8}$$

with, in (2.8), α a *positive* constant. If (1.2) holds, (1.1) has a unique solution in $H^1(\Omega)/R$, implying the uniqueness of \mathbf{p} . An alternative formulation of (1.1) is provided by

$$\left\{ \begin{array}{l} \text{Find } u \in L^2(\Omega), \mathbf{p} \in H(\Omega; \text{div}) \text{ such that} \\ \mathbf{p} \cdot \mathbf{n} = g \text{ on } \Gamma, \\ \int_{\Omega} (\nabla \cdot \mathbf{p} + f)v \, dx = 0, \forall v \in L^2(\Omega), \\ \int_{\Omega} (A^{-1}\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) \, dx = 0, \forall \mathbf{q} \in P_0. \end{array} \right.$$

The L^p and Sobolev spaces used above provide a convenient setup to study the various decomposition principles and related algorithms described below. For details about Sobolev spaces see e.g. [6-9].

We shall consider in detail the first class of domain decomposition methods and the associated algorithms, and then later more briefly, since there is much similarity between them, a second class of methods.

3. Solving (1.1), via (2.9), by Domain Decomposition.

3.1. *An equivalent formulation of problem (2.9) using domain decomposition.*

For simplicity, we consider a 2-domain decomposition like the ones depicted below. If we denote by $\{u_i, \mathbf{p}_i\}$ the restriction of $\{u, \mathbf{p}\}$ to Ω_i , there is clearly equivalence between (2.9) and

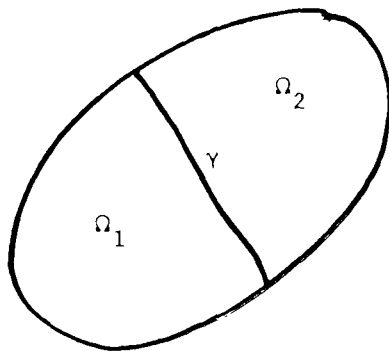


Figure 3.1 (a)

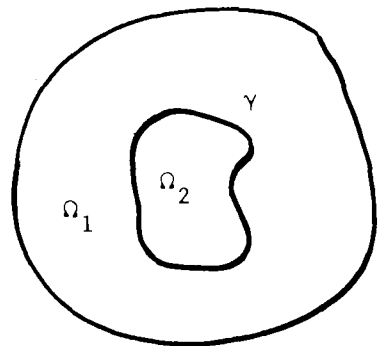


Figure 3.1 (b)

$$\begin{cases} \int_{\Omega_i} (\nabla \cdot \mathbf{p}_i + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q}_i + u_i \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i\circ}, \quad \forall i = 1, 2, \end{cases} \quad (3.1)$$

$$\mathbf{p}_i \cdot \mathbf{n}_i = g \quad \text{on } \Gamma \cap \partial\Omega_i, \quad \forall i = 1, 2, \quad (3.2)$$

$$\sum_{i=1}^2 \mathbf{p}_i \cdot \mathbf{n}_i = 0 \quad \text{on } \gamma, \quad (3.3)$$

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q} + u_i \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_\circ, \quad (3.4)$$

with

$$P_{i\circ} = \{\mathbf{q}_i \mid \mathbf{q}_i \in H(\Omega_i; \text{div}), \mathbf{q}_i \cdot \mathbf{n}_i = 0 \text{ on } \partial\Omega_i\}.$$

Since $P_\circ = P_{1\circ} \oplus P_{2\circ} \oplus P_{\gamma\circ}$ (where $P_{\gamma\circ}$ is a complementary subspace of $P_{1\circ} \oplus P_{2\circ}$ in P_\circ), it follows from (3.1), (3.4) that (3.4) can be replaced by the less demanding condition

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q} + u_i \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_{\gamma\circ}. \quad (3.5)$$

In addition to (3.1)-(3.5), $\{u_i, \mathbf{p}_i\}$ has to satisfy

$$\int_{\Omega_i} f \, dx + \int_{\partial\Omega_i \cap \Gamma} g \, d\Gamma + \int_{\gamma} \mathbf{p}_i \cdot \mathbf{n}_i \, d\gamma = 0. \quad (3.6)$$

3.2. Principle of iterative methods solving (2.9) via (3.1)-(3.6).

(i) Consider $\lambda^\circ \in H(\Omega; \text{div})$ and satisfying, $\forall i = 1, 2$,

$$\lambda^\circ \cdot \mathbf{n} \big|_{\gamma} = \Lambda^\circ \cdot \mathbf{n} \big|_{\gamma} \quad \text{where } \Lambda^\circ \in H(\Omega; \text{div}), \Lambda^\circ \cdot \mathbf{n} = g \quad \text{on } \Gamma, \quad (3.7)$$

$$\int_{\Omega_i} f \, dx + \int_{\partial\Omega_i \cap \Gamma} g \, d\Gamma + \int_{\gamma} \lambda^\circ \cdot \mathbf{n}_i \, d\gamma = 0. \quad (3.8)$$

(ii) Solve for $i = 1, 2$,

$$\begin{cases} \int_{\Omega_i} (\nabla \cdot \mathbf{p}_i^\circ + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \mathbf{p}_i^\circ \cdot \mathbf{q}_i + u_i^\circ \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i\circ}, \end{cases} \quad (3.9)$$

$$\mathbf{p}_i^\circ \cdot \mathbf{n}_i = g \quad \text{on } \Gamma \cap \partial\Omega_i, \quad \mathbf{p}_i^\circ \cdot \mathbf{n}_i = \lambda^\circ \cdot \mathbf{n}_i \quad \text{on } \gamma. \quad (3.10)$$

Since u_i is defined only within an arbitrary constant, the constants in u_1^o and u_2^o are adjusted in such a way that

$$\int_{\Omega_1} u_1^o \, dx = 0, \tag{3.11}$$

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i^o \cdot \boldsymbol{\pi}_o + u_i^o \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0, \tag{3.12}$$

where $\boldsymbol{\pi}_o \in P_{\gamma_o}$, but $\int_{\gamma} \boldsymbol{\pi}_o \cdot \mathbf{n} \, d\gamma \neq 0$.

(iii) Define now

$$P_{\gamma_o}^o = \{ \boldsymbol{\mu} \mid \boldsymbol{\mu} \in P_{\gamma_o}, \int_{\gamma} \boldsymbol{\mu} \cdot \mathbf{n} \, d\gamma = 0 \}.$$

If

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i^o \cdot \mathbf{q} + u_i^o \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_{\gamma_o}^o, \tag{3.13}$$

and since (3.12) already holds, it follows from (3.5) that $u_i^o = u_i$, $\mathbf{p}_i^o = \mathbf{p}_i$. If (3.13) does not hold we have to correct λ^o . Such a correction can be done through a steepest descent or a *conjugate gradient* algorithm as we shall see below.

3.3 Solving (3.1)-(3.6) via a variational problem in $P_{\gamma_o}^o$.

First of all, let us define a bilinear form over $P_{\gamma_o}^o \times P_{\gamma_o}^o$. If we denote it by $a(\cdot, \cdot)$, $a(\cdot, \cdot)$ is defined as follows:

Consider $\boldsymbol{\mu} \in P_{\gamma_o}^o$; we associate to $\boldsymbol{\mu}$, $\mathbf{p}_i(\boldsymbol{\mu})$ and $u_i(\boldsymbol{\mu})$ by solving

$$\begin{cases} \int_{\Omega_i} \nabla \cdot \mathbf{p}_i(\boldsymbol{\mu}) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{q}_i + u_i(\boldsymbol{\mu}) \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i_o}, \end{cases} \tag{3.14}$$

$$\mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_i = 0 \text{ on } \Gamma \cap \partial \Omega_i, \quad \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_i = \boldsymbol{\mu} \cdot \mathbf{n}_i \text{ on } \gamma. \tag{3.15}$$

Since $\int_{\partial \Omega_i} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_i \, d\Gamma_i = 0$, the above problem is well posed in $H(\Omega_i, \text{div}) \times L^2(\Omega_i)/R$.

Let us adjust now $u_1(\boldsymbol{\mu})$ and $u_2(\boldsymbol{\mu})$ by

$$\int_{\Omega_1} u_1(\boldsymbol{\mu}) \, dx = 0, \quad \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \boldsymbol{\pi}_o + u_i(\boldsymbol{\mu}) \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0. \tag{3.16}$$

Finally, we define $a(\cdot, \cdot)$ by

$$\begin{cases} a(\boldsymbol{\mu}, \boldsymbol{\mu}') = \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}' + u_i(\boldsymbol{\mu}) \nabla \cdot \boldsymbol{\mu}') \, dx, \\ \forall \boldsymbol{\mu}' \in P_{\gamma_0}^o. \end{cases} \quad (3.17)$$

THEOREM 3.1. *The bilinear form $a(\cdot, \cdot)$ is symmetric and positive semi definite over $P_{\gamma_0}^o$; it is moreover elliptic for the norm induced by $H(\Omega; \text{div})$ over the quotient space $P_{\gamma_0}^o/\hat{R}$, where \hat{R} is the equivalence relation defined by*

$$\boldsymbol{\mu} \hat{R} \boldsymbol{\mu}' \Leftrightarrow (\boldsymbol{\mu} - \boldsymbol{\mu}') \cdot \mathbf{n} = 0 \quad \text{on } \gamma.$$

Proof.

(1) *Symmetry of $a(\cdot, \cdot)$:* Taking $\boldsymbol{\mu} = \boldsymbol{\mu}'$ in (3.14), (3.15), (3.16) define $\mathbf{p}_i(\boldsymbol{\mu}')$ and $u_i(\boldsymbol{\mu}')$. If we denote by $\mathbf{p}(\boldsymbol{\mu}')$ the element of $(L^2(\Omega))^N$ such that

$$\mathbf{p}(\boldsymbol{\mu}')|_{\Omega_i} = \mathbf{p}_i(\boldsymbol{\mu}')$$

we have that

$$\mathbf{p}(\boldsymbol{\mu}') \in P_o,$$

and also that

$$\mathbf{p}(\boldsymbol{\mu}') = \sum_{i=1}^2 \mathbf{p}_{i_0}(\boldsymbol{\mu}') + \boldsymbol{\mu}' \quad (3.18)$$

with $\mathbf{p}_{i_0}(\boldsymbol{\mu}') \in P_{i_0}$.

It follows from (3.14), (3.17), (3.18) that

$$\begin{cases} a(\boldsymbol{\mu}, \boldsymbol{\mu}') = \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{p}_i(\boldsymbol{\mu}') + u_i(\boldsymbol{\mu}) \nabla \cdot \mathbf{p}_i(\boldsymbol{\mu}')) \, dx \\ - \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{p}_{i_0}(\boldsymbol{\mu}') + u_i(\boldsymbol{\mu}) \nabla \cdot \mathbf{p}_{i_0}(\boldsymbol{\mu}')) \, dx. \end{cases} \quad (3.19)$$

The second term in the right hand side of (3.19) vanishes from (3.14); on the other hand, it follows from (3.14) that $\nabla \cdot \mathbf{p}_i(\boldsymbol{\mu}') = 0$. Therefore, (3.19) reduces to

$$a(\boldsymbol{\mu}, \boldsymbol{\mu}') = \sum_{i=1}^2 \int_{\Omega_i} A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{p}_i(\boldsymbol{\mu}') \, dx. \quad (3.20)$$

The symmetry of $a(\cdot, \cdot)$ is obvious from (3.20).

(2) *Positivity of $a(\cdot, \cdot)$:* It follows from (3.20) that

$$a(\boldsymbol{\mu}, \boldsymbol{\mu}) = \sum_{i=1}^2 \int_{\Omega_i} A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{p}_i(\boldsymbol{\mu}) \, dx.$$

From the properties of A we have

$$a(\boldsymbol{\mu}, \boldsymbol{\mu}) \geq 0, \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^o;$$

moreover, if $a(\boldsymbol{\mu}, \boldsymbol{\mu})=0$ then $\mathbf{p}_i(\boldsymbol{\mu})=0$, implying in turn that $\mathbf{p}(\boldsymbol{\mu})=0$ and therefore that $\boldsymbol{\mu} \cdot \mathbf{n}=0$. Hence, $a(\cdot, \cdot)$ is positive definite over $P_{\gamma_0}^o/\hat{R}$.

(3) $P_{\gamma_0}^o$ - ellipticity of $a(\cdot, \cdot)$: It is easily shown that

$$\boldsymbol{\mu} \rightarrow \mathbf{p}(\boldsymbol{\mu})$$

is an isomorphism from $P_{\gamma_0}^o/\hat{R}$ onto $\{\boldsymbol{\mu} \mid \boldsymbol{\mu} \in P_o, \nabla \cdot \boldsymbol{\mu}=0\}$; the ellipticity of $a(\cdot, \cdot)$ for the $H(\Omega; \text{div})$ -norm follows then easily from

$$a(\boldsymbol{\mu}, \boldsymbol{\mu}) = \int_{\Omega} A^{-1} \mathbf{p}(\boldsymbol{\mu}) \cdot \mathbf{p}(\boldsymbol{\mu}) \, dx$$

and from the fact that $\nabla \cdot \mathbf{p}(\boldsymbol{\mu})=0$.

From the above result, it is not too difficult to interpret (3.1)-(3.6) as a *linear variational problem* in $P_{\gamma_0}^o$. To formulate this latter problem consider $\boldsymbol{\Lambda}_o \in H(\Omega; \text{div})$ such that

$$\boldsymbol{\Lambda}_o \cdot \mathbf{n} = g \quad \text{on } \Gamma, \tag{3.21}$$

$$\int_{\Omega_i} f \, dx + \int_{\Gamma \cap \partial \Omega_i} g \, d\Gamma + \int_{\Gamma} \boldsymbol{\Lambda}_o \cdot \mathbf{n}_i \, d\gamma = 0; \tag{3.22}$$

solve then, for $i = 1, 2$,

$$\left\{ \begin{array}{l} \int_{\Omega_i} (\nabla \cdot \mathbf{p}_{oi} + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \mathbf{p}_{oi} \cdot \mathbf{q}_i + u_{oi} \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{io}, \end{array} \right. \tag{3.23}$$

$$\mathbf{p}_{oi} \cdot \mathbf{n}_i = g \quad \text{on } \Gamma \cap \partial \Omega_i, \quad \mathbf{p}_{oi} \cdot \mathbf{n}_i = \boldsymbol{\Lambda}_o \cdot \mathbf{n}_i \quad \text{on } \gamma. \tag{3.24}$$

The constants for u_{oi} are adjusted as follows:

$$\int_{\Omega_1} u_{o1} \, dx = 0, \tag{3.25}$$

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_{oi} \cdot \boldsymbol{\pi}_o + u_{oi} \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0. \tag{3.26}$$

Let us denote by \mathbf{p}_o this element of $H(\Omega; \text{div})$ such that $\mathbf{p}_o|_{\Omega_i} = \mathbf{p}_{oi}$. If we define now $\bar{\mathbf{p}}$ by

$$\bar{\mathbf{p}} = \mathbf{p} - \mathbf{p}_o \tag{3.27}$$

we clearly have

$$\bar{\mathbf{p}} \in P_o. \quad (3.28)$$

Denoting $\bar{\lambda} \in P_{\gamma_o}$ as the component of $\bar{\mathbf{p}}$ in the decomposition $P_o = P_{1_o} \oplus P_{2_o} \oplus P_{\gamma_o}$ we have from (3.6), (3.22), (3.24) that

$$\int_{\gamma} \bar{\lambda} \cdot \mathbf{n}_i \, d\gamma = 0, \quad \text{i.e. } \bar{\lambda} \in P_{\gamma_o}^o; \quad (3.29)$$

define similarly \bar{u}_i by $\bar{u}_i = u_i - u_{oi}$. We have then

$$\begin{cases} \int_{\Omega_i} \nabla \cdot \bar{\mathbf{p}}_i v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \bar{\mathbf{p}}_i \cdot \mathbf{q}_i + \bar{u}_i \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i_o}, \end{cases} \quad (3.30)$$

$$\bar{\mathbf{p}}_i \cdot \mathbf{n}_i = 0 \text{ on } \partial\Omega_i \cap \Gamma, \quad \bar{\mathbf{p}}_i \cdot \mathbf{n}_i = \bar{\lambda} \cdot \mathbf{n}_i \text{ on } \gamma, \quad (3.31)$$

$$\int_{\Omega_1} \bar{u}_1 \, dx = 0, \quad (3.32)$$

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \bar{\mathbf{p}}_i \cdot \boldsymbol{\pi}_o + \bar{u}_i \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0. \quad (3.33)$$

It follows from (3.5) that

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \boldsymbol{\mu} + u_i \nabla \cdot \boldsymbol{\mu}) \, dx = 0, \quad \forall \boldsymbol{\mu} \in P_{\gamma_o}^o. \quad (3.34)$$

From the definition of $\bar{\mathbf{p}}_i, \bar{u}_i$ and from (3.34) we obtain

$$\begin{cases} \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \bar{\mathbf{p}}_i \cdot \boldsymbol{\mu} + \bar{u}_i \nabla \cdot \boldsymbol{\mu}) \, dx = \\ - \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_{oi} \cdot \boldsymbol{\mu} + u_{oi} \nabla \cdot \boldsymbol{\mu}) \, dx, \quad \forall \boldsymbol{\mu} \in P_{\gamma_o}^o. \end{cases} \quad (3.35)$$

It follows then from (3.17) and (3.29) that $\bar{\lambda}$ is the unique solution of the linear variational equation

$$\begin{cases} \text{Find } \bar{\lambda} \in P_{\gamma_o}^o \text{ such that} \\ a(\bar{\lambda}, \boldsymbol{\mu}) = - \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_{oi} \cdot \boldsymbol{\mu} + u_{oi} \nabla \cdot \boldsymbol{\mu}) \, dx, \quad \forall \boldsymbol{\mu} \in P_{\gamma_o}^o. \end{cases} \quad (E)$$

4. Iterative Solution of (E) by Conjugate Gradient Algorithms.

4.1. *Generalities on the conjugate gradient solution of linear variational problems.*

Consider the following problem

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) = L(v), \quad \forall v \in V \end{cases} \quad (\text{P})$$

where:

- (1) V is an Hilbert space for (\cdot, \cdot) and $\|\cdot\|$,
- (2) $a: V \times V \rightarrow R$ is bilinear, continuous, V -elliptic, and is also *symmetric*,
- (3) $L: V \rightarrow R$ is linear and continuous.

With the above hypotheses, problem (P) has a *unique* solution.

Description of a conjugate gradient algorithm for solving (P):

$$u^0 \in V, \quad \text{given}; \quad (4.1)$$

$$\begin{cases} \text{Find } g^0 \in V \text{ such that} \\ (g^0, v) = a(u^0, v) - L(v), \quad \forall v \in V; \end{cases} \quad (4.2)$$

if $\|g^0\| \leq \epsilon_0$, with ϵ_0 "small", then $u^0 \simeq u$; if the contrary holds, then set

$$w^0 = g^0. \quad (4.3)$$

For $n \geq 0$, assuming that u^n, g^n, w^n are known compute

$$\rho_n = \frac{(g^n, w^n)}{a(w^n, w^n)} = \left\{ \frac{\|g^n\|^2}{a(w^n, w^n)} \right\}, \quad (4.4)$$

$$u^{n+1} = u^n - \rho_n w^n, \quad (4.5)$$

$$\begin{cases} g^{n+1} \in V, \\ (g^{n+1}, v) = (g^n, v) - \rho_n a(w^n, v), \quad \forall v \in V. \end{cases} \quad (4.6)$$

$$\text{If } \frac{\|g^{n+1}\|}{\|g^0\|} \leq \epsilon_1 \text{ then } u \simeq u^{n+1}; \text{ if not go to (4.8)} \quad (4.7)$$

$$\gamma_n = \frac{\|g^{n+1}\|^2}{\|g^n\|^2}, \quad (4.8)$$

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \quad (4.9)$$

Do $n = n + 1$ and go to (4.4).

4.2. *Application to the Solution of Problem (E).*

We shall equip $P_{\gamma_0}^{\circ}$ with the L^2 scalar product

$$(\boldsymbol{\mu}, \boldsymbol{\mu}') = \int_{\Omega} A^{-1} \boldsymbol{\mu} \cdot \boldsymbol{\mu}' \, dx, \quad \forall \boldsymbol{\mu}, \boldsymbol{\mu}' \in P_{\gamma_0}^{\circ}.$$

Unless $\nabla \cdot \boldsymbol{\mu} = 0$, $\nabla \cdot \boldsymbol{\mu}' = 0$ the above scalar product is not equivalent to the one induced by $H(\Omega; \text{div})$.

The following conjugate gradient algorithm is partly formal, but will make sense for the mixed finite element variants of problem (2.9). It follows then from (4.1)-(4.9) that a conjugate gradient algorithm for solving (E) is provided by the following method:

Step 0: Initialization.

$$\left\{ \begin{array}{l} \text{Consider } \boldsymbol{\Lambda}_o \in H(\Omega; \text{div}) \text{ such that} \\ \boldsymbol{\Lambda}_o \cdot \mathbf{n} = g \text{ over } \Gamma, \\ \int_{\Omega_i} f \, dx + \int_{\partial\Omega_i \cap \Gamma} g \, d\Gamma + \int_{\gamma} \boldsymbol{\Lambda}_o \cdot \mathbf{n}_i \, d\gamma = 0, \quad \forall i = 1, 2; \end{array} \right. \quad (4.10)$$

solve then, $\forall i = 1, 2$,

$$\left\{ \begin{array}{l} \int_{\Omega_i} (\nabla \cdot \mathbf{p}_i^{\circ} + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \mathbf{p}_i^{\circ} \cdot \mathbf{q}_i + u_i^{\circ} \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i_0}, \end{array} \right. \quad (4.11)$$

$$\mathbf{p}_i^{\circ} \cdot \mathbf{n}_i = g \text{ on } \partial\Omega_i \cap \Gamma, \quad \mathbf{p}_i^{\circ} \cdot \mathbf{n}_i = \boldsymbol{\Lambda}_o \cdot \mathbf{n}_i \text{ on } \gamma, \quad (4.12)$$

with

$$\int_{\Omega_1} u_1^{\circ} \, dx = 0, \quad \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i^{\circ} \cdot \boldsymbol{\pi}_o + u_i^{\circ} \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0. \quad (4.13)$$

Define then \mathbf{g}° by

$$\left\{ \begin{array}{l} \mathbf{g}^{\circ} \in P_{\gamma_0}^{\circ}, \\ \int_{\Omega} A^{-1} \mathbf{g}^{\circ} \cdot \boldsymbol{\mu} \, dx = \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i^{\circ} \cdot \boldsymbol{\mu} + u_i^{\circ} \nabla \cdot \boldsymbol{\mu}) \, dx, \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^{\circ}; \end{array} \right. \quad (4.14)$$

if $\mathbf{g}^{\circ} = 0$ (or is small) then $\mathbf{p}_i = \mathbf{p}_i^{\circ}$, $u_i = u_i^{\circ}$; if not set

$$\mathbf{w}^o = \mathbf{g}^o . \quad (4.15)$$

For $n \geq 0$, suppose that $\mathbf{p}_i^n, u_i^n, \mathbf{g}^n, \mathbf{w}^n$ are known; we compute then $\mathbf{p}_i^{n+1}, u_i^{n+1}, \mathbf{g}^{n+1}, \mathbf{w}^{n+1}$ as follows:

Step 1: Descent.

Solve the mixed problems

$$\begin{cases} \int_{\Omega_i} \nabla \cdot \delta \mathbf{p}_i^n v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \\ \int_{\Omega_i} (A^{-1} \delta \mathbf{p}_i^n \cdot \mathbf{q}_i + \delta u_i^n \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i0}, \end{cases} \quad (4.16)$$

with

$$\delta \mathbf{p}_i^n \cdot \mathbf{n}_i = 0 \text{ on } \delta \Omega_i \cap \Gamma, \quad \delta \mathbf{p}_i^n \cdot \mathbf{n}_i = \mathbf{w}^n \cdot \mathbf{n}_i \text{ on } \gamma, \quad (4.17)$$

$$\int_{\Omega_1} \delta u_1^n \, dx = 0, \quad \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \delta \mathbf{p}_i^n \cdot \boldsymbol{\pi}_o + \delta u_i^n \nabla \cdot \boldsymbol{\pi}_o) \, dx = 0 . \quad (4.18)$$

Using the fundamental relation

$$a(\mathbf{w}^n, \boldsymbol{\mu}) = \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \delta \mathbf{p}_i^n \cdot \boldsymbol{\mu} + \delta u_i^n \nabla \cdot \boldsymbol{\mu}) \, dx; \quad \forall \boldsymbol{\mu} \in P_{\gamma o}^o, \quad (4.19)$$

compute

$$\rho_n = \frac{\int_{\Omega} A^{-1} \mathbf{g}^n \cdot \mathbf{g}^n \, dx}{a(\mathbf{w}^n, \mathbf{w}^n)}, \quad (4.20)$$

and then

$$\begin{cases} \mathbf{p}_i^{n+1} = \mathbf{p}_i^n - \rho_n \delta \mathbf{p}_i^n, \\ \mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \rho_n \delta \mathbf{u}_i^n. \end{cases} \quad (4.21)$$

Using again (4.19), solve the linear variational problem

$$\begin{cases} \text{Find } \mathbf{g}^{n+1} \in P_{\gamma_0}^o \text{ such that} \\ \int_{\Omega} A^{-1} \mathbf{g}^{n+1} \cdot \boldsymbol{\mu} \, dx = \int_{\Omega} A^{-1} \mathbf{g}^n \cdot \boldsymbol{\mu} \, dx - \rho_n a(\mathbf{w}^n, \boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^o. \end{cases} \quad (4.22)$$

If $\mathbf{g}^{n+1} = 0$ (or is "small") then $\mathbf{u}_i^{n+1} = \mathbf{u}_i$, $\mathbf{p}_i^{n+1} = \mathbf{p}_i$; if not, define

Step 2: New descent direction.

$$\gamma_n = \frac{\int_{\Omega} A^{-1} \mathbf{g}^{n+1} \cdot \mathbf{g}^{n+1} \, dx}{\int_{\Omega} A^{-1} \mathbf{g}^n \cdot \mathbf{g}^n \, dx}, \quad (4.23)$$

and finally

$$\mathbf{w}^{n+1} = \mathbf{g}^{n+1} + \gamma_n \mathbf{w}^n. \quad (4.24)$$

Do $n = n+1$ and go to (4.16)-(4.18).

In view of practical implementations of algorithm (4.10)-(4.24) we define $\boldsymbol{\pi}_o$ by

$$\begin{cases} \boldsymbol{\pi}_o \in P_{\gamma_0} \\ \int_{\Omega} A^{-1} \boldsymbol{\pi}_o \cdot \mathbf{q} \, dx = \int_{\gamma} \mathbf{n}_{12} \cdot \mathbf{q} \, d\gamma, \quad \forall \mathbf{q} \in P_{\gamma_0}, \end{cases} \quad (4.25)$$

with \mathbf{n}_{12} as below (see Figure 4.1). From this definition, we clearly have

$$P_{\gamma_0} = P_{\gamma_0}^o \oplus \{\mathbf{q} \mid \mathbf{q} = t \boldsymbol{\pi}_o, \quad t \in R\},$$

moreover $\{\mathbf{q} \mid \mathbf{q} = t \boldsymbol{\pi}_o, \quad t \in R\} = (P_{\gamma_0}^o)^{\perp}$ for the scalar product $\int_{\Omega} A^{-1} \mathbf{q} \cdot \mathbf{q}' \, dx$.

Consider now the solution of

$$\begin{cases} \boldsymbol{\lambda} \in P_{\gamma_0}^o \\ \int_{\Omega} A^{-1} \boldsymbol{\lambda} \cdot \boldsymbol{\mu} \, dx = L(\boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^o \end{cases} \quad (4.26)$$

in the special case where $L(\boldsymbol{\pi}_o) = 0$; we observe that this condition is satisfied by the linear functionals occurring in the right hand sides of (4.14) and (4.22).

To solve (4.26), we shall proceed as follows:

- (i) Solve

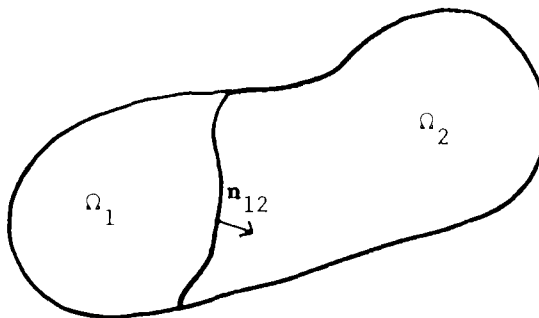


Figure 4.1

$$\begin{cases} \tilde{\lambda} \in P_{\gamma_0}, \\ \int_{\Omega} A^{-1}\tilde{\lambda} \cdot \mu \, dx = L(\mu), \quad \forall \mu \in P_{\gamma_0}. \end{cases} \tag{4.27}$$

(ii) Compute the component of $\tilde{\lambda}$ in $(P_{\gamma_0}^{\circ})^{\perp}$; it is clearly given by

$$\frac{\int_{\Omega} A^{-1}\tilde{\lambda} \cdot \pi_0 \, dx}{\int_{\Omega} A^{-1}\pi_0 \cdot \pi_0 \, dx} \pi_0. \tag{4.28}$$

(iii) We have then

$$\lambda = \tilde{\lambda} - \frac{\int_{\Omega} A^{-1}\tilde{\lambda} \cdot \pi_0 \, dx}{\int_{\Omega} A^{-1}\pi_0 \cdot \pi_0 \, dx} \pi_0. \tag{4.29}$$

5. Mixed Finite Element Implementation.

5.1 *Synopsis.* The computer implementation of the solution technique discussed in Sections 3 and 4 can be achieved through mixed finite element approximations like those already discussed in [10, 13].

We shall first consider the mixed finite element approximation of the global problem (1.1) through the equivalent mixed formulation (2.9) and then apply the

decomposition principles described in Section 3 to the global discrete problem in order to obtain a discrete equivalence of Problem (E) of Section 3.3. We shall also describe a finite dimensional variant of the conjugate gradient algorithm (4.10)-(4.24).

5.2 *Mixed Finite Element Approximations of Problem (1.1).* For convenience we consider only two-dimensional problems with Ω a rectangular domain. Set $\Omega = (0, x_L) \times (0, y_L)$ and let $\Delta_x: 0 = x_0 < x_1 < \dots < x_{N_x} = x_L$ and $\Delta_y: 0 = y_0 < y_1 < \dots < y_{N_y} = y_L$ be partitions of $[0, x_L]$ and $[0, y_L]$ respectively. For Δ a partition, define the piecewise polynomial space

$$M_s^r(\Delta) = \{v \in C^s([0, L]): v \text{ is a polynomial of degree } \leq r \text{ on each subinterval of } \Delta\} \quad (5.1)$$

where $s = -1$ refers to the discontinuous functions. Let us introduce now the following approximations of $L^2(\Omega)$, $H(\Omega; \text{div})$ and P_o (cf. (2.6)), respectively

$$W_h^{s,r} = M_s^r(\Delta_x) \otimes M_s^r(\Delta_y), \quad (5.2)$$

$$P_h^{s,r} = [M_{s+1}^r(\Delta_x) \otimes M_s^r(\Delta_y)] \times [M_s^r(\Delta_x) \otimes M_{s+1}^r(\Delta_y)] \quad (5.3)$$

and

$$P_{oh}^{s,r} = P_h^{s,r} \cap \{q: q \cdot n = 0 \text{ on } \partial\Omega\}, \quad (5.4)$$

where $h = \max_{ij} \{(x_{i+1} - x_i), (y_{j+1} - y_j)\}$. We remark that these spaces satisfy for $q \in P_h^{s,r}$ $\nabla \cdot q \in W_h^{s,r}$, i.e. $\nabla \cdot P_h^{s,r} \subset W_h^{s,r}$.

We shall denote by Q_h the set of elementary rectangles associated with grid $\Delta_x \times \Delta_y$. The mixed finite element formulation involves the solution pair $u_h \in W_h^{s,r}$ and $p_h \in P_h^{s,r}$ satisfying

$$\begin{cases} \int_{\Omega} (\nabla \cdot p_h + f)v \, dx = 0, \quad \forall v \in W_h^{s,r}, \\ \int_{\Omega} (A^{-1}p_h \cdot q + u_h \nabla \cdot q) \, dx = 0, \quad \forall q \in P_{oh}^{s,r} \end{cases} \quad (5.5a)$$

and

$$\int_{\partial\Omega} (p_h \cdot n - g) z \cdot n \, d\Gamma = 0, \quad \forall z \in P_{oh}^{s,r}. \quad (5.5b)$$

The discrete problem (5.5) is clearly the finite dimensional analogue of the mixed problem (2.9). The analysis of [10,11] shows that $p_h - p$ and $u_h - u$ are $O(h^{r+1})$ in $L^2(\Omega)$.

5.3 *Domain Decomposition for the Discrete Problem.* In this section we consider a two domain decomposition of the discrete problem with γ parallel to the y axis and $x = x_{I_\gamma}$ with I_γ between 1 and N (see Figure 5.1). Decompositions involving more than two subdomains will be discussed in Section 6. For simplicity we shall consider the case where $s = -1$ and r is arbitrary; the elements in these spaces are in general discontinuous. The spaces $W_h^{-1,r}$, $P_h^{-1,r}$ and $P_{oh}^{-1,r}$ should be denoted by W_h , P_h , and P_{oh} respectively.

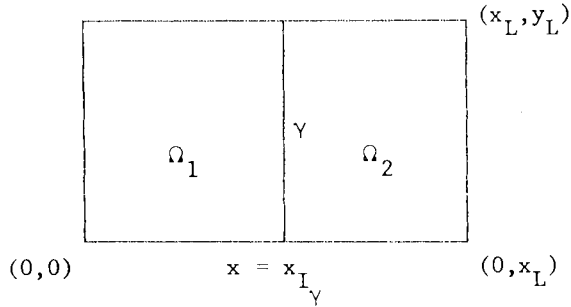


Figure 5.1
Two Domain Decomposition

Following Section 3.1, it is easily proven that the discrete mixed problem (5.5) is equivalent to finding $(\mathbf{p}_{h,i}, u_{h,i})$, $i = 1, 2$, satisfying

$$\int_{\Omega_i} (\nabla \cdot \mathbf{p}_{h,i} + f) v_i dx = 0, \quad \forall v_i \in W_{h,i}, \tag{5.6a}$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_{h,i} \cdot \mathbf{q}_i + u_{h,i} \nabla \cdot \mathbf{q}_i) dx = 0, \quad \forall \mathbf{q}_i \in P_{oh,i}, \tag{5.6b}$$

$$\int_{\partial \Omega_i \cap \Gamma} (\mathbf{p}_{h,i} \cdot \mathbf{n} - g) \mathbf{z} \cdot \mathbf{n} d\Gamma = 0, \quad \forall \mathbf{z} \in P_{oh,i}, \tag{5.6c}$$

$$\sum_{i=1}^2 \mathbf{p}_{h,i} \cdot \mathbf{n}_i = 0 \quad \text{on} \quad \gamma, \tag{5.6d}$$

$$\sum_{i=1}^2 \int_{\Omega} (A^{-1} \mathbf{p}_{h,i} \cdot \mathbf{q} + u_{h,i} \nabla \cdot \mathbf{q}) dx = 0, \quad \forall q \in P_{oh}. \tag{5.6e}$$

As in the continuous case, we associate to γ a complementary subspace $P_{oh,\gamma}$ of $P_{oh,1} \oplus P_{oh,2}$ in P_{oh} ; that is

$$P_{oh} = P_{oh,1} \oplus P_{oh,2} \oplus P_{oh,\gamma}.$$

It follows then from (5.6a), (5.6b) that (5.6e) can be replaced by the less demanding condition

$$\sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_{h,i} \cdot \mathbf{q} + u_{h,i} \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_{oh,\gamma}. \tag{5.7}$$

In addition to (5.6), (5.7), $\{\mathbf{p}_{h,i}, u_{h,i}\}$ has to satisfy the compatibility conditions

$$\int_{\Omega_i} f \, dx + \int_{\partial\Omega_i \cap \Gamma} g \, d\Gamma + \int_{\gamma} \mathbf{p}_{h,i} \cdot \mathbf{n}_i \, d\gamma = 0, \tag{5.8}$$

for $i = 1, 2$.

Several possibilities exist for $P_{oh,\gamma}$. From a practical point of view we shall take

$$P_{oh,\gamma} = P_{oh,\gamma}^x \oplus P_{oh,\gamma}^y \tag{5.9}$$

where

$$P_{oh,\gamma}^x = \left\{ \mathbf{q}_h = \{q_h^x, q_h^y\} \mid \mathbf{q}_h \in P_{oh}, \right. \tag{5.10}$$

$$\left. q_h^y = 0, \, q_h^x|_K = 0, \, \forall K \in Q_h \text{ such that } \gamma \cap \partial K = \phi \right\}.$$

In an analogous fashion, $P_{oh,\gamma}^y$ is defined. From (5.9) and (5.10) if \mathbf{q}_h belongs to $P_{oh,\gamma}$ then it vanishes outside the rectangle union of those elements of Q_h whose boundary touches γ . Moreover the role of $P_{o\gamma}^0/\hat{R}$ of the continuous case (see Section 3.3) is played here by the subspace of $P_{oh,\gamma}^x$ consisting of those functions \mathbf{q}_h satisfying $\int_{\gamma} \mathbf{q}_h \cdot \mathbf{n} \, d\gamma = 0$.

Solving the discrete system (5.6), (5.7) by a conjugate gradient method which is a discrete analogue of algorithm (4.10)-(4.24) of Section 4.2 is fairly straightforward and therefore will not be included in this discussion.

6. Generalization to Strip and Patch Decompositions.

6.1. *Generalities.* We consider in this section the generalization of the results and methods of the above sections to the case where the decomposition of Ω involves more than two subdomains. Here we concentrate on patch decompositions (see Figure 6.1) since the strip decomposition can be observed to be an easy case of the former.

For simplicity we shall consider two dimensional problems where Ω is a rectangle and the case where the decomposition is a tensor product of one-dimensional decompositions as in Figure 6.1. Generalizations to three dimensions and more complicated decompositions are possible and will be treated in a later paper.

Let M denote the number of subdomains and consider again Problem (1.1) and its mixed formulation (2.9). The decomposition discussed in Section 3 can be generalized in the sense that Problems (1.1) and (2.9) are equivalent (with obvious notation) to

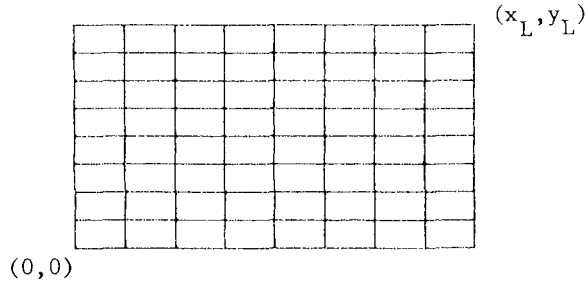


Figure 6.1
Patch Decompositions

$$\int_{\Omega_i} (\nabla \cdot \mathbf{p}_i + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \tag{6.1a}$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q}_i + u_i \nabla \cdot \mathbf{q}_i) \, dx = 0, \quad \forall \mathbf{q}_i \in P_{i0}, \quad \forall i = 1, 2, \dots, M, \tag{6.1b}$$

$$\mathbf{p}_i \cdot \mathbf{n}_i = g \quad \text{on} \quad \Gamma \cap \partial \Omega_i, \tag{6.1c}$$

$$\mathbf{p}_i \cdot \mathbf{n}_{ij} + \mathbf{p}_j \cdot \mathbf{n}_{ji} = 0 \quad \text{on} \quad \gamma_{ij}, \tag{6.1d}$$

where, for $i \neq j$, $\gamma_{ij} = \gamma_{ji} = \partial \Omega_i \cap \partial \Omega_j$ and \mathbf{n}_{ij} is the normal to γ_{ij} pointing outward from Ω_i . We set $\gamma = \bigcup_{ij} \gamma_{ij}$; we only consider those γ_{ij} whose measure is positive.

To relations (6.1)-(6.4) we must add the additional compatibility condition

$$\sum_{i=1}^M \int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q} + u_i \nabla \cdot \mathbf{q}) \, dx = 0, \quad \forall \mathbf{q} \in P_{\gamma_0}, \tag{6.1e}$$

where P_{γ_0} is a complementary subspace of $\bigoplus_{i=1}^M P_{i0}$ in P_0 . Obviously the $\{u_i, \mathbf{p}_i\}$ have to satisfy

$$\int_{\Omega_i} f \, dx + \int_{\partial \Omega_i \cap \Gamma} g \, d\Gamma + \sum_{j=1}^M \int_{\gamma_{ij}} \mathbf{p}_i \cdot \mathbf{n}_{ij} \, d\gamma = 0. \tag{6.1f}$$

To solve (1.1), (2.9) using the decomposition properties (6.6a) -(6.6f) we follow basically the same approach as in Section 2 for the two domain case. Therefore, we shall proceed as follows:

- (i) Define the subspace $P_{\gamma_0}^o$ of P_{γ_0} ;
- (ii) define the bilinear form $a(\cdot, \cdot)$ on $P_{\gamma_0}^o \times P_{\gamma_0}^o$;
- (iii) reformulate problem (1.1), (2.9) as variational problem in $P_{\gamma_0}^o$;
- (iv) solve the above using conjugate gradients;
- (v) define a convenient mixed finite element implementation of the above process.

6.2 *The Space $P_{\gamma_0}^o$.* From the compatibility conditions (6.1f), we are motivated to consider the subspace $P_{\gamma_0}^o$ of P_{γ_0} consisting of those functions \mathbf{q} satisfying the relation

$$\sum_{i=1}^M \int_{\gamma_{ij}} \mathbf{q} \cdot \mathbf{n}_{ij} d\gamma = 0, \quad i = 1, 2, \dots, M. \quad (6.2)$$

One can easily verify that the set of linear functionals

$$\mathbf{q} \rightarrow \sum_{j=1}^M \int_{\gamma_{ij}} \mathbf{q} \cdot \mathbf{n}_{ij} d\gamma, \quad i = 1, 2, \dots, M,$$

is of rank $M-1$ over P_{γ_0} implying that the codimension of $P_{\gamma_0}^o$ in P_{γ_0} is $M-1$. Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots, \boldsymbol{\pi}_{M-1}$ be $M-1$ elements of P_{γ_0} which are linearly independent and which span a complementary subspace of $P_{\gamma_0}^o$ in P_{γ_0} .

6.3 *The Bilinear Form $a(\cdot, \cdot)$.* Generalizing Section 3.3 let us define a bilinear form $a(\cdot, \cdot)$ over $P_{\gamma_0}^o \times P_{\gamma_0}^o$ as follows: Consider $\boldsymbol{\mu} \in P_{\gamma_0}^o$; we associate to $\boldsymbol{\mu}$, M pairs $(\mathbf{u}_i(\boldsymbol{\mu}), \mathbf{p}_i(\boldsymbol{\mu}))$ by solving

$$\int_{\Omega_i} \nabla \cdot \mathbf{p}_i(\boldsymbol{\mu}) v_i dx = 0, \quad \forall v_i \in L^2(\Omega_i), \quad (6.3a)$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{q}_i + u_i(\boldsymbol{\mu}) \nabla \cdot \mathbf{q}_i) dx = 0, \quad \forall \mathbf{q}_i \in P_{i\sigma}, \quad (6.3b)$$

$$\mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_i = 0 \quad \text{on} \quad \Gamma \cap \partial \Omega_i, \quad (6.3c)$$

$$\mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_{ij} = \boldsymbol{\mu} \cdot \mathbf{n}_{ij} \quad \text{on} \quad \gamma_{ij}. \quad (6.3d)$$

Since $\boldsymbol{\mu} \in P_{\gamma_0}^o$ implies that

$$\int_{\partial \Omega_i} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_i d\Gamma_i = \sum_{j=1}^M \int_{\gamma_{ij}} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \mathbf{n}_{ij} d\gamma = 0$$

the above problems are well posed in $H(\Omega_i, \text{div}) \times L^2(\Omega_i)/R$. Let us adjust now $\mathbf{u}_i(\boldsymbol{\mu})$, $i = 1, 2, \dots, M$, by

$$\int_{\Omega_M} u_M(\boldsymbol{\mu}) dx = 0, \quad (6.4a)$$

$$\sum_{j=1}^M \int_{\Omega_j} (A^{-1} \mathbf{p}_j(\boldsymbol{\mu}) \cdot \boldsymbol{\pi}_i + u_j(\boldsymbol{\mu}) \nabla \cdot \boldsymbol{\pi}_i) dx = 0, \quad i = 1, 2, \dots, M-1. \quad (6.4b)$$

The constants associated to $u_1(\boldsymbol{\mu}), u_2(\boldsymbol{\mu}), \dots, u_{M-1}(\boldsymbol{\mu})$ are therefore solutions of a linear system with a matrix whose general element is $\int_{\Omega_j} \nabla \cdot \boldsymbol{\pi}_i dx$, $1 \leq i, j \leq M-1$. The linear independence of the $\boldsymbol{\pi}$'s implies that this matrix is regular implying in turn that the above constants are uniquely determined.

Finally, we define $a(\cdot, \cdot)$ by

$$a(\boldsymbol{\mu}, \boldsymbol{\mu}') = \sum_{i=1}^M \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}' + u_i(\boldsymbol{\mu}) \nabla \cdot \boldsymbol{\mu}') dx, \quad \forall \boldsymbol{\mu}' \in P_{\gamma_0}^o. \quad (6.5)$$

By a variant of the proof of Theorem 3.1 in Section 3.3, it follows that $a(\cdot, \cdot)$ is symmetric and positive semi-definite over $P_{\gamma_0}^o \times P_{\gamma_0}^o$, and also strongly elliptic over $P_{\gamma_0}^o/\hat{R}$ where \hat{R} is the equivalence relation defined as in Section 3.3 by

$$\boldsymbol{\mu} \hat{R} \boldsymbol{\mu}' \Leftrightarrow (\boldsymbol{\mu} - \boldsymbol{\mu}') \cdot \mathbf{n} = 0 \text{ on } \gamma.$$

6.4 Reformulation of (1.1), (2.9) as a Variational Problem on $P_{\gamma_0}^o$. Using the same approach as in Section 3.3 we can reduce the solution of the Neumann problem (1.1) to the solution of the linear variational equation

$$\begin{cases} \text{Find } \bar{\lambda}^o \in P_{\gamma_0}^o \text{ such that} \\ a(\bar{\lambda}, \boldsymbol{\mu}) = - \sum_{i=1}^M \int_{\Omega_i} (A^{-1} \mathbf{p}_{oi} \cdot \boldsymbol{\mu} + u_{oi} \nabla \cdot \boldsymbol{\mu}) dx, \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^o, \end{cases} \quad (6.6)$$

where \mathbf{p}_{oi} and u_{oi} are defined almost as in Section 3.3. The modification consists of adjusting the u_{oi} by (6.4a) and (6.4b).

6.5 Conjugate Gradient Solution of (6.6). Starting again from the general conjugate gradient algorithm (4.1)-(4.9) of Section 4 we can easily solve the linear variational problem (6.6) by an algorithm which is a simple generalization of Algorithm (4.10)-(4.24) of Section 4.2. Again the scalar product used over $P_{\gamma_0}^o$ is defined by

$$(\boldsymbol{\mu}, \boldsymbol{\mu}') = \int_{\Omega} A^{-1} \boldsymbol{\mu} \cdot \boldsymbol{\mu}' dx. \quad (6.7)$$

As in Section 4.2 the conjugate gradient implementation leads to the solution of linear variational problems of the following type:

$$\begin{cases} \lambda \in P_{\gamma_0}^o, \\ \int_{\Omega} A^{-1} \lambda \cdot \boldsymbol{\mu} dx = L(\boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in P_{\gamma_0}^o. \end{cases} \quad (6.8)$$

with $L(\boldsymbol{\pi}_i) = 0, \forall i = 1, 2, \dots, M-1$. To simplify the solution of problem (6.8), we

we may define $\pi_1, \pi_2, \dots, \pi_{M-1}$ by

$$\begin{cases} \pi_i \in P_{\gamma_0}, \\ \int_{\Omega} A^{-1} \pi_i \cdot \mathbf{q} \, dx = \sum_{j=1}^M \int_{\gamma_j} \mathbf{q} \cdot \mathbf{n}_{ij} \, d\gamma, \quad \forall \mathbf{q} \in P_{\gamma_0}, \end{cases} \quad (6.9)$$

for $i = 1, 2, \dots, M-1$. From (6.9) it is quite clear that the subspace of P_{γ_0} generated by $\{\pi_i\}_{i=1}^{M-1}$ is precisely the orthogonal complement of $P_{\gamma_0}^o$ in P_{γ_0} for the inner product (6.7).

To solve (6.8) we shall proceed as follows:

(i) Solve

$$\begin{cases} \tilde{\lambda} \in P_{\gamma_0}, \\ \int_{\Omega} A^{-1} \tilde{\lambda} \cdot \mu \, dx = L(\mu), \quad \forall \mu \in P_{\gamma_0}. \end{cases}$$

(ii) Compute the component of $\tilde{\lambda}$ in $P_{\gamma_0}^\perp$; it is given by $\sum_{j=1}^{M-1} c_j \pi_j$, where the c_j are the solution of the linear system

$$\sum_{j=1}^{M-1} \left(\int_{\Omega} A^{-1} \pi_i \cdot \pi_j \, dx \right) c_j = \int_{\Omega} A^{-1} \tilde{\lambda} \cdot \pi_i \, dx, \quad i = 1, 2, \dots, M-1, \quad (6.10)$$

whose matrix is block tridiagonal, symmetric and positive definite. Actually if the π_i 's are defined by (6.9) one can easily verify using Green's Formula that the above matrix coincides with the one occurring in (6.4b) to adjust the $u_i(\mu)$.

(iii) We have then

$$\lambda = \tilde{\lambda} - \sum_{j=1}^{M-1} c_j \pi_j.$$

6.7 Mixed Finite Element Implementation. The mixed finite element implementation of the above patch decomposition techniques are analogous to the procedure described in Section 5.3 for the two domain problem.

We define γ^x (resp. γ^y) as the union of those faces of γ which are parallel to the x direction (resp. y direction). Then we approximate the space $P_{\gamma_0}^o/\hat{R}$ of Section 6.3 by $P_{\sigma_h, \gamma}^x \oplus P_{\sigma_h, \gamma}^y$ where in this context

$$\begin{aligned} P_{\sigma_h, \gamma}^x = & \left\{ q_h = \{q_h^x, q_h^y\} \mid q_h \in P_{\sigma_h}, \quad q_h^y = 0, \quad q_h^x|_K = 0, \right. \\ & \left. \forall K \in Q_h \text{ such that } \gamma^x \cap \partial K = \phi \right\} \end{aligned} \quad (6.11)$$

and $P_{\sigma_h, \gamma}^y$ is defined in a similar fashion. In (6.11) the space P_{σ_h} is the same as in Section 5.

The finite element implementation of the conjugate gradient algorithm discussed in Section 6.5 is straightforward.

7. A Second Decomposition Principle for the Mixed Formulation of Problem 1.1.

7.1. *Generalities.* We consider again the Neumann problem (1.1) with the data f and g satisfying the compatibility condition (1.2). In Section 2, we formulated the mixed problem (2.9) which is equivalent to (1.1).

In this section we would like to discuss a domain decomposition method which can be seen as the *dual* of the one defined in Section 3. Here duality is implied by the fact that the master unknown is no longer the flux $A \nabla u \cdot \mathbf{n}$ on γ , but instead is the trace of u on γ .

Applying the following to a Dirichlet or a Neumann-Dirichlet problem will be even less complicated and therefore will not be discussed here. Anticipating the next sections we remark that this second decomposition method is simpler than the first since no constant adjustment is needed; however, we have the feeling that the first method is easier to precondition than the second. We are presently running numerical experiments to substantiate these conjectures (which will be reported in a forthcoming paper).

7.2 *Another Equivalent Formulation of Problem (2.9) Using Domain Decomposition.* For simplicity we consider again a two domain decomposition as depicted in Figures 3.1a and Figure 3.1b of Section 3.1 (whose notation has been retained).

Let us denote by λ the trace over γ of the solution u of Problem 1.1. We have then for $i = 1, 2$,

$$\int_{\Omega_i} (\nabla \cdot \mathbf{p}_i + f) v_i \, dx = 0, \quad \forall v_i \in L^2(\Omega_i), \quad (7.1)$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_i \cdot \mathbf{q}_i + u_i \nabla \cdot \mathbf{q}_i) \, dx = \int_{\gamma} \lambda \mathbf{q}_i \cdot \mathbf{n}_i \, d\gamma, \quad \forall \mathbf{q}_i \in H_i, \quad (7.2)$$

$$\mathbf{p}_i \cdot \mathbf{n}_i = g \quad \text{on } \Gamma \cap \partial \Omega_i, \quad (7.3)$$

$$\int_{\gamma} \left(\sum_{i=1}^2 \mathbf{p}_i \cdot \mathbf{n}_i \right) \mu \, d\gamma = 0, \quad \forall \mu \in \Lambda, \quad (7.4)$$

where

$$H_i = \{ \mathbf{q}_i \mid \mathbf{q}_i \in H(\Omega_i, \text{div}), \quad \mathbf{q}_i \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma \cap \partial \Omega_i \},$$

and \mathbf{p}_i, u_i are the restrictions to Ω_i of \mathbf{p} and u respectively, and where

$$\Lambda = \{ \mu \mid \mu \in L^2(\gamma), \quad \mu = \bar{\mu} \mid_{\gamma} \quad \text{where } \bar{\mu} \in H^1(\Omega) \}. \quad (7.5)$$

7.3 *Solving (7.1)-(7.4) Via a Variational Problem on Λ .* We define a bilinear form over $\Lambda \times \Lambda$, which is denoted by $b(\cdot, \cdot)$ and given by

$$b(\mu, \mu') = \int_{\gamma} (\mathbf{p}_1(\mu) \cdot \mathbf{n}_1 + \mathbf{p}_2(\mu) \cdot \mathbf{n}_2) \mu' \, d\gamma \quad (7.6)$$

where $\mathbf{p}_1(\mu)$ and $\mathbf{p}_2(\mu)$ are obtained through the solution of the two local mixed problems

$$\int_{\Omega_i} \nabla \cdot \mathbf{p}_i(\mu) v_i dx = 0, \quad \forall v_i \in L^2(\Omega_i), \tag{7.7a}$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_i(\mu) \cdot \mathbf{q}_i + u_i(\mu) \nabla \cdot \mathbf{q}_i) dx = \int_{\gamma} \mu \mathbf{q}_i \cdot \mathbf{n}_i d\gamma, \quad \forall \mathbf{q}_i \in H_i, \tag{7.7b}$$

$$\mathbf{p}_i(\mu) \in H_i. \tag{7.7c}$$

Combining (7.6) and (7.7b) we easily obtain that, for $\mu, \mu' \in \Lambda$

$$b(\mu, \mu') = \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} \mathbf{p}_i(\mu') \cdot \mathbf{p}_i(\mu) + u_i(\mu') \nabla \cdot \mathbf{p}_i(\mu)) dx$$

which combined with (7.7a) yields

$$b(\mu, \mu') = \sum_{i=1}^2 \int_{\Omega_i} A^{-1} \mathbf{p}_i(\mu) \cdot \mathbf{p}_i(\mu') dx. \tag{7.8}$$

From (7.8) the bilinear form b is *symmetric positive semi-definite*. Indeed b is positive definite over Λ/R . To prove this result suppose that $b(\mu, \mu) = 0$. It follows from (7.8) that $\mathbf{p}_i(\mu) \equiv 0$ which in turn implies from (7.7b) that

$$\int_{\Omega_i} u_i(\mu) \nabla \cdot \mathbf{q}_i dx = \int_{\gamma} \mu \mathbf{q}_i \cdot \mathbf{n}_i d\gamma, \quad \forall \mathbf{q}_i \in H_i. \tag{7.9}$$

Consider now for $i = 1, 2$ the local Neumann problems

$$\begin{aligned} \Delta \phi_i &= 0 \quad \text{on } \Omega_i, \\ \frac{\partial \phi_i}{\partial \mathbf{n}_i} &= 0 \quad \text{on } \Gamma \cap \partial \Omega_i, \\ \frac{\partial \phi_i}{\partial \mathbf{n}_i} &= z \quad \text{on } \gamma, \end{aligned} \tag{7.10}$$

where $z \in L^2(\gamma)$ with $\int_{\gamma} z d\gamma = 0$. Problem (7.10) has a solution which is unique modulo constant. Moreover, $\nabla_{\gamma} \phi_i$ belongs to H_i and is divergence free. Thus setting $\mathbf{q}_i = \nabla \phi_i$ in (7.9) we obtain that

$$\int_{\gamma} \mu z d\gamma = 0. \tag{7.11}$$

Therefore μ belongs to the orthogonal in $L^2(\gamma)$ of the closed subspace of the functions whose average value is zero on γ ; consequently μ is a constant. In fact it can be easily shown that the bilinear form $b(\cdot, \cdot)$ is strongly elliptic over Λ/R . We define \mathbf{p}_{i0}, u_{i0} as the solutions of the local mixed problems

$$\int_{\Omega_i} (\nabla \cdot \mathbf{p}_{i0} + f) v_i dx = 0, \quad \forall v_i \in L^2(\Omega_i), \tag{7.12}$$

$$\int_{\Omega_i} (A^{-1} \mathbf{p}_{i0} \cdot \mathbf{q}_i + u_{i0} \nabla \cdot \mathbf{q}_i) dx = 0, \quad \forall q_i \in H_i, \quad (7.13)$$

$$\mathbf{p}_{i0} \cdot \mathbf{n}_i = g \quad \text{on} \quad \Gamma \cap \partial \Omega_i, \quad (7.14)$$

with

$$u_{i0} \in L^2(\Omega_i) \quad \text{and} \quad \mathbf{p}_{i0} \in H(\Omega_i, \text{div}).$$

Now we define $\bar{u}_i, \bar{\mathbf{p}}_i$ by

$$\bar{u}_i = u_i - u_{i0}, \quad \bar{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{p}_{i0}, \quad (7.15)$$

where u_i, \mathbf{p}_i are as in Section 7.2. Subtracting (7.12), (7.13), (7.14) from (7.1), (7.2) and (7.3) respectively, we obtain that \bar{u}_i and $\bar{\mathbf{p}}_i$ satisfy

$$\int_{\Omega_i} \nabla \cdot \bar{\mathbf{p}}_i v_i dx = 0, \quad \forall v_i \in L^2(\Omega_i), \quad (7.16)$$

$$\int_{\Omega_i} (A^{-1} \bar{\mathbf{p}}_i \cdot \mathbf{q}_i + \bar{u}_i \nabla \cdot \mathbf{q}_i) dx = \int_{\gamma} \lambda \mathbf{q}_i \cdot \mathbf{n}_i d\gamma, \quad \forall q_i \in H_i, \quad (7.17)$$

$$\bar{\mathbf{p}}_i \cdot \mathbf{n}_i = 0 \quad \text{on} \quad \Gamma \cap \partial \Omega_i. \quad (7.18)$$

Also combining (7.4) and (7.15), we have that

$$\int_{\gamma} \left(\sum_{i=1}^2 \mathbf{p}_i \cdot \mathbf{n}_i \right) \mu d\gamma = - \int_{\gamma} \left(\sum_{i=1}^2 \bar{\mathbf{p}}_{i0} \cdot \mathbf{n}_i \right) \mu d\gamma, \quad \forall \mu \in \Lambda. \quad (7.19)$$

From the definition of $\mathbf{p}_1(\mu)$ and $\mathbf{p}_2(\mu)$, we see from (7.7) that

$$\bar{\mathbf{p}}_i = \mathbf{p}_i(\lambda), \quad \bar{u}_i = u_i(\lambda). \quad (7.20)$$

Combining (7.19), (7.20) it follows from the definition of the bilinear form $b(\cdot, \cdot)$ given by (7.6) that

$$\begin{cases} b(\lambda, \mu) = - \int_{\gamma} \sum_{i=1}^2 (\bar{\mathbf{p}}_{i0} \cdot \mathbf{n}_i) \mu d\gamma, \quad \forall \mu \in \Lambda, \\ \lambda \in \Lambda. \end{cases} \quad (7.21)$$

From (7.21) the trace λ of u on γ appears as the solution of a variational problem in Λ . Since the bilinear form $b(\cdot, \cdot)$ is symmetric and strongly elliptic over Λ/R problem (2.1) can be solved by the conjugate gradient algorithm (4.1)-(4.9).

Due to page limitation no details of the conjugate gradient implementation will be discussed here; we mention however that the mixed finite element implementation of the algorithm is easy if one uses as scalar product on the discrete equivalent of Λ

$$\{\mu, \mu'\} \rightarrow \int_{\gamma} \mu \mu' d\gamma.$$

Actually one could use a more sophisticated scalar product in order to obtain an efficient preconditioner of the above algorithm. This important point will be discussed in a later paper together with the mixed finite element discretization of the second class of methods.

8. Numerical Experiments.

8.1. *Generalities.* In this section we would briefly like to describe some numerical results obtained using the method discussed in Sections 3-6 where the master unknown is the flux at the subdomain interfaces. Comparison with results obtained by the method of Section 7 will be reported elsewhere.

The various test problems we have been considering involve smooth and nonsmooth coefficients, right-hand sides and solutions.

We have also been comparing the performance of our algorithm on strip and patch decompositions. In view of engineering applications for which efficiency is essential we have been using as an initial guess of our conjugate gradient algorithm a predicted solution from a coarse grid calculation.

The local mixed problems have been solved using the preconditioned conjugate gradient MINV technique due to Concus, Golub and Meurant [14]. For details of application of this to mixed finite element method procedures see [13].

For all of the experiments to be described below the stopping criteria of our conjugate gradient algorithm, using the notation of Section 4, was

$$\frac{\int_{\Omega} A^{-1} \mathbf{g}^n \cdot \mathbf{g}^n \, dx}{\int_{\Omega} A^{-1} \mathbf{g}^0 \cdot \mathbf{g}^0} < 10^{-12} . \tag{8.1}$$

In addition, the region Ω was assumed to be the unit square. The mixed finite element approximating space chosen was the Raviart-Thomas tensor space $r = 1$ and $s = -1$ given by (5.2) - (5.4).

8.2. *First Test Problem.* The first test problem is the Neumann problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g & \text{on } \partial\Omega, \end{aligned} \tag{8.2}$$

where f and g have been chosen in such a way that the exact solution of problem (8.2) is $u(x, y) = \sin \pi x \sin \pi y$. We used a uniform mesh with 20×20 , 40×40 and 80×80 elementary squares. The number of unknowns is roughly 5,000, 20,000, and 80,000 for each case respectively.

The domain decompositions depicted in Figures 8.1 - 8.4 have been considered (indeed we used also a (16, 16) decomposition).

Table 8.1 depicts the number of conjugate gradient iterations required to satisfy the stopping test (8.1) according to the value of h and the decomposition of Ω .

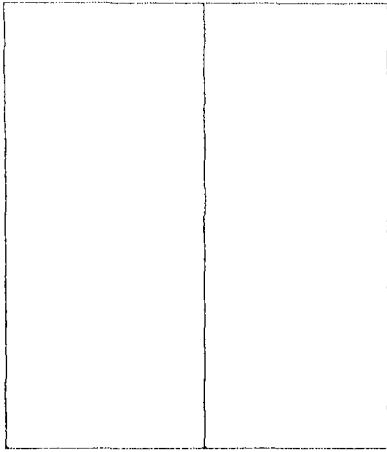


Figure 8.1
(2,1) Decomposition

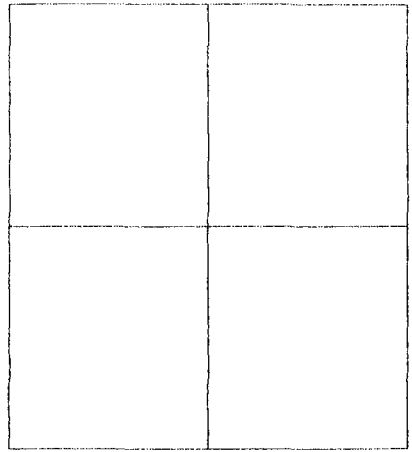


Figure 8.2
(2,2) Decomposition

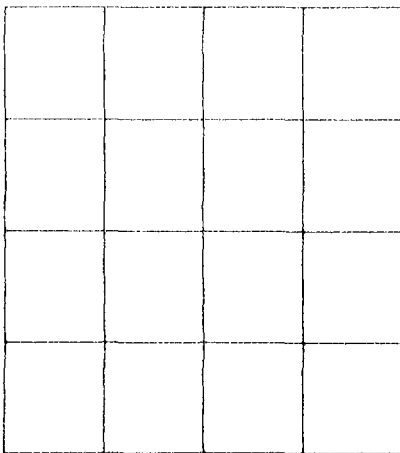


Figure 8.3
(4,4) Decomposition

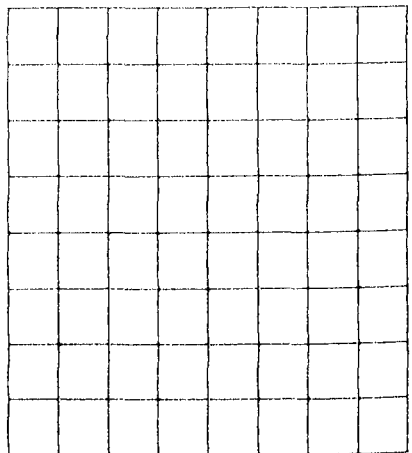


Figure 8.4
(8,8) Decomposition

Decompositions	$h^{-1} = 20$	$h^{-1} = 40$	$h^{-1} = 80$
(2,1)	3	8	
(2,2)	5	8	
(4,4)	23	23	
(8,8)		27	41
(16,16)			31

Table 8.1
 Number of Conjugate Gradient Iterations
 Smooth Problem

We observe that the number of iterations is slightly increasing with h^{-1} but definitely not proportional to h^{-1} implying that the preconditioning properties of the scalar product $\int_{\Omega} A^{-1} \mu \cdot \mu' dx$ are quite good.

8.3. *Nonsmooth Test Problems.* Motivated by applications in reservoir engineering we are considering now the following class of test problems:

$$\begin{aligned}
 -\nabla \cdot (A \nabla u) &= \delta_{(1,0)} - \delta_{(0,1)} \quad \text{in } \Omega, \\
 A \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega
 \end{aligned}
 \tag{8.3}$$

where A is defined by either

- (i) $A = A_1 = I,$
- (ii) $A = A_2 = \frac{1}{1 + 100(x^2 + y^2)} I,$

or

- (iii) $A = A_3 = \alpha I$ where

$$\alpha = \begin{cases} 1, & 0 \leq x \leq .5, \\ .01, & .5 < x \leq 1. \end{cases}$$

Tables 8.2-8.4 depict the number of iterations required by the conjugate gradient with the same stopping criteria as before.

Clearly the slope dependence on h still holds, roughly showing a $h^{-1/2}$ influence; moreover the speed of convergence is practically insensitive to the roughness of the source term and coefficients. This is a most interesting property in view of practical applications. It is our opinion that this property originates from the *harmonic averaging* associated to the mixed method and to the scalar product used for the conjugate gradient iteration.

A natural question arising from the above numerical tests is how accurate must be the solution of the local problems; actually the results reported here were obtained by solving these subproblems within machine precision ($\sim 10^{-12}$ on the Cray-XMP). Indeed the final precision and the global performances of our conjugate gradient algorithm were practically identical when the subproblems were solved within a 10^{-3} precision on the local residuals.

Decompositions	$h^{-1} = 20$	$h^{-1} = 40$	$h^{-1} = 80$
(2,1)	15	20	
(2,2)	15	20	26
(4,4)	20	29	43
(8,8)		29	43
(16,16)			33

Table 8.2
Number of Conjugate Gradient Iterations
 $A = A_1$

Decompositions	$h^{-1} = 20$	$h^{-1} = 40$
(2,1)	19	24
(2,2)	20	27
(4,4)	21	34
(8,8)		30
(4,1)		38
(8,1)		> 50

Table 8.3
Number of Conjugate Gradient Iterations
 $A = A_2$

Decompositions	$h^{-1} = 20$	$h^{-1} = 40$
(2,1)	19	24
(2,2)	18	21
(4,4)	20	33
(8,8)		29

Table 8.4
Number of Conjugate Gradient Iterations
 $A = A_3$

9. Conclusion.

Domain decomposition, combined with mixed finite element methods of approximation, seems to provide efficient techniques for elliptic problems with discontinuous and rapidly varying coefficients which arise in many important engineering applications. This combination seems to be particularly well suited for box decomposition since the traditional difficulty associated with vertices, when classical C^0 -conforming finite element methods are used, does not hold here. Indeed, the same comment applies to large classes of time dependent linear and nonlinear problems. One of the attractive features of this

method is that it readily lends itself to exploiting parallelism. In fact, we think it is the most interesting field in which computer science and numerical analysis can merge to produce efficient tools for scientific computing.

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REFERENCES

- (1) R. GLOWINSKI, J. PERIAUX, G. TERRASSON, and M.F. WHEELER, *On the coupling of finite element and domain decomposition methods for the numerical solution of partial differential equations*, to appear in Proceedings of Antibes (France) Conference on Finite Element Methods, June 1986.
- (2) R. GLOWINSKI and M.F. WHEELER, *Domain decomposition for mixed finite element methods*, Abstract from Mathematical Sciences Institute, Cornell University, Workshop on Preconditioned Iterative Methods and Domain Decomposition for Solving Partial Differential Equations, August 1986, pp. 23-24.
- (3) J.H. BRAMBLE, J.E. PASCIAK, and A.H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring, I*, *Mathematics of Computation* 47, 175 (1986), pp. 103-134.
- (4) P.E. BJORSTAD and O.B. WIDLUND, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, *SIAM J.N.A.* 23, 6 (1986), pp. 1097-1120.
- (5) A. BENSANSSAN, J.L. LIONS, and G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.
- (6) J. NECAS, *Les Methodes Directes on Theorie des Equations Elliptiques*, Masson, Paris, 1967.
- (7) J.L. LIONS and E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications, I*, Springer-Verlag, Berlin, 1972.
- (8) R.A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
- (9) J.T. ODEN and J. REDDY, *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley & Sons, New York, 1976.
- (10) P.A. RAVIART and J.M. THOMAS, *A mixed finite element method for second order elliptic problems*, in *Mathematical Aspects of the Finite Element Method*. Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 1977.

- (11) J.M. THOMAS, *Sur l'analyse numerique des methodes d'elements finis hybrides et mixtes*, Doctoral Thesis, Universite Pierre et Marie Curie, Paris, 1977.
- (12) R.E. EWING and M.F. WHEELER, *Computational aspects of mixed finite element methods*, in Numerical Methods for Scientific Computing, R.S. Stepleman, ed., North Holland, Amsterdam, pp. 163-172.
- (13) M.F. WHEELER and R. GONZALEZ, *Mixed finite element methods for petroleum reservoir engineering problems*, in Computing Methods in Applied Sciences and Engineering VI, R. Glowinski and J.L. Lions, eds., North Holland, New York, 1984, pp. 639-658.
- (14) P. CONCUS, G. GOLUB and G. MEURANT, *Block preconditioning for the conjugate gradient method*, SIAM Journal on Scientific and Statistical Computing 6 (1985), pp. 220-252.