Mixing Finite Elements and Finite Differences in a Subdomain Method

M. BOULBRACHENE*, Ph. CORTEY-DUMONT†, AND J. C. MIELLOU*

Abstract. The interest of subdomain decomposition principle is well established by an algorithm point of view, as well as for the storage management. Another possibility offered by this technics gives some flexibilities for the choice of the discretization of each subdomain. For example, finite element approximation can be used in the part of the domain with irregular data, but finite difference working for the regular one.

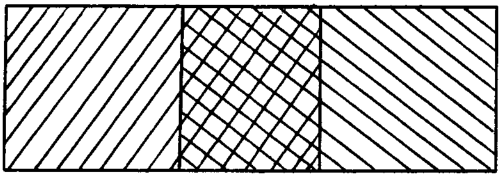
The purpose of this article is, firstly, to justify and clarify this empirical approach in the case of the Schwarz method, secondly to point out some constraints when using it.

Introduction. The interest of subdomain decomposition principle is well established by an algorithm point of view, as well as for the storage management. Another possibility offered by this technic gives some flexibilities for the choice of the discretization of each subdomain. For example, finite element approximation can be used in the part of the domain with irregular data, but finite difference working for the regular one.

*Laboratoire de Calcul Scientifique, Faculté des Sciences, 16, Route de Gray, 25030 Besançon Cedex, France.

†I. B. M. Département du Marketing Scientifique, Tour Septentrion, La Défense, 92800 Puteaux, France.

Actually, this approach is intensively used in industrial fluid mechanic simulation.



$\Omega = \Omega_1 \cup \Omega_2$

$\partial\Omega$ the boundary of Ω

 subdomain I Ω_1

 subdomain II Ω_2

 the overlapping domain

$\Gamma_1 = \partial\Omega_1 \cap \Omega_2$

$\Gamma_2 = \partial\Omega_2 \cap \Omega_1$.

The purpose of this article is, firstly, to justify and clarify this empirical approach in the case of the Schwarz algorithm method secondly to point out some contraints when using it.

- The overlapping subdomain must be independent od the mesh size h.
- The intersection of Γ_1 and Γ_2 is empty.
- The constant of approximation is directly dependent on the regularity of the solution z_i of the following problems in neighbourhood of Γ_j, $j \neq i$

$$(I) \begin{cases} -\Delta z_i + k z_i = 0 & \text{in } \Omega_i \\ z_i = 0 & \text{on } \Omega_i - \Gamma_i \\ z_i = 1 & \text{on } \Gamma_i \end{cases}$$

Consequently, as soon as the overlapping becomes thinner, the constant grows while the approximation get to worth.

This paper consists of two parts :

1 - the finite element approach

2 - the mixed discretization case.

The first part deals with the subdomain decomposition for the standard Poisson's problem. A standard finite element approximation (P_1 by triangle) for the two subdomains is considered. Each subdomain supports an independent mesh. Our main result is to get the usual order of approximation under suitable regularity properties, the regularity of the solution of problem (I) being of crucial interest.

In the second section, we adapt the methodology previously developed to the case where one subdomain is discretized with finite elements, and the other with finite difference. We get the same kind of approximation order as in the previous section.

<u>An approximation problem about some domain decomposition methods</u>

1. <u>Notations - Basic assumptions.</u>

(1. 1) $\begin{cases} \text{Let } \Omega \text{ be a bounded polyhedral domain of } \mathbb{R}^2 \text{ (or } \mathbb{R}^3\text{) with, in} \\ \text{both cases, acute angles only, for the boundary of } \Omega \text{ noted } \partial\Omega; \\ \text{let } k \in \mathbb{R},\ k>0, \text{ and } f \text{ a regular function on } \Omega. \end{cases}$

(1. 2) $\begin{cases} \text{Let } \Omega \text{ be decomposed into two overlapping subdomains } \Omega_1,\ \Omega_2, \\ \text{each } \Omega_i\ i = (1, 2) \text{ satisfying (1. 1)} \end{cases}$

$$\Omega = \Omega_1 \cup \Omega_2.$$

In all the sequels we shall use two indexes $i, j \in \{1, 2\}$ with the following convention i is different from j whenever i and j are both present.

(1. 3) Let $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$.
The intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$ is supposed to be empty.

Let z be the solution of the following problem :

(1. 4) $$\begin{cases} -\Delta z + kz = f & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

We associate to this problem a system; the solution is $\{z_1, z_2\}$

(1. 5) $$\begin{cases} -\Delta z_i + k z_i = f_i = f & \text{in } \Omega_i \\ z_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ z_{i/\Gamma_i} = z_{j/\Gamma_i} & \end{cases}$$

and y_i is the solution of the Dirichlet problem :

(1.6)
$$\begin{cases} -\Delta y_i + k\, y_i = 0 & \text{in } \Omega_i \\ y_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ y_i = v_i & \text{on } \Gamma_i \,. \end{cases}$$

Moreover let us introduce the following uncoupled problems :

(1.7)
$$\begin{cases} -\Delta \tilde{y}_i + k\tilde{y}_i = f_i = f & \text{in } \Omega_i \\ \tilde{y}_i = 0 & \text{on } \partial\Omega_i \end{cases}$$

It's easy to identify z_i $i = (1, 2)$, $z_i = z_{/\Omega_i}$.

The fixed point application in the continuous framework :

(1.8) Let $E = E_1 \times E_2$ with $E_i = C(\Gamma_i)$

We consider the following application :

(1.9)
$$\bar{T}_j : E_i \longrightarrow E_j$$
$$v_i \longmapsto \bar{T}_j(v_i)$$

where : $\bar{T}_j(v_i) = y_{i/\Gamma_j}$.

Let $t_i = \tilde{y}_{i/\Gamma_j}$

obviously $z_i = y_i + \tilde{y}_i$ on Ω_i .

The fixed point application :

Let T be the following application :

$$T : E \longrightarrow E$$

(1.10)
$$v = (v_1, v_2) \longmapsto T(v) = (\bar{T}_1(v_2) + t_1 \,;\, \bar{T}_2(v_1) + t_2)$$
$$= (T_1(v_2) \,;\, T_2(v_1))$$

We associate to T the fixed point problem :

(1.11)
$$u_j = \bar{T}_j(u_i) + t_j = T_j(u_i)$$

by [3] we know that :

$$\|\bar{T}_j\|_{\mathcal{L}(L^\infty(\Gamma_i), L^\infty(\Gamma_j))} = q_j < 1 .$$

Moreover $\bar{T}_j$ is isotone with respect to the natural order T being a contraction, the system (1.11) admits a unique fixed point $u = \{u_1, u_2\}$ which can be easily identified as the solution of (1.5).

More precisely, we get that $z_i = z/_{\Omega_i}$ is the solution of the following problems :

(1.12)
$$\begin{cases} -\Delta z_i + k z_i = f_i = f/_{\Omega_i} & \\ z_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ z_i = u_i & \text{on } \Gamma_i \end{cases}$$

2. The discrete case.

(2.1) Upon each subdomain Ω_i we consider a quasi-uniform regular triangulation.
The two meshes being mutually independent On $\Omega_1 \cap \Omega_2$ a triangle belonging to one triangulation does not necessarily belong to the other.

(2.2) We consider the usual basis of affine functions φ_ℓ, $\ell = \{1, 2, \ldots, m(h)\}$ defined by : $\varphi_\ell(M_k) = \delta_{\ell k}$ where M_k is a summit of the considered triangulation. Let $r_{i,h}$ be the usual interpolation on Γ_i.

We consider the following discrete spaces :

(2.3)
$$\mathcal{V}_{i,h} = \{v_h \in C(\bar{\Omega}_i) \,/\, v_{h/K_h} \in P_1\}.$$

For every $w_i \in (\bar{\Gamma}_i)$ we set :

(2.4)
$$\mathcal{V}_{i,h}^{(w_i)} = \{v_h \in \mathcal{V}_{ih} / \, v_h = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \ \ v_h = r_{i,h}(w_i) \text{ on } \Gamma_i\}.$$

Thus :

(2.5)
$$\mathcal{V}_{i,h}^{(0)} = \{v_h \in \mathcal{V}_h / \, v_h = 0 \text{ on } \partial\Omega_i\}.$$

The space $E_{i,h}$ will be the range of the trace application γ_i defined on $\mathcal{V}_{j,h}$.

(2.6) Let $E_h = E_{1,h} \times E_{2,h}$

(2.7) Let $a_i(u,v) = \int_{\Omega_i} \left(\sum_{\ell,s} \frac{\partial u}{\partial x_\ell} \cdot \frac{\partial v}{\partial x_s} + k u v \right) dx$

(2.8) $(f,v)_i = \int_{\Omega_i} f v \, dx.$

We then consider the following system :

(2.9) Let $\{u_{1h}, u_{2h}\} \in E_{1h} \times E_{2h}$ and $z_{ih} \in \mathcal{V}_{ih}^{(u_{jh})}$ such that :

(2.10) $$\begin{cases} a_i(z_{ih}, v_h) = (f, v_h)_i & \forall\, v_h \in \mathcal{V}_{ih}^{(0)} \\ z_{ih/\Gamma_j} = u_{jh} \end{cases}$$

<u>The discrete fixed point application</u> :

For w_j belonging to $C(\bar{\Gamma}_j)$ we consider $\tilde{y}_{ih}$ and y_{ih} defined by :

(2.11) $$\begin{cases} a_i(\tilde{y}_{ih}, v_h) = (f_i, v_h)_i & \forall\, v_h \in \mathcal{V}_{ih}^{(0)} \\ \tilde{y}_{ih} \in \mathcal{V}_{ih}^{(0)} \end{cases}$$

and :

(2.12) $$\begin{cases} y_{jh} \in \mathcal{V}_{j,h}^{(w_j)} \text{ such that} \\ a_j(y_{ih}, v_h) = 0 & \forall\, v_h \in \mathcal{V}_{j,h}^{(0)} \end{cases}$$

Let $\bar{T}_{i,h}$ the mapping which associates to w_j

(2.13) $$\bar{T}_{i,h}(w_j) = r_{i,h}(\gamma_i(y_{jh})).$$

The mapping $T_{i,h}$ is defined by :

(2.14) $$\begin{cases} T_{i,h} : C(\bar{\Gamma}_j) \longrightarrow E_{i,h} \\ \qquad w_j \longmapsto T_{i,h}(w_j) = \bar{T}_{i,h}(w_j) + t_{ih} \\ \text{with } t_{ih} = r_{i,h}(\gamma_i(\tilde{y}_{jh})). \end{cases}$$

Then the system (2.10) is equivalent to the fixed point problem :

(2.15) $$u_{ih} = T_{i,h}(u_{jh}) = \bar{T}_{i,h}(u_{jh}) + t_{ih} .$$

<u>Remark</u> :

(2.16) As the two meshes are independent over the overlapping subdomains, it is impossible to formulate a global approximate problem which would be the direct discrete analogue of problem (1.4).

(2.17) Let $T_h = \{T_{1h}, T_{2h}\}$.

<u>The discrete maximum principle assumption</u> :

(2.18) The matrices whose coefficients are $a_i(\varphi_\ell, \varphi_s)$ are supposed to be M-matrices. For convenience in all the sequels, C will be a generic constant independent on h.

Theorem :

Under the assumption (1. 1) to (1. 4), (2. 1) (2. 18) it exists $\bar{h}>0$ such that for all $0\leq h\leq\bar{h}$, the system (2. 10) admits a unique solution $\{z_{1h}, z_{2h}\}$. Let $\{z_1, z_2\}$ be the solution of (1. 12) or equivalently z_i be the restriction to Ω_i of z the solution of the global problem (1. 4). Then for h small enough

$$\max_{i\in\{1,2\}} \|z_i - z_{ih}\|_{L^\infty(\Omega_i)} \leq Ch^2|\mathrm{Log}\ h|. \tag{2.19}$$

The proof lies upon several lemmas.

Lemma 1 :

(2.20) Under the previous notations and assumptions the application $T_{j,h}$ is isotone on $C(\bar{\Gamma}_i)$ endowed with the natural order.

Proof :

The discrete analogous problem (1. 6) can be defined by the following system :

$$\forall P\in\Omega_{j,h} \quad a_j\Big(\sum_{M\in\Omega_{j,h}} y_{j,h}\,\varphi_M(x), \varphi_p(x)\Big)$$

$$= -a_j\Big(\sum_{M\in\Gamma_{j,h}} w_j(M), \varphi_p(x)\Big)$$

or equivalently by :

$$\tag{2.21} \left\{\begin{aligned} &\forall P\in\Omega_{jh} \quad \sum_{M\in\Omega_{j,h}} y_{jh}(M)\, a_j(\varphi_M, \varphi_p) = \\ &\qquad -\sum_{M\in\Gamma_{j,h}} w_j(M)\; a_j(\varphi_M, \varphi_p) \end{aligned}\right.$$

Let us consider v_i^1 and v_i^2 such that for natural order :

$$v_i^1 \geq v_i^2 \tag{2.22}$$

Then if we consider y_i^1 and y_i^2 solutions of problems (1. 6) with respective boundary values v_i^1 and v_i^2 on Γ_i, their corresponding discrete analog gives rise to the following systems.

$$\forall P\in\Omega_{ih}, \sum_{M\in\Omega_{ih}} y_{ih}^1(M)\, a_i(\varphi_M, \varphi_p) = -\sum_{M\in\Gamma_{ih}} v_i^1(M)\; a_i(\varphi_M, \varphi_p)$$

and :

$$\forall P\in\Omega_{ih}, \sum_{M\in\Omega_{ih}} y_{ih}^2(M)\, a_i(\varphi_M, \varphi_p) = -\sum_{M\in\Gamma_{ih}} v_i^2(M)\, a_i(\varphi_M, \varphi_p)$$

and by substraction :

$$\forall P \in \Omega_{ih}, \sum_{M \in \Omega_{ih}} (y_{ih}^2(M) - y_{ih}^1(M))\, a_i(\varphi_M, \varphi_p)$$
$$= \sum_{M \in \Gamma_{ih}} (v_i^1(M) - v_i^2(M))\, a_i(\varphi_M, \varphi_p).$$

Following (2.22), we have :

$$v_i^1(M) \geq v_i^2(M) \qquad \forall M \in \Gamma_{ih}$$

and by (2. 18) we have :

$$a_i(\varphi_M, \varphi_p) \leq 0 \qquad \forall M \in \Omega_{ih},\ \forall P \in \Gamma_{i,h}.$$

These inequalities imply :

$$\sum_{M \in \Omega_{i,h}} (y_{ih}^2(M) - y_{ih}^1(M))\, a_i(\varphi_M, \varphi_p) \leq 0 \qquad \forall P \in \Omega_{ih}$$

Hence, (2. 18) implies :

$$y_{ih}^2(M) - y_{ih}^1(M) \leq 0 \qquad \forall M \in \Omega_{ih}$$

and so :

$$T_{jh}(v_i^1) \geq T_{j,h}(v_i^2)$$

<u>Lemma 2</u> :

(2.23) Under assumptions (1. 1) to (1. 4), (2. 17), (2. 18) $\forall\ \varepsilon > 0$ there exists a constant M_ε and a function $h \to \theta_1(h)$ with $0 \leq \theta_1(h) \leq M_\varepsilon\, h^2 |\mathrm{Log}\, h| + \varepsilon$ such that :

$$\|\bar{T}_{jh}\|_{\mathcal{L}(L^\infty(\Gamma_i), L^\infty(\Gamma_j))} \leq q_j + \theta_1(h)$$

where $q_j = \|\bar{T}_j\|_{\mathcal{L}(L^\infty(\Gamma_i)\,;\, L^\infty(\Gamma_j))}$.

<u>Proof</u> :

χ_{Γ_i} being the characteristic function of the set Γ_i, we introduce the following auxiliary problem :

(2.24)
$$\begin{cases} -\Delta z_i^{\chi_i} + k\, z_i^{\chi_i} = 0/\Omega_i \\ z_i^{\chi_i}/\partial\Omega_i = \chi_{\Gamma_i} \end{cases}$$

Using the maximum principle it is classical that :

$$q_j = \max_{\Gamma_j} \; z_i^{\chi_i}/\Gamma_j .$$

In the same way, taking into account the discrete maximum principle (lemma 1) if $z_{ih}^{\chi_i}$ is the solution of the discrete analog of (2.24), then

$$\|T_{j,h}\|_{\mathcal{L}(L^\infty(\Gamma_i)\,;\,L^\infty(\Gamma_j))} = \sup_{\Gamma_j} \; z_{ih}^{\chi_i}/\Gamma_j$$

which has now to be estimated.

We know (cf. [2]) that with our assumptions, $T_j \in \mathcal{L}(L^1(\Gamma_i), L^\infty(\Gamma_j))$ and we note :

(2.25) $$C = \max_j \|T_j\|_{\mathcal{L}(L^1(\Gamma_i)\,;\,L^\infty(\Gamma_j))} .$$

Let now $\hat{\chi}_{\Gamma_i}^{\tilde{\varepsilon}}$ and $\check{\chi}_{\Gamma_i}^{\tilde{\varepsilon}}$ be such that :

(2.26) $$\hat{\chi}_{\Gamma_i}^{\tilde{\varepsilon}} \geq \chi_{\Gamma_i} \geq \check{\chi}_{\Gamma_i}^{\tilde{\varepsilon}} \quad \text{with } \hat{\chi}_{\Gamma_i}^{\tilde{\varepsilon}} \text{ and } \check{\chi}_{\Gamma_i}^{\tilde{\varepsilon}} \text{ belonging to}$$

$C^\infty(\partial\Omega_i)$, and such that :

(2.27) $$\|\hat{\chi}_{\Gamma_i}^{\tilde{\varepsilon}} - \check{\chi}_{\Gamma_i}^{\tilde{\varepsilon}}\|_{L^1(\partial\Omega_i)} \leq \frac{\tilde{\varepsilon}}{C}$$

Let $\hat{z}_i$ and $\check{z}_i$ be the respective solutions of :

(2.28) $$-\Delta \hat{z}_i + k\,\hat{z}_i = 0/_{\Omega_i} \; ; \; \hat{z}_{i/\partial\Omega_i} = \hat{\chi}_{\Gamma_i}^{\varepsilon}$$

and

(2.29) $$-\Delta \check{z}_i + k\,\check{z}_i = 0/_{\Omega_i} \; ; \; \check{z}_{i/\partial\Omega_i} = \check{\chi}_{\Gamma_i}$$

The maximum principle implies :

(2.30) $$\hat{z}_i \geq z_i^{\chi_i} \geq \check{z}_i$$

and moreover, by (2.25), (2.27) and (2.30) :

(2.31) $$\|(\hat{z}_i - z^{\chi_i})/\Gamma_j\|_{L^\infty(\Gamma_j)} \leq \|(\hat{z}_i - \check{z}_i)/\Gamma_j\|_{L^\infty(\Gamma_j)} \leq \tilde{\varepsilon}$$

$z_{ih}^{\chi_i}$, $\hat{z}_{ih}$, $\check{z}_{ih}$ being the respective solution of the discrete analogs of (2.24), (2.28), (2.29) then using (2.18), we get :

(2.32) $$\hat{z}_{ih} \geq z_{ih}^{\chi_i} \geq \check{z}_{ih}$$

by the regularity of $\hat{\chi}^{\tilde{\epsilon}}_{\Gamma_i}$ and $\check{\chi}^{\tilde{\epsilon}}_{\Gamma_i}$ assumptions (1. 1), (1. 2) we get :

$$\|\hat{z}_{ih} - \hat{z}_i\|_{L^\infty(\Omega_i)} \leq M_{\tilde{\epsilon}}\, h^2 |\mathrm{Log}\, h| \tag{2.33}$$
$$\|\check{z}_{i,h} - \check{z}_i\|_{L^\infty(\Omega_i)} \leq M_{\tilde{\epsilon}}\, h^2 |\mathrm{Log}\, h|$$

where the constant $M_{\tilde{\epsilon}}$ depends on $\tilde{\epsilon}$.

In order to simplifi our notations, we write :

$$\hat{z}_{i/\Gamma_j} = a\ ;\ z_i^{\chi_i}/_{\Gamma_j} = b\ ;\ \check{z}_{i/\Gamma_j} = c$$
$$\hat{z}_{ih/\Gamma_j} = a'\ ;\ z^{\chi_i}_{ih/\Gamma_j} = b'\ ;\ \check{z}_{ih/\Gamma_j} = c'$$

and also $|\ \ |_\infty = \|\ \ \|_{L^\infty(\Gamma_j)}$.

Then by (2.30), (2.32), we have :

$$a \geq b \geq c \quad \text{and} \quad a' \geq b' \geq c'$$

which imply :

$$|b - b'|_\infty \leq |b - a|_\infty + |a - a'|_\infty + |a - b'|_\infty$$
$$\leq |a - c|_\infty + |a - a'|_\infty + |a' - c'|_\infty$$

where :

$$|a' - c'|_\infty \leq |a' - a|_\infty + |a - c|_\infty + |c - c'|_\infty$$

and so :

$$|b - b'|_\infty \leq 2|a - c|_\infty + 2|a - a'|_\infty + |c - c'|_\infty$$

and analogously :

$$|b - b'|_\infty \leq 2|a - c|_\infty + |a - a'|_\infty + 2|c - c'|_\infty$$

after adding each members of the above inequalities :

$$|b - b'|_\infty \leq 2|a - c|_\infty + \frac{3}{2}|a - a'|_\infty + \frac{3}{2}|c - c'|_\infty.$$

Then taking into account (2.30), (2.32), we get that :

$$|z_i^{\chi_i} - z_{ih}^{\chi_i}|_\infty \leq 2\tilde{\epsilon} + 3M_{\tilde{\epsilon}}\, h^2 |\mathrm{Log}\, h|$$

Let us write now $\epsilon = 2\tilde{\epsilon}$ and $3M_{\tilde{\epsilon}} = M_\epsilon$, we get (2.23).

Remark :

(2.34) In lemma 2 we can choose $\bar{h}$ such that for all $h \leq \bar{h}$, $M_\varepsilon\, h^2|\mathrm{Log}\,h| \leq \varepsilon = \frac{1-q}{4}$ and then $\theta_1(h) \leq \frac{1-q}{2}$.

So, (2.23) asserts that :

$$\|\bar{T}_{j,h}\|_{\mathcal{L}(L^\infty(\Gamma_i)\,;\,L^\infty(\Gamma_j))} \leq \ell = q + \frac{1-q}{2} = \frac{1+q}{2} < 1$$

or :

$$\|T_{j,h}\|_{\mathcal{L}(L^\infty(\Gamma_i)\,;\,L^\infty(\Gamma_j))} < 1.$$

Lemma 3 :

With the assumptions (1. 1) to (1. 4), (2. 1), (2. 2) $\{u_1, u_2\}$ being the solution of (1. 11), then :

$$\|T_{j,h}(u_i) - T_j(u_i)\|_{L^\infty(\Gamma_j)} \leq \theta^o(h) = C_1\, h^2|\mathrm{Log}\,h| \tag{2.35}$$

Proof :

$\{z_1, z_2\}$ being the solution of (1. 5) ; z_1, z_2 are also solution of (1. 12), that we approximate by their discrete analog :

Search $\bar{z}_{ih}$ such that :

$$\begin{cases} a_i(\bar{z}_{ih}, v_h) = (f, v_h)_i = \int_{\Omega_i} f\, v_h\, dx\ ;\ \forall\, v_h \in \mathcal{V}_{ih} \\ \bar{z}_{ih/\Gamma_i} = r_h(u_i)\ ;\ \bar{z}_{i,h} = 0_{/\partial\Omega_i \cap \partial\Omega} \end{cases} \tag{2.36}$$

Then with our regularity assumptions :

$$\|z_i - \bar{z}_{ih}\|_{L^\infty(\Omega_i)} \leq \theta^o(h) = Ch^2|\mathrm{Log}\,h|$$

and as

$$|z_{i/\Gamma_j} - \bar{z}_{ih/\Gamma_j}| = |T_{jh}(u_i) - T_j(u_i)|$$

we get (2.35).

Lemma 4 :

Under assumptions of the lemma 3 , we have :

$$\|u_i - u_{ih}\|_{L^\infty(\Gamma_i)} \leq \frac{2\,\theta_o(h)}{1-\ell}. \tag{2.37}$$

Proof :

$$\|u_i - u_{ih}\|_{L^\infty(\Gamma_i)} \leq \|u_i - T_{ih}(u_j)\|_{L^\infty(\Gamma_i)} + \|T_{ih}(u_j) - u_{ih}\|_{L^\infty(\Gamma_i)}$$

$$\leq \|T_i(u_j) - T_{ih}(u_j)\|_{L^\infty(\Gamma_i)} + \|T_{ih}(u_j) - T_{ih}(u_{jh})\|_{L^\infty(\Gamma_i)} .$$

Following (2.34), (2.35)

$$\|u_i - u_{ih}\|_{L^\infty(\Gamma_i)} \leq \theta^o(h) + \ell \|u_j - u_{jh}\|_{L^\infty(\Gamma_j)}$$

and by symmetry we obtain :

$$\|u_j - u_{jh}\|_{L^\infty(\Gamma_j)} \leq \theta^o(h) + \ell \|u_i - u_{ih}\|_{L^\infty(\Gamma_i)}$$

so, we get (2.37).

Proof of the theorem :

z_i and z_{ih} being respective solution of the following problem :

$$\text{(2.38)} \quad \begin{cases} a_i(z_i, v) = (f, v)_i = \int_{\Omega_i} f.\, v\, dx & \forall\, v \in H_o^1(\Omega) \\ z_i \in H^2(\Omega)\ ;\ z_{i/\Gamma_i} = u_i\ ;\ z_{i/\partial\Omega \cap \partial\Omega_i} = 0 \end{cases}$$

$$\text{(2.39)} \quad \begin{cases} a_i(z_{ih}, v_h) = (f, v_h)_i & \forall\, v_h \in \mathscr{V}_{ih}^{(o)} \\ z_{ih} \in \mathscr{V}_{i,h}^{(u_{jh})} \end{cases}$$

Let us also consider the following continuous auxiliary problem :

$$\text{(2.40)} \quad \begin{cases} a_i(\tilde{z}_{ih}, v_h) = (f, v_h)_i & \forall\, v_h \in \mathscr{V}_{i,h}^{(o)} \\ \tilde{z}_{i,h} \in \mathscr{V}_{i,h}^{(u_i)} \end{cases}$$

with our regularity assumptions, classical approximation results in L^∞ norm, assert that :

$$\|\tilde{z}_{ih} - z_i\|_{L^\infty(\Omega_i)} \leq Ch^2 |\text{Log } h| = \theta_o(h)$$

$$\|r_h(u_{ih}) - r_h(u_i)\|_{L^\infty(\Gamma_i)} \leq \|u_i - u_{ih}\|_{L^\infty(\Gamma_i)} \leq \frac{2\,\theta_o(h)}{1-\ell}$$

and by the discrete maximum principle (closed related to lemma 2) we have also :

$$\|z_{i,h} - \tilde{z}_{i,h}\|_{L^\infty(\Omega_i)} \leq \|r_h(u_i) - r_h(u_{ih})\|_{L^\infty(\Omega_i)} \leq \frac{2\,\theta_o(h)}{1-\ell}$$

so, we get :

$$\|z_{ih} - z_i\|_{L^\infty(\Omega_i)} \leq \theta_o(h) + \frac{2\,\theta_o(h)}{1-\ell} \leq \frac{3\,\theta_o(h)}{1-\ell}$$

and if we replace here $\frac{3C}{1-\ell}$ by C, we achieve the proof.

3. Mixing finite elements and finite differences.

(3.1) On Ω_1 we consider the discretization by finite elements previously used in section (2).

(3.2) We consider a regular square (or cubic in 3D cases) grid with size step h on $\mathbb{R}^n$; let $\Omega_{2,h}$ be the points of this grid which are inside Ω_2 ; we assume that $\partial\Omega_{2,h}$ (which is the set of the points of the grid which ly at a distance strictly less than h of Ω_2) is included in $\partial\Omega_2$.
Let $\bar{\Omega}_{2,h} = \Omega_{2,h} \cup \partial\Omega_{2,h}$ and $\Gamma_{2,h} = \partial\Omega_{2,h} \cap \Omega_1$.
Then of course $\Gamma_{2,h} \subset \Gamma_2 = \partial\Omega_2 \cap \Omega_1$.
On $\Omega_{2,h}$ we consider the usual five points (or seven points in the 3D cases) finite difference scheme.

(3.3) Moreover we assume that the nodes if triangles of Ω_1 (tetrahedrons in 3D cases) lying in Γ_1, belong to $\Omega_{2,h}$.
Let then $\Gamma_{1,h} = \Omega_{2,h} \cap \Gamma_1$.

(3.4) Let $\tilde{r}_{i,h}$ be the restriction from Γ_i to $\Gamma_{i,h}$ and p_h^i the prolongation operator from $\Gamma_{i,h}$ by piecewise linear interpolation to a continuous function defined on Γ_i.

Then we can get here the operator $r_{1,h}$ used in the previous section by :

(3.5) $$r_{1,h} = p_h^1 \tilde{r}_{1,h} .$$

Then taking for the Ω_1 part of our system notations very close to the ones of the previous section, we consider the system :

(3.6)
Find $z_{1,h} \in \mathcal{V}_{1,h}^{(p_h^1 u_{1,h})}$ such that :

$$a_1(z_{1,h}, v_h) = (f, v_h) \qquad \forall\, v_h \in \mathcal{V}_{1,h}^{(o)}$$

and Δ_5 being the discrete five points analog of the Laplacian Δ.
Find the grid function $z_{2,h}$ such that :

$$-\Delta_5(z_{2,h})(M) + k z_{2,h}(M) = f(M) \ \forall\, M \in \Omega_{2,h}$$

$$z_{2,h/\partial\Omega_{2,h} \cap \partial\Omega} = 0 \ ; \ z_{2,h/\Gamma_{2,h}} = u_{2,h}$$

$$u_{2,h} = \tilde{r}_{2,h} \circ \gamma_2(z_{1,h}).$$

The discrete fixed point application associated to system (3. 6) :

On Ω_1, we consider first :

(3. 7)
$$\begin{cases} \tilde{y}_{1,h} \in \mathscr{V}^{(o)}_{1,h} \text{ such that} \\ a_1(\tilde{y}_{1,h}, v_h) = (f, v_h)_i \quad ; \quad \forall\, v_h \in \mathscr{V}^{o}_{1,h} \end{cases}$$

and then :

(3. 8)
$$\begin{cases} w_{1,h} \text{ being a grid function on } \Gamma_{1,h} \\ \text{Find } y_{1,h} \in \mathscr{V}^{(p^1_h w_{1,h})}_{1,h} \text{ such that} \\ a_1(y_{1,h}, v_h) = 0 \qquad \forall\, v_h \in \mathscr{V}^{o}_{1,h} \end{cases}$$

On $\Omega_{2,h}$, we consider first :

Find $\tilde{y}_{2,h}$ defined on $\Omega_{2,h}$ such that :

(3. 9)
$$\begin{cases} -\Delta_5(\tilde{y}_{2,h})(M) + k\tilde{y}_{2,h}(M) = f(M) \\ \tilde{y}_{2,h}(M) = 0 \qquad \forall\, M \in \partial\Omega_{2,h} \end{cases}$$

and then :

(3.10)
$$\begin{cases} w_2 \text{ being a continuous function on } \Gamma_2 \\ \text{Find } y_{2,h} \text{ defined on } \Omega_{2,h} \text{ such that :} \\ -\Delta_5(y_{2,h})(M) + k\, y_{2,h}(M) = 0 \qquad \forall\, M \in \Omega_{2,h} \\ y_{2,h}(M) = 0 \quad \forall\, M \in \partial\Omega_{2,h} \cap \partial\Omega \,; \\ y_{2,h}(M) = \tilde{r}_{2,h}(w_2)(M) \text{ on } \Gamma_{2,h} \end{cases}$$

We define the linear application :

$$\bar{T}'_h = \{\bar{T}'_{1,h} \,;\, \bar{T}'_{2,h}\}$$

$\bar{T}'_{2,h}$ which to $w_{1,h}$ associates :

(3.11)
$$\bar{T}'_{2,h}(w_{1,h}) = \tilde{r}_{2,h} \circ \gamma_2(y_{1,h})$$

where $\gamma_2(y_{1,h}) = y_{1,h}/\Gamma_2$

(3. 12) $\bar{T}'_{1,h}$ which to w_2 associates :

$$\bar{T}'_{1,h}(w_2) = \gamma_{1,h}(y_{2,h}) = y_{2,h}/\Gamma_{1,h}$$

and then the affine application $T'_h = \{T'_{1,h} \,;\, T'_{2,h}\}$ with :

$$\Big(T'_{1,h}(w_2) = \bar{T}'_{1,h}(w_2) + t_{1,h} \,;\; T'_{2,h}(w_{1,h}) = \bar{T}'_{2,h}(w_{1,h}) + t_{2,h}$$

(3.13) with :

$$t_{1,h} = \tilde{y}_{2,h/\Gamma_{1,h}} = \gamma_{1,h}(\tilde{y}_{2,h}) \; ; \; t_{2,h} = \tilde{y}_{1,h/\Gamma_{2,h}} = \tilde{r}_{2,h} \circ \gamma_2(\tilde{y}_{1,h}).$$

Then, the system (3.6) is equivalent to :

(3.14)

Search the fixed point $\{u_{1h} \; ; \; u_{2h}\}$:

$$u_{i,h} = T_{i,h}(u_{j,h})$$

and then find :

$$z_{1,h} \in \mathcal{V}_{1,h}^{(p_h^1 u_{1h})} \quad \text{such that :}$$

$$a_1(z_{1h}, v_h) = (f, v_h)_1 \qquad \forall\, v_h \in \mathcal{V}_{1,h}^{(o)}$$

and :

$$-(\Delta_5 z_{2,h})(M) + k\, z_{2,h}(M) = f(M) \qquad \forall\, M \in \Omega_{2,h}$$

$$z_{2,h/\partial\Omega_{2,h} \cap \partial\Omega} = 0 \; ; \; z_{2,h/\Gamma_{2,h}} = u_{2,h}$$

Theorem 2 :

Under the assumptions (1.1) to (1.4) and (3.1) to (3.3), it exists $\bar{h}>0$, such that, for all $h \leq \bar{h}$, $h>0$ the system admits a single solution $\{z_{1,h} \; ; \; z_{2,h}\}$ and z_i being the restriction of z solution of problem (1.4) to Ω_i.

Then :

(3.15)
$$\begin{cases} \|z_1 - z_{1h}\|_{L^\infty(\Omega_1)} \leq C h^2 |\mathrm{Log}\, h| \\ \|z_{2/\Omega_{2,h}} - z_{2,h}\|_{L^\infty(\Omega_{2,h})} \leq C h^2 |\mathrm{Log}\, h| \end{cases}$$

where C is a constant independent of h.

The proof lies upon several lemmas.

Lemma 5 :

Under assumptions (1.1) to (1.4), (3.1) to (3.3), $\forall\, \varepsilon>0$ there exists a constant M_ε and a function $h \to \theta_1(h)$ with $0 \leq \theta_1(h) \leq M_\varepsilon h^2 |\mathrm{Log}\, h| + \varepsilon$, such that :

(3.16)
$$\|\bar{T}_{j,h}\|_{\mathcal{L}(L^\infty(\Gamma_{i,h}) \; ; \; L^\infty(\Gamma_{j,h}))} \leq q_j + \theta_1(h)$$

where, here also, $q_j = \|\bar{T}_j\|_{\mathcal{L}(L^\infty(\Gamma_i); \; L^\infty(\Gamma_j))}$.

Proof :

For $\bar{T}'_{2,h}$ the situation is quite similar to the case of lemma 2.

For $\bar{T}'_{1,h}$, using the fact that the matrix associated to the usual five points scheme, is an M-matrix, we associate to problems (2.24), (2.28), (2.29), for $i=2$, their finite difference analogs and we get on $\Omega_{1,h}$ inequalities similar to (2.32).

With our regularity assumptions we get in $L^\infty(\Omega_{i,h})$ ($\hat{z}_i$, $\check{z}_i$ being restricted to $\Omega_{i,h}$) the analog of (2.33) with their orders of approximation, the sequel of the proof is now the same as in the case of lemma 2.

Lemma 6 :

With the assumptions of lemma 5 $\{u_1, u_2\}$ being the solution of (1.11), then :

(3.17) $$\|T'_{1,h}(\tilde{r}_{2,h}(u_2)-\tilde{r}_{1,h}(u_1))\|_{L^\infty(\Gamma_{1,h})} \leq \theta_o(h) = Ch^2|\mathrm{Log}\, h|$$

(3.18) $$\|T'_{2,h}(\tilde{r}_{1,h}(u_1)-\tilde{r}_{2,h}(u_2))\|_{L^\infty(\Gamma_{2,h})} \leq \theta_o(h) = Ch^2|\mathrm{Log}\, h|.$$

Proof :

On Ω_1 we consider the problem :

$$\begin{cases} -\Delta_5(\bar{z}_{2,h})(M) + k\bar{z}_{2,h}(M) = f(M), & \forall M \in \Omega_{2,h} \\ \bar{z}_{2,h/\Gamma_{2,h}} = \tilde{r}_{2,h}u_2 \ ; \ \bar{z}_{2,h/\partial\Omega_{1,h}\ominus\Gamma_{2,h}} \end{cases}$$

and with our regularity assumption, we get here, by classical L^∞ approximation results :

$$\|z_{2/\Omega_{2,h}} - \bar{z}_{2,h}\|_{L^\infty(\Omega_{2,h})} \leq Ch^2|\mathrm{Log}\, h|$$

which implies (3.17).

For $T'_{2,h}$ with assumption (3.2)

$$p_h^2\, T'_{2,h}\tilde{r}_1^h u_1 = T_{2,h}u_1$$

where $T_{2,h}$ is the same as in section (II).

By lemma 3, we know that :

$$\|T_{2,h}(u_1) - u_2\|_{L^2(\Gamma_2)} \leq \theta_o(h)$$

which $\tilde{r}_{1,h}$ being non expensive implies :

$$\|\tilde{r}_{2,h} T_{2,h} u_1 - \tilde{r}_{2,h} u_2\|_{L^\infty(\Gamma_{2,h})} \leq \theta_o(h)$$

but $\tilde{r}_{2,h} T_{2,h} = \tilde{r}_{2,h} p_h^2 \quad T'_{2,h} = T_{2,h}$

Then we get (3. 18).

<u>Proof of the theorem 2</u> :

As in the case of <u>theorem 1</u>, we can choose, here by <u>lemma 5</u>, $\bar{h}$ such that for $h \leq \bar{h}$

$$(3.19) \qquad \|\bar{T}'_{j,h}\|_{\mathcal{L}(L^\infty(\Gamma_{i,h})\,;\,L^\infty(\Gamma_{j,h}))} \leq \frac{1+q}{2} = \ell < 1$$

$$\max_{j\in\{1,2\}} \|\tilde{r}_{j,h} u_j - u_{j,h}\|_{L^\infty(\Gamma_{j,h})} \leq \|\tilde{r}_{j,h} u_j - T_{j,h}(\tilde{r}_{i,h} u_i)\|_{L^\infty(\Gamma_{j,h})}$$

$$+ \|T_{j,h}(\tilde{r}_{i,h} u_i) - T_{j,h}(u_{i,h})\|_{L^\infty(\Gamma_{j,h})}$$

$$\leq \theta_o(h) + \ell \max_{i\in\{1,2\}} \|\tilde{r}_{i,h} u_i - u_{i,h}\|_{L^\infty(\Gamma_{i,h})}$$

donc :

$$(3.20) \qquad \max_{j\in\{1,2\}} \|\tilde{r}_{j,h} - u_{jh}\|_{L^\infty(\Gamma_{j,h})} \leq \frac{2\theta_o(h)}{1-\ell}$$

We consider now the auxiliary problems :

$$a_1(\tilde{z}_{1,h}, v_h) = (f, v_h)_1 \qquad \forall\, v_h \in \mathcal{V}^o_{1,h}$$

where $\tilde{z}_{1,h} \in \mathcal{V}^{(u_1)}_{1,h} = \mathcal{V}^{(p_h^1 \tilde{r}_{1,h} u_1)}_{1,h}$

and :

$$(-\Delta_5 \tilde{z}_{2,h})(M) + k\tilde{z}_{2,h}(M) = f(M) \ , \qquad \forall\, M \in \Omega_{2,h}$$

$$\tilde{z}_{2,h/\partial\Omega_{2,h}\cap\partial\Omega} = 0 \ ; \ \tilde{z}_{2,h/\Gamma_{2,h}} = \tilde{r}_{2,h}\, u_2(M)$$

and we have the estimates :

$$(3.21) \qquad \begin{cases} \|\tilde{z}_{1,h} - z_1\|_{L^\infty(\Omega_1)} \leq \theta_o(h) \\ \|\tilde{z}_{2,h} - z_{2/\Omega_{2,h}}\|_{L^\infty(\Omega_{2,h})} \leq \theta_o(h) \end{cases}$$

we have also :

$$(3.22)\begin{cases} \|z_1-z_{1,h}\|_{L^\infty(\Omega_1)} \le \|z_{1,h}-\tilde{z}_{1,h}\|_{L^\infty(\Omega_1)} + \|\tilde{z}_{1,h}-z_{1,h}\|_{L^\infty(\Omega_1)} \\ \|z_{2}/_{\Omega_{2,h}}-z_{2,h}\|_{L^\infty(\Omega_{2,h})} \le \|z_{2,h}-\tilde{z}_{2,h}\|_{L^\infty(\Omega_{2,h})} + \\ \qquad + \|\tilde{z}_{2,h}-z_{2}/_{\Omega_{2,h}}\|_{L^\infty(\Omega_{2,h})} \end{cases}$$

by (3.21) (3.22) and the use of themaximum principle, which is valid for our two distinct discrete problems, we get (3.15).

Remark :

All our results, that is to say as well in the case of theorem 1 of the previous section, as in the case of theorem 2 our results are presented in the context of the regularity assumption (1.1), (1.2), (1.3) which implies standard $h^2|\mathrm{Log}\ h|$ order of approximation for Poisson's problem.

If such regularity assumption is not satisfied, the methodology presented here acts as well, provided that we can ensure that $\theta_1(h)$ converges to zero with h ; but then this would lead to a reduction of order of approximation by this methodology.

REFERENCES

[1] Krasnosel'skill et Alt, Approximate solution of operator equation, Walters Noordhoff, (1972).

[2] J. C. Miellou, Variantes synchrones et asynchrones de la méthode alternée de Schwarz, Rapport de contrat S. T. P. A. et E. R. A. de Mathématiques n°070654, (1982).

[3] R. Courant, D. Hilbert, Methods of mathematical physics, vol. 1, J. Wiley Interscience, New-York, (1962).

[4] Ph. Cortey-Dumont, Approximation numérique de l'I. Q. V. de gestion de stock, R. A. I. R. O. Anal. Num., (1980).

[5] M. Boulbrachêne, Sur quelques questions d'approximations de problèmes à frontière libre, de sous-domaines et d'erreurs d'arrondi, Thèse, Université de Franche-Comté, Juin 1987.

[6] M. Boulbrachêne, Ph. Cortey-Dumont, J. C. Miellou, La notion de a-contraction. Applications, C. R. Acad. Sc. Paris, t. 302, série I, n°16, (1986).

[7] Ph. Cortey-Dumont, Sur les inéquations variationnelles à opérateurs non coercifs, M^2AN, 19, n°2, (1985).

[8] Ph. Cortey-Dumont, On finite element approximation of variational inequalities with non linear operator, Num. Math., (1985).

[9] Ph. Cortey-Dumont, Thèse de Doctorat ès Sciences, Université de Paris VI, (1985).

[10] P.G. Ciarlet, P.A. Raviart, Maximum principle and uniform convergence for the finite element method, Comp. Math. in Appl. Mech. and Eng., 2, p. 1-20, (1973).