On Some Difficulties Occurring in the Simulation of Incompressible Fluid Flows by Domain Decomposition Methods

J. CAHOUET*

ABSTRACT. This paper deals with the main problems encountered when applying domain decomposition methods to the solution of NAVIER-STOKES or Shallow Water equations. After a brief summary of the difficulties related to industrial fluid dynamics computations using the F.E.M. we analyse on a simplified STOKES model the theoretical formulation of the matching operator and the main obstacles raised when implementation of domain decomposition methods is considered. Those considerations lead us to inquire about the relevancy of such methods, to the industrial problems we face.

INTRODUCTION

Accurate simulation of industrial flows governed by NAVIER-STOKES or SAINT-VENANT type equations requires a great number of discretization points which can lead to saturation of computer's mass storage. To get along with this restraint a first approach is to devise solution algorithms which decouples the linear system for each unknown (e.g. GLOWINSKI-PIRONNEAU [1] or preconditionned UZAWA's algorithm [2]). This allows the numerical treatment of turbulent models involving heat transfer, for 3-D finite element computation of NAVIER-STOKES equations with up to 100,000 degrees of freedom (N3S-code [3] for example).

A second complementing approach is to split the geometrical domain into sub-regions which can be dealt with separately. Quite naturally this partition implies the use of a coupling operator to define boundary conditions on each sub-domain.

Three more attractive properties back up this method which applies to a great number of fluid mechanics problems :

- One can take advantage of geometric smoothness of certain sub-domains which can represent up to 80 % of the whole domain (e.g. in marine or external flows,...) ;

* LABORATOIRE NATIONAL D'HYDRAULIQUE, ELECTRICITE DE FRANCE, 6, QUAI WATIER 78401 CHATOU CEDEX FRANCE.

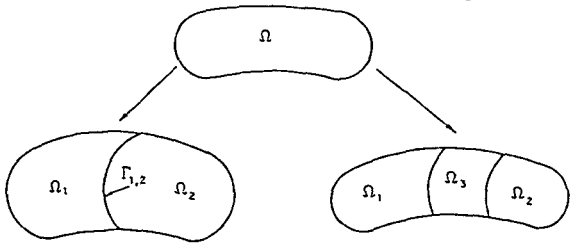
this smoothness can be taken into account algebraically (adequation of the linear system solution) or through the discretizations of equations (e.g. use of finite differences, multigrids or spectral methods).
- The algorithmic structure is well-adapted to multi-processor parallel computers such as CRAY XMP.
- Distinct physical models, based on the relative importance of phenomena in each sub-domain, can be coupled.

This paper is an overview of major difficulties one encounters when implementing the domain decomposition methods applied to incompressible or weakly compressible flows. The theorical results coming along in this note are, in their great extent, taken from J.P. MARTINAUD'S thesis [4] .

In the first part we thoroughly present the theoretical formulation of the coupling operator for a generalized STOKES-type problem. The presentation of numerical results is the scope of the second part. Two sets of benchmarks give the validation basis of the choice we made and a first analysis of the resulting cost of the decomposition method. As a conclusion, the relevancy of the application of such methods to the presented problems are investigated.

A) THEORETICAL FORMULATION

The principle of domain decomposition is quite general (cf. BENSOUSSAN, LIONS, TEMAM [7] LEMONNIER [8]) so that many choices can be made. For the sake of simplicity, we can divide the approaches according to two criteria :
- the level at which the decomposition is taken into account ; the most internal corresponds to sub-structuring of the linear system (LICHNEWSKY [9]) or of the elementary elliptic operators (DINH [10]); the most external involves the choice of the model for the physical phenomenon.
- the decomposition of the initial domain, either by partitioning the domain (non-overlapping sub-domains (Fig 1a, MARTINAUD [4] ,GOUTAL [17]), or by overlapping sub-domains (fig 1b, GLOWINSKI, LIONS, PERIAUX [13] ,DINH [10]).



$\Omega_1 \cup \Omega_2 = \Omega$
$\overline{\Omega_1} \cap \overline{\Omega_2} = \Gamma_{1,2}$
$\Omega_1 \cap \Omega_2 = \phi$

$\Omega_1 \cup \Omega_2 = \Omega$
$\Omega_1 \cap \Omega_2 = \Omega_3$

Fig. 1a : Non-overlapping sub-domains Fig. 1b : Overlapping sub-domains

Figure 1 : Possible decompositions of domain Ω .

In this paper, we will restrict ourselves to the a priori costless case of non-overlapping sub-domains, since this decomposition do not require the solution of an adjoint problem (cf. DINH [10]). After having chosen a weak formulation and a time-discretization which decouple the hyperbolic and parabolic parts (cf. CAHOUET [5]), it can be shown that the solution of the NAVIER-STOKES equations requires at each time-step :

- the computation of the right-hand-side terms taking the non-linearities into account,
- the solution of a (generalized) STOKES problem in the velocity-pressure (V,p) non-dimensionalized formulation :

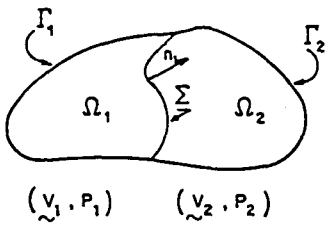
$$\begin{cases} \alpha \underset{\sim}{V} - \nu \Delta \underset{\sim}{V} + \underset{\sim}{\nabla} p = \underset{\sim}{f} \quad \text{ON } \Omega, \\ \underset{\sim}{\nabla}^t . \underset{\sim}{V} = 0 \quad \text{ON } \Omega \,, \quad \underset{\sim}{V} = 0 \quad \text{ON } \Gamma, \end{cases}$$

where α and ν are time-dependent, non-negative functions defined in Ω (α involves time-discretization and loss of charges contributions, ν embodies laminar or turbulent viscous effects). This problem will be subsequently referred to as the initial or global problem, its solution as the global solution .

2. Sub-structuring of the STOKES operator

2.1 Notations

For the sake of simplicity, we only consider here the partition of Ω into two sub-domains Ω_1 and Ω_2, lying on each side of the coupling boundary Σ . With each sub-domain Ω_i is associated variables with subscript $_i$ (see fig. 2).



Γ_i boundary of Ω_i
$\underset{\sim}{n}_i$ exterior unit normal
$\underset{\sim}{V}_i$, p_i velocity field and pressure distribution on Ω_i

Fig. 2 - Partition of Ω and main notations.

The jump of variables along Σ will be denoted by $[\,.\,]$; we have mainly

$$[\underset{\sim}{u}] = \underset{\sim}{u}_1\big|_\Sigma - \underset{\sim}{u}_2\big|_\Sigma \quad ,$$

$$[\underset{\sim}{\sigma}] = \underset{\sim}{\sigma}_1\big|_\Sigma - \underset{\sim}{\sigma}_2\big|_\Sigma \quad , \quad \underset{\sim}{\sigma}_i = - p_i \underset{\sim}{n}_j + \nu \frac{\partial \underset{\sim}{u}_i}{\partial n_i}$$

The stress vector appears naturally as the dual variable of the velocity on Σ in the case of the variational formulation of the STOKES problem.

The two following sections, which are taken from MARTINAUD'S thesis [4] concern the theoretical results and require the following additional notations for non-standard functional spaces :

- $\underset{\sim}{V}$ the velocity field of STOKES problem lies in $H_o^1(\Omega)$; this implies, taking into account the incompressibility conditions ;

$$[\underset{\sim}{V}] \in \overset{\circ}{H}{}^{1/2}_{oo}(\Sigma) = \left\{ \underset{\sim}{V} \in H^{1/2}_{oo}(\Sigma)^n \,,\ \int_\Sigma \underset{\sim}{V} . \underset{\sim}{n} \, d\Gamma = 0 \right\}$$

where $H^{1/2}_{oo}$ is the half-interpolated space between $H_o(\Sigma)^N$ and $L^2(\Sigma)^N$ (this allows to take the decay of $\underset{\sim}{V}$ at the ends of Σ into account).

- p_i is defined within an additive constant, in $L^2(\Omega_i)$; this implies

$$[\underset{\sim}{\sigma}] \in T \ , \ T = H^{-1/2}(\Sigma)^N / \mathcal{R}$$

where the equivalence relation $\mathcal{R}$ is defined over $H^{-1/2}(\Sigma)^N$ by

$$\forall (\underset{\sim}{\tau}, \underset{\sim}{\sigma}) \in H^{-1/2}(\Sigma)^{2N} \quad \underset{\sim}{\tau} \mathcal{R} \underset{\sim}{\sigma} \Leftrightarrow \underset{\sim}{\tau} - \underset{\sim}{\sigma} = c \underset{\sim}{n} \ , \ c \in \mathbb{R}$$

2.2 Principles of sub-structuring

If one wishes to deal with the solution of STOKES problem separately on each Ω_i, one requires the boundary values on each side of Σ.
If one wishes the solutions on each Ω_i, those boundary values must involve the continuity of velocity vectors and of stresses across Σ, to be the restrictions of the global solution (cf. MARTINAUD [4]).

It is thus natural :
- either to look for values of $\underset{\sim}{v}$ along Σ which imply the null value of the jump $[\sigma]$ across Σ ;
- or to look for a value of stress with imply the vanishing of the jump of velocity $[\underset{\sim}{v}]$,

the coupling of $\underset{\sim}{u}_{|\Sigma}$ and $\underset{\sim}{\sigma}_{|\Sigma}$ resulting in the STOKES operator.
Those two intuitive approaches which can be given the names primal and dual (with respect to the functional spaces $\underset{\sim}{u}_{|\Sigma}$ and $\underset{\sim}{\sigma}_{|\Sigma}$ lie in), are developed in the next two sections.

2.3 Primal formulation

Consider the following operator M

$$M : \overset{\circ}{H}{}^{1/2}_{oo} \longmapsto T$$

$$\underset{\sim}{u}_\Sigma \longmapsto M(\underset{\sim}{u}_\Sigma)$$

$$M(\underset{\sim}{u}_\Sigma) = [\underset{\sim}{\sigma}(\underset{\sim}{u}_\Sigma)] = \underset{\sim}{\sigma}_1(\underset{\sim}{u}_\Sigma)|_\Sigma - \underset{\sim}{\sigma}_2(\underset{\sim}{u}_\Sigma)|_\Sigma$$

where the stress vector $\underset{\sim}{\sigma}_i(\underset{\sim}{u}_\Sigma)$ is associated with the following (quasi-) homogeneous STOKES problem :

$$\begin{cases} \alpha \underset{\sim}{v}_i - \nu \Delta \underset{\sim}{v}_i + \underset{\sim}{\nabla} p_i = 0 \quad \text{ON} \quad \Omega_i \ , \\ \underset{\sim}{\nabla}^t . \underset{\sim}{v}_i = 0 \ , \ \underset{\sim}{v}_j = 0 \ \text{ON} \ \Gamma_i \ , \ \underset{\sim}{v}_i = \underset{\sim}{u}_\Sigma \ \text{ON} \ \Sigma \ . \end{cases}$$

This is a well-posed problem so that M defines an isomorphism to which we associate a symmetric bilinear form m coercive on $\overset{\circ}{H}{}^{1/2}_{oo}(\Sigma)$. This classical extension of the POINCARE-STEKLOV operator (cf AGOSKOV [18]) allows us to compute the jump of stress associated with a velocity-boundary value, with flux vanishing on Σ. On the other hand, the linearity of equations makes it possible to find the boundary condition by solving :

$$M\,(u\Sigma) = -\left[\underset{\sim}{\sigma}^{\circ}\right]$$

where σ°_{i} results from the solution of STOKES problem taking into account right-hand sides and non-homogeneous boundary conditions of the initial problem, that is :

$$\begin{cases} \alpha\ \underset{\sim}{v}_i^{\circ} - \nu\ \Delta \underset{\sim}{v}_i^{\circ} + \underset{\sim}{\nabla} p_i^{\circ} = \underset{\sim}{f}_i \\ \underset{\sim}{\nabla}^{t}.\underset{\sim}{v}_i = 0,\ \underset{\sim}{v}_i = \underset{\sim}{0}\ \text{ON}\ \Gamma_i,\ \underset{\sim}{v}_i = \underset{\sim}{u}_{FN}\ \text{ON}\ \Sigma, \end{cases}$$

where the boundary value u_{FN} on Σ is any function which ensures the compatibility condition resulting from fluid incompressibility :

$$\int_{\Sigma} \underset{\sim}{u}_{FN} \cdot \underset{\sim}{n}\ d\Gamma = 0$$

It is readily seen that the boundary value $V\Sigma$ defined by :

$$\underset{\sim}{V}\Sigma = M^{-1}\left(-\left[\underset{\sim}{\sigma}^{\circ}(\underset{\sim}{u}_{FN})\right]\right) + \underset{\sim}{u}_{FN}$$

does not actually depend on $\underset{\sim}{u}_{FN} \in \overset{\circ}{H}{}^{1/2}_{oo}(\Sigma)$. Existence and uniqueness result in the application of LAX-MILGRAM lemma to operator M.

2.4 Dual formulation

We consider now the following operator K :

$$K : T \longrightarrow \overset{\circ}{H}{}^{1/2}_{oo}(\Sigma)$$

$$\underset{\sim}{\lambda} \longrightarrow K(\underset{\sim}{\lambda})$$

$$K(\underset{\sim}{\lambda}) = \left[\underset{\sim}{u}(\underset{\sim}{\lambda})\right] = \underset{\sim}{u}_1(\underset{\sim}{\lambda})\Big|_{\Sigma} - \underset{\sim}{u}_2(\underset{\sim}{\lambda})\Big|_{\Sigma}$$

where $\underset{\sim}{u}_i(\underset{\sim}{\lambda})\big|_{\Sigma}$ is the trace of the solution of the stress-formulation of STOKES problem :

$$\begin{cases} \alpha\ \underset{\sim}{u}_i - \nu\ \Delta \underset{\sim}{u}_i + \underset{\sim}{\nabla} p_i = 0 \quad \text{ON}\ \Omega_i, \\ \underset{\sim}{\nabla}^{t}.\underset{\sim}{u}_i = 0\ \text{ON}\ \Omega_i,\ \underset{\sim}{u}_i = 0\ \text{ON}\ \Gamma_i,\ \underset{\sim}{\sigma}_i = \underset{\sim}{\lambda}\ \text{ON}\ \Sigma. \end{cases}$$

As in the preceding section, this operator is an isomorphism associated with a symmetric bilinear form k coercive on T.

Operator K, which makes the coupling between $\underset{\sim}{\sigma}\Sigma$ and $\underset{\sim}{u}(\underset{\sim}{\sigma})\Sigma$ explicit, via the homogeneous STOKES problem, yields the boundary value of the stress vector on Σ. In order to do this, one inverts this operator as follows :

$$K(\lambda) = -\left[u^{\circ}\right] = -u_1^{\circ}\Big|_{\Sigma} + u_2^{\circ}\Big|_{\Sigma}$$

where u_i° is the solution to

$$\begin{cases} \alpha\ u_i^{\circ} - \nu\ \Delta u_i^{\circ} + \underset{\sim}{\nabla} p_i^{\circ} = \underset{\sim}{f}_i \quad \text{ON}\ \Omega_i, \\ \underset{\sim}{\nabla}^{t}.\underset{\sim}{u}_i^{\circ} = 0 \quad \text{ON}\ \Omega_i,\ \underset{\sim}{u}_i^{\circ} = \underset{\sim}{0}\ \text{ON}\ \Gamma_i,\ \underset{\sim}{\sigma}_i^{\circ} = \underset{\sim}{0}\ \text{ON}\ \Sigma. \end{cases}$$

Those two formulations allow to find the trace of the velocity on Σ (primal form.) or of the stress (dual form.) of the solution of the initial problem by inverting a

boundary operator defined on Σ (referred to as the coupling operator). The solutions in each sub-domain, computed independently, are the restrictions of the initial problem solution to the corresponding sub-domains.

2.5 Finite element discretization

2.5a Space discretization

For the numerical solution of each problem in each sub-domain we use a so-called TAYLOR-HOOD lagrangian-type finite element discretization which assures continuity of pressure (see [14]).

Figure 3 : Definition of compatible meshes

Two compatible meshes $\tau_h^{e=1,2}$ (see fig. 3) being defined, the approximation spaces are :

$$H_h^{1\ell} = \left\{ \underset{\sim}{v}_h \in \mathcal{C}^0(\Omega\ell)^2 ; \underset{\sim}{v}_n^\ell \Big|_k \in P_2(k), \forall k \in \tau_n^\ell \right\} \quad \text{for the velocity } \underset{\sim}{V}_1$$

$$Q_h^\ell = \left\{ q_h^\ell \in \mathcal{C}^0(\Omega\ell) ; q_n^\ell \Big|_k \in P_1(k), \forall k \in \tau_n^\ell \right\} \quad \text{for the pressure} \in P_1 \left(\int_{\Omega\ell} q_n^\ell \, dx = 0 \right)$$

P_2 - interpolation for velocities, P_1 - interpolation for pressures ensure the conservation of discrete coupling operators properties since they satisfy BREZZI'S discrete 'inf-sup' condition (see GLOWINSKI-PIRONNEAU [1]), from which existence and uniqueness for the discrete STOKES problem are derived.

2.5b Discrete coupling operator

The choice of space-discretization induces quite naturally the discretization of the coupling operator. Velocity and stress space is spanned by the traces of trial functions of H^{11} (or H^{21} since their traces coincide by constructions). With each degree of freedom is associated a column of the coupling operator through the discrete (coupling) bilinear forms m_h and k_h, discrete analogous of form m and k.
The discrete form of the coupling operator is a ND-dimensional symmetric matrix (ND is the number of P2 - nodes on Σ times the dimension N of the physical space) and of rank R = ND - 1, since we did not introduced in our space of approximation the compatibilty condition associated with $H_{oo}^{1/2}$ and T spaces :

- zero mean-flux for the primal formulation ;
- uniform stress quotient for the dual formulation.

2.6 Solution to the coupling equations

In these formulations, sub-structuring into non-overlapping sub-domains is equivalent to the solution of a linear system which dimension is proportionnal to the number of nodes on the coupling boundary. Two efficient solution algorithms can be used according to the physical origin aspect of the problem.

2.6a Assembly and factorization

When a (relatively) small number of coupling nodes is involved and in the case of time-independent coefficients, the coupling operator can be derived numerically at a small cost. This requires, in the first place, the solution of a STOKES problem for each coupling degree of freedom while elementary computations are made on each sub-domain independently. The assembly of all these contributions yields the columns of the operator associated with an unfortunately non-sparse though symmetric matrix.
With kernel of rank 1 eliminated, the factorization of the matrix can be retained to compute at each time-step the boundary values on the coupling boundary by inversion of a small boundary system. The computing time of the solution method (incomplete Cholevski factorizations preconditionning of conjugate gradient method) can be neglected in front of the computing time of the velocity-pressure solution process.

2.6b Iterative solution

On the other hand, one will better use an iterative solution process for the coupling operator whenever either one deals with time-dependent problem, domains involving a large number of coupling nodes (as in the 3-D case) or when coefficients α and ν are time-dependent (as with case of turbulent flows). The matrix properties (symmetric and positive definiteness) allow the use of efficient method such as the conjugate gradient method. A solution of STOKES problem in each sub-domain is then required at each iteration.

2.7 Conclusion

The choice of either approach depends highly on the type of applications one has in sight. In our case (large number of unknowns, time dependent P.D.E. coefficients) as well as for technical considerations (definition of normal vectors, handling of operator kernel ; see ref. [5]), the primal formulation associated with an iterative solution process of the coupling operator is best-suited.
In this frame, we shall study in the second part of this paper the main features of the method for two types of benchmarks : numerical simulation of the flow in a square cavity and around a cylinder.
A first estimation of the comparison of computing costs in the cases of domain decomposition methods and of global solutions results in the quantitative analysis of these first tests.

B) NUMERICAL RESULTS

1 - Introduction

As we mentioned earlier, the coupling operator discretization is a rather natural consequence of the finite element discretization of the velocity-pressure variables and implies no particular difficulties.

The reader is invited to refer to [5] in order to deepen the problems occuring with implementation and convergence acceleration method. Thoroughful flow diagram, numerical treatment of the zero-flux condition applied to the gradient, possible preconditionning as well as refined algorithms to ensure the continuity of the pressure across the coupling boundary are presented there.

We will, in what follows, focus on the presentation and analysis of a few numerical simulations.

2. Moving cavity benchmark

2.1 Physical problem and discretisation

We study the flow confined in a unit square cavity, set in motion by the displacement of the upper boundary ($\underset{\sim}{v}.\underset{\sim}{i}$ = 1 m/s). A rotation motion of the fluid is induced by the shear along the upper boundary, and a stationary phase is obtained once all of the input energy is dissipated by viscosity.

We do not consider here the efficiency of UZAWA'S methods which are presented by LABADIE [15] , CAHOUET and CHABARD [2] , but we rather focus on the comparison between global and domain decomposition methods.

The features of F.E. meshes of the whole domain and of the sub-domains are summed up in table 1 and displayed on fig. II.1. We choose a domain decomposition into two balanced sub-domains, one of them not interfering with the global boundary, in order to have a large number of coupling nodes (48 P_2 - nodes, viz. 96 degrees of freedom). As a matter of comparison, the rank of the coupling operator is the same of the one MARTINAUD [4] used in a realistic case (study of storm surges in the Northern sea) involving a 4 sub-domains decomposition.

	Number of triangles	Number of P2-nodes	Number of P1-nodes	Number of unknowns
global region	200	441	121	1003
outer sub-region 1	128	320	96	736
inner sub-region 2	72	169	49	387

Table 1 : Features of the finite element meshes

2.2 Study of the stationary problem

Though we are generally faced with time-dependent problems (in order to take hyperbolic terms into account), we prefer to present first the results of a stationary problem which allow a straight forward analysis of the influence of the various parameters involved in the algorithm, leaving aside time discretization. A conjugate gradient has been used for all these tests.

A null vector is chosen to initialize the iterative process for the velocity field along the coupling boundary. As soon as the 8th step of the iterations is reached, a rather good approximation of the vortex is obtained ; subsequent steps lead to a refinement of the flow in the corners of the cavity. The limit of the solution is reached after 17 steps, slight discrepancies of the continuity of pressure (cf. fig. II.2) can be pointed out which are emphasized in the vicinity of high gradient regions.

Stopping-criterion of the iterative process and comparison of costs

One is immediately faced with the problem of choice of a stopping criterion in the conjugate gradient iterative process bearing on the coupling operator. The acuteness of the criterion has a great influence on computing times and on the conclusion concerning the ratio between quantities involved in sub-structuring and global methods.

We propose a stopping criterion, based on the variations of two evolutive quantities, to study both the convergence of the iterative process and the quality of results ; these two quantities are :

- Relative jump of stress along Σ , viz $SC^{\ell} = \dfrac{\| \bar{\sigma}_1 \underset{\sim}{n}_1 + \sigma_2 \underset{\sim}{n}_2 \|}{\| \sigma_{REF} \|}_{L^2(\Sigma)}$

  (ℓ referring to the number of the step of the iterative process)

- Relative velocity variation on Σ from one step to the other

$$\Delta u^{\ell} = \frac{\| \underset{\sim}{u}_{\Sigma}^{\ell+1} - \underset{\sim}{u}_{\Sigma}^{\ell} \|}{\| \underset{\sim}{u}_{REF} \|}_{L^2(\Sigma)}$$

which can be alternatively read

$$\Delta u^{\ell} = |\rho| \frac{\| [\underset{\sim}{\sigma}]^{\ell} \|}{\| \underset{\sim}{u}_{REF} \|} , [\underset{\sim}{\sigma}]^{\ell} \quad = \text{direction of descent at step } \ell.$$

In these definitions $\| \sigma \|_{ref}$ and $\| u \|_{ref}$ are estimations of the norm of both stress and velocity vectors along Σ.

In the stationary case a good approximation is known as soon as the 5th step of the iterative process is reached ; with time-dependent case, we can take updated values. Table 2 sums up the evolution of both quantities as well as the ratio of CPU time associated with sub-structuring, to CPU time associated with the global method. Those results were obtained with a quite sharp stopping criterion for the iterative solution of STOKES problem via UZAWA'S method :

$$\frac{\| \underset{\sim}{\nabla}^{t} \underset{\sim}{v} \|}{\| \underset{\sim}{v} \|}_{L^2(\Omega)} < 2\ 10^{-6} \qquad \text{or} \quad \| \underset{\sim}{\nabla}^{t} . \underset{\sim}{v} \| < 2\ 10^{-6}$$

Such a degree of accuracy is only needed when one wishes to refine the quality of the coupling because the incompressibility condition works as a control over the pressure field, which is directly involved in the estimation of the stress jump. (An insufficient precision might lead to divergence of the conjugate gradient iterative process while computing the coupling operator).

Step of the iterative process	SC stress relative jump	V relative variation of velocity	Ratio Sub.Str. / global CPU time
2	-	-	1.05
4	0.96	0.35	1.84
6	0.91	0.18	2.55
8	0.74	0.06	3.33
10	0.19	0.08	4.04
12	0.22	0.007	4.52
14	0.11	0.013	5.08
16	0.04	0.010	5.56
18	0.02	0.002	6.00
20	0.01	0.001	6.15

Table 2

The conclusion from this first (small) stationary test are :

- an important increase (up to a factor 5 or 6) of CPU time is implied when sub-structuring is used, when one does not take the smoothness of sub-region meshes into account in any way.
- a stopping criterion on the iterative solution of the coupling terms of the form

  $SC < 1\ \%$ or $\Delta u < 0{,}5\ \%$

  ensures a satisfactory solution whenever the UZAWA'S stopping criterion is coherent with those criteria.

2.3 The time dependent case

a The limiting stationary state

In running this test we have set $\alpha = 0.01$ and $DT = 0.1$ s which corresponds to C.F.L. element-built number of order 2 in the regions where the velocity is maximum.
The stationary state is reached after 7 seconds (that is 70 time steps). In our analysis, we only consider the 50 first time-steps since the evolution of solution is quite insensitive afterwards. The reference solution of this test was obtained after convergence of the coupling operator at each time-step according to the stop criteria :

$$SC < 10^{-2} \text{ or } \Delta u < 10^{-2} \ (\text{or } \Delta u < 5.10^{-3})$$

b Variations of the number of iterations with respect to time

At each time-step the initial values for the iterative process are the final values of the preceding time-steps for both the coupling terms and preconditioned UZAWA's method. In doing so, we lessen as much as possible the number of iterations on the coupling terms when a steady state limit exists.

The following table shows up the variation, with respects to time of the number of iterations of the conjugate gradient induced by the aforementioned stopping criteria.

T in second		0.1	0.2	0.3	0.4	0.5	1	2	> 3
NUMBER OF G.C.ITERATIONS	$U < 10^{-2}$	10	8	7	7	7	5 to 6	2 to 5	1 or 2
	$U < 510^{-3}$	11	11	10	7	7	6	2 to 5	1 to 3

The overcost induced by substructuring, computed on the time interval from 0 to 5 s., is given by :

$$\frac{Ts.s}{Tglob} = 6.5 \text{ for } \Delta u < 10^{-2} \quad \text{or} \quad \frac{Ts.s}{Tglob} = 7.7 \text{ for } \Delta u < 5.10^{-3}$$

for a computational test we consider as a benchmark. In other configurations, it is possible to lessen this overcost (while, of course, losing some of the quality of the results at each time-step).

c Limiting the number of iterations

A first method is to bound the number of iterations in the computation of the coupling independently of the stopping-criteria. This approach is correct if one is only interested in the stationary state and it gives the following ratios (when applied together with the preceding stop criteria, viz. $SC < 10^{-2}$ and $\Delta u < 10^{-2}$).

4 iterations limit : $\frac{Tss}{Tglob} = 5.3$

3 iterations limit : $\frac{Tss}{Tglob} = 4.8$

and the solution at time 5 s is still in good agreement.

Another method is to choose a more coarser stopping-criteria for which the incompressibility condition can be relaxed as to lessen CPU time at each step. The adaptation of the various parameters is then quite tricky, and small improvement in CPU time is balanced by a noticeable loss on global accuracy.

d Conclusions

Those series of numerical tests run on a mesh involving a relatively few number of nodes allowed us to validate the primal coupling technique and to have a good idea of how influent the different parameters are.
About stopping-criteria, we consider that the limit with recurrent process in the computations of the coupling terms is reached when either the relative jump of the stress vector or the relative velocity variation on the coupling boundary is less than one percent, while the relative norm of the divergence over each sub-domain is of order 10^{-5}.
When all these hypotheses hold, sub-structuring and classical finite element solution in each sub-domain yield an increased CPU time with a scaling factor of about 5 with respect to the classical solution on the global domain.

In the following those comparisons are made on a larger scale problem in which the number of unknowns is of the order of the ones one meets in 3-D problems and in which one hopes to take the geometrical smoothness of the sub-domain meshes into account.

3. Flow around a cylinder test

3.1 Physical problem and discretization

In what follows, we are interested in external flows (such as aerodynamics, weather forecast,...), in which the refined part of the mesh covering a small domain nearby the obstacle, is small with respect to the physical domain relevant for the computation of the flow.

We have chosen the 2-D flow around a cylinder lying in a uniform flow of unit-velocity field at infinity. The discretization of the global domain (of length 25 D, where D denotes the diameter and of width 8 D, see fig. II.3) is made after IBLER [16] .

In the case of sub-structuring, we have chosen a 2 sub-domains decomposition with the same number of nodes as in the global discretization. In the acute mesh of the inner sub-region (enclosing the obstacle) a finite element method is carried out while the outer region is regularly gridded by a finite difference type of mesh (fig. II.3). The features of the discretization of the 3 regions are summed up in the following table;

	Number of triangles	Number of P2-nodes	Number of P1-nodes	Number of unknowns
global region	1158	2440	641	5521
outer sub-region 1	840	1804	482	4090
inner sub-region 2	318	684	183	1551

3.2 Convergence of stationary and transient time-dependent and independent solutions

We display on figure II.4 the velocity field and pressure distribution of the domain decomposition time independent solution after 8 iteration steps on the coupling term computations.

3.3 Stationary and transient CPU time comparisons

In order to refine our CPU times estimations, we will distinguish between them in each sub-region computation. Several reckonings can thus be made while supposing that the smoothess in the outer sub-domain implies an improvement (speed up) time factor denoted by g. We can then follow the evolution of ratio between sub-structuring and global CPU time when factor g goes from 1 (present computation) up to 50 (expected speed up resulting from finite differences computations)(cf Table 3).

a Steady case

One can see that 88 % of CPU time is required in the solution of STOKES problems in the outer region (which only involves 2.6 times as much unknowns as the inner region).
The slightest speed up due to smoothness improving brings a noticeable improvement of the ratios. A speed up factor of order 5 (expected through the use of frontal methods) makes the sub-structuring computation competitive as far as CPU time is concerned.

	global	Sub-structuring	Smoothness influence (factor g) on the ratio Sub-St. CPU / global CPU				
		inner + outer = both	g=1	g=2	g=5	g=10	g=50
STEADY	350	8it 122 + 955 = 1077	3.1	1.7	0.9	0.62	0.4
		12it 176 + 1366 = 1542	4.4	2.5	1.3	0.9	0.58
UNSTEADY							
1^{st} time step	24.2	40 + 138 = 178	7.3	4.5	2.8	2.2	1.8
usual time step (30^{th})	6.6	5.1 + 14.4 = 19.5	3	1.9	1.2	1	0.81
overall up to 5s of physical time	390	300 + 900 = 1200	3.1	1.9	1.2	1	0.81

Table 3 : Comparisons of global and sub-structuring solution CPU times according to expected speed-up computation (g) due to outer mesh smoothness.

One can also note an appreciable improvement of the ratio for classical finite element computation (g = 1), with respect to the mobile cavity problem ; this improvement results from an enlargement of the rank of the linear systems involved. Under a 1,000 P2 - nodes limit (mobile cavity test), the increase in CPU time for each UZAWA iteration is linear, then behaves as a power function n^{γ}, $\gamma = 1.3$, as the 2,000 nodes limit is reached. We may conclude that the "gross" overcost implied by direct domain decomposition (g = 1) decreases as a function of the number of unknowns. In doing so we assume the number of UZAWA iterations does not decrease proportionately with respect to the number of unknowns ! though this is the practical case since the number of iterations is always less than 5.

b Transient case

The speed up factor is much less sensitive in this case (75 % of CPU time spent on the outer region which represent 72 % of the unknowns) and the resulting overcost ratio is of order 3 in the standard solutions (g = 1) when the stopping criteria are

$$SC < 10^{-2} \quad \text{or} \quad \Delta u < 10^{-2}$$

Those results agree with the nearly-linear variation ($\gamma = 1.1$) of CPU time for each preconditionned UZAWA iteration (two linear system solutions for the velocity, one for the pressure) VS. the number of unknowns. This linear variation of CPU time (-VS. the number of nodes) for STOKES solution by preconditionned UZAWA method has been recently confirmed by large sized transient 3-D computations (of about 80,000 unknowns, see CAHOUET-CHABARD [2]).

c Conclusions

As the overcosts implied in the domain decomposition decreases as the number of involved unknowns increases, the importance of this reduction depends straight forwardly on the influence of the number of unknowns on the CPU time solution of STOKES problem.

In the case of external flows, an estimated minimal speed up factor of order 10 is expected for the solution of 3-D time-independent problems. On the contrary when applied to transient problems, domain decomposition methods are hopelessly harmed by the efficiency of STOKES problem solution.

3.4 Use of parallel processing computers

In the beginning the attraction borne to domain decompositions methods resulted in their possibilities to get rid of mass-storage bounds on the computer. For instance, in the beginning of the 80's, the bound was of about 80,000 unknowns on a 17,000 triangle mesh for a 2-D problem on E.D.F.'s CRAY 1 S. To day, with the increase in mass-storage, those bounds are obsolete, and the limitations bear on CPU times which are required for large computations. To overcome these limitations one naturally thinks of parallel computers.

We have, quite intentionally, checked that the algorithm we use, have an important rate of "parallelism-ness" (> 99 %) and is thus quite suitable for multiprocessor computers. A thoroughful study of computing times shows that the theoretical speed-up factor, decreasing time factor for resuming the job (though accumulated CPU time is slightly increased), strongly depends on the balance between the different sub-domain size as the following table demonstrates :

	DRIVEN CAVITY		CYLINDER TEST					
	STEADY	UNST.	STEADY			UNST.		
	g = 1	g = 1	g = 1	g = 3	g = 10	g = 1	g = 3	g = 10
Theoretical Speed-up	1.81	1.62	1.12	1.38	1.78	1.33	1.99	1.3

Table 4 : Theoretical speed-up using a 2-domains decomposition computation run on a 2 - processor computer ($\Rightarrow$ speed-up < 2).

Since every elementary computation requires more than 1 s of CPU time, we can neglect the overhead elapsed time so that the estimations presented here are quite realistic ; and thus a balance of each elementary task, which is required for an optimal efficiency, is an additional criterion in the domain decomposition process which we feel as extremely constraining.

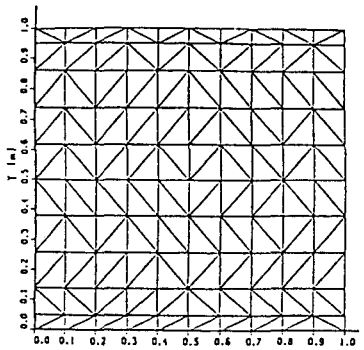
CONCLUSION

Starting as soon as 1980, the members of the L.N.H., quite aware of the growing interest in domain decompositions methods in fluids mechanics, have developped a great amount of work concerning the numerical solution of incompressible (NAVIER-STOKES) or weakly compressible (St-Venant) flows. This work led the frame to devise the notion of STEKLOV-POINCARE coupling operators for those mixed-type problems and to analyze the efficiency of several solutions methods involving finite-element discretizations.

Those methods are proved adaptative and reliable on many applications and seem quite competitive in the case of time-independent geometries and P.D.E's coefficients. In many cases the overcost in the computation of the coupling operator is balanced by the speeding up of the iterative algorithm process.

Nevertheless, the increase of computer powerfulness, the growing efficiency of numerical methods and the complexity of embraced P.D.E's (time-dependent coefficients) do not seem to justify the use of such methods in that context ; as matter of fact their overcost (almost as 5 times as high as an equivalent global solution) make it appear quite improbably balanced by the use of the smoothness of the mesh of certain sub-domains. Moreover, for the time being, the parallelization of such algorithms is rather out of question ; the modification into parallelism of a 3-D industrial code (tens of thousands of FORTRAN instructions) in a multitasking structure would require many years of an engineer for an hypothetical result, and, the running of such a code which requires a computer dedicated to the user, is not compatible with the constraints of an industrial computing center.

Thus, those considerations lead us, for the time being, to inquire about the relevancy of such methods to the industrial problems which we are faced to.



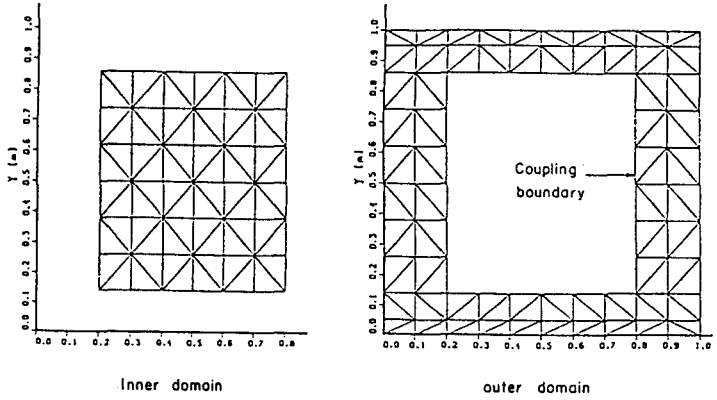
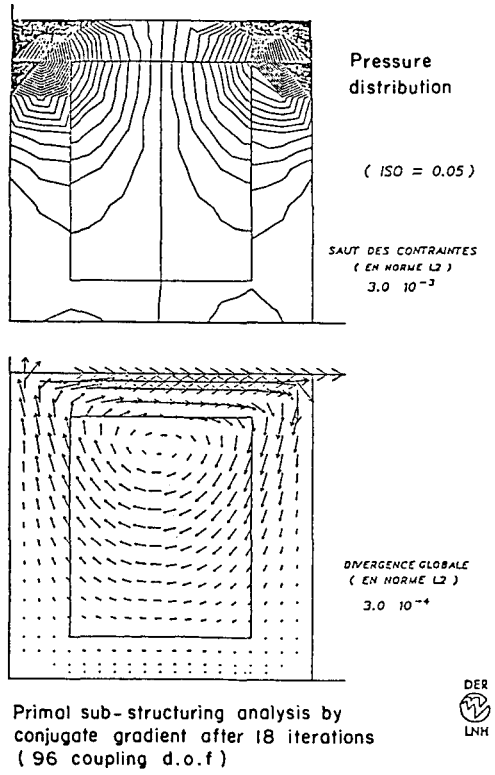


Figure II.1 : F.E. Meshes.



Primal sub-structuring analysis by conjugate gradient after 18 iterations (96 coupling d.o.f)

Figure II.2 : Pressure and velocity fields at convergence.

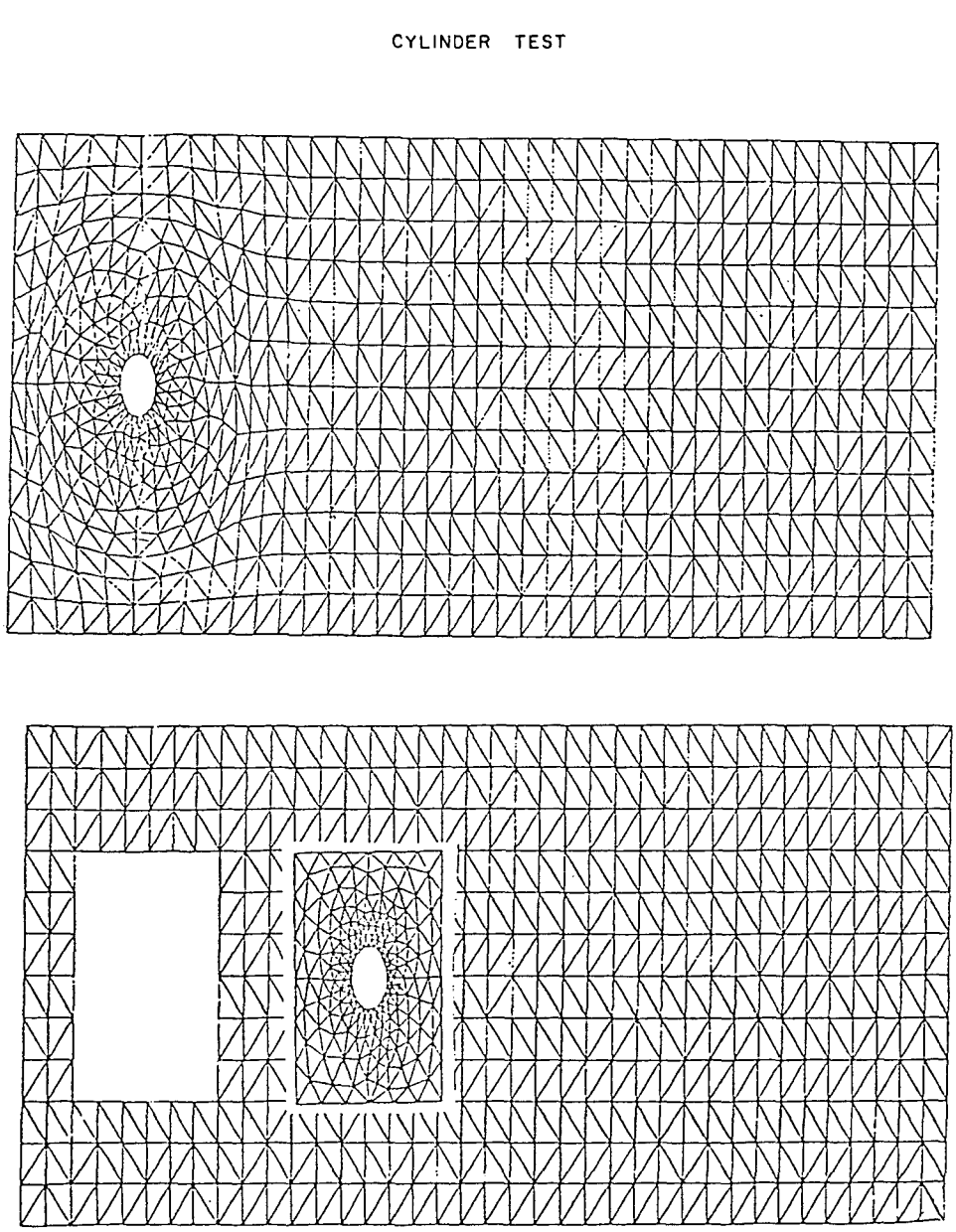
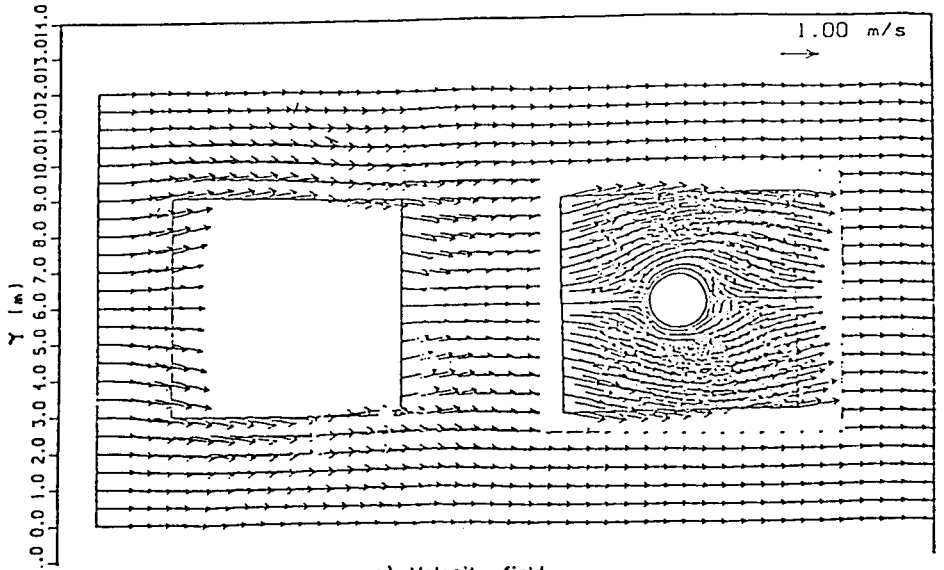
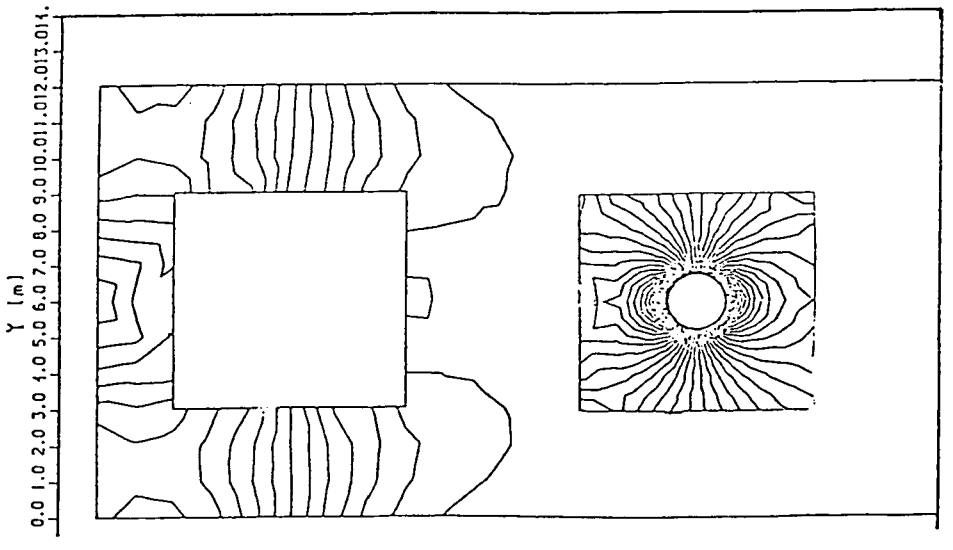


Figure II.3 : Global and sub-structured meshes.



a) Velocity field



b) Pressure distribution

Figure II.4 : Velocity field and pressure distribution obtained after 8th iteration.

REFERENCES

(1) GLOWINSKI R., PIRONNEAU O. On a mixed finite element approximation of the Stokes problem, Num. Math., 33 (1979) p. 397-424.

(2) CAHOUET J, CHABARD J.P.C. Somme Fast 3.D Finite Element Solvers For Generalized Stokes Problem to appear in Numerical Methods in Fluids (1987).

(3) CHABARD J.P.C., GREGOIRE J.P. Avancement du projet N3S de mécanique des fluides Rapport EDF/HE41/86.04 (1985), (in French).

(4) MARTINAUD J.P. Approximation éléments finis d'équations aux dérivées partielles par une méthode de décomposition de domaines. Applications aux équations de Stokes et Saint-Venant Thèse - Université PARIS VI - 20 juin 1984. And report EDF/HE41, (in French).

(5) CAHOUET J.(1985) Méthodes de sous-structurations de domaine pour les équations de Navier-Stokes Rapport EDF HE41/85.03, (in French).

(6) CAHOUET J., CHABARD J.P. Multi-domains and multi-solvers finite element approach for the Stokes problems 4th International Symposium on numerical methods in engineering, Atlanta, 1986, Springer Verlag, New-York (1986).

(7) BENSOUSSAN A., LIONS J.L., TEMAM R. Sur les méthodes de décomposition, de décentralisation, de coordination et applications dans méthodes numériques en sciences physiques et économiques LIONS J.L., MARCHOUK G.I. ed, Dunod-Bordas PARIS 1974, (in French).

(8) LEMONNNIER P. Résolution numérique d'équations aux dérivés partielles par décomposition et coordination Cahier de l'IRIA 1972, (in French).

(9) LICHNEWSKY A. Sur la résolution des systèmes linéaires issus de la méthode des éléments finis par une machine "multiprocesseur", Rapport de recherche n° 119 INRIA - février 1982, (in French).

(10) DINH Q.V. Simulation numérique en éléments finis d'écoulements de fluides visqueux incompressibles par une méthode de décomposition de domaines sur processeurs vectoriels, Thèse, université P. et M. Curie PARIS VI soutenue le 11/03/1982, (in French).

(11) CAMBIER L., GHAZZI W., VEUILLOT J.P., et VIVIAND H. Une approche par domaines pour le calcul d'écoulements compressibles 5ème colloque international sur les méthodes de calcul scientifique et technique de l'INRIA VERSAILLES 14-18 décembre 1981, (in French).

(12) FISCHLER A. (1985) Résolution du problème de Stokes par une méthode de décomposition de domaine Thèse, Université PARIS VI, mai 1985, (in French).

(13) GLOWINSKI R., LIONS P.L., PERIAUX J. (à paraître).

(14) HOOD P., TAYLOR C. A numerical solution of Navier-Stokes equation using the finite element technique Computers and fluids 1 pp. 73-100 (1973).

(15) LABADIE G., LASBLEIZ P. Quelques méthodes de résolution du problème de Stokes en éléments finis Rapport EDF.HE41/83.01 (1983),(in French).

(16) IBLER B. Résolution des équations de Navier-Stokes par une méthode d'éléments finis Thèse de 3ème cycle - Université PARIS-SUD 1981, (in French).

(17) GOUTAL N. Résolution de l'équation de transport par une méthode de décomposition de domaine Rapport de stage D.E.A. analyse numérique PARIS VI 1984, (in French).

(18) AGOSHKOV V.I. Poincaré-Steklov's operators and domain decomposition methods in finite dimensional spaces, (in this book).