On the Coupling of Viscous and Inviscid Models for Incompressible Fluid Flows Via Domain Decomposition

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Abstract. We discuss in this paper the coupling between the Navier-Stokes equations for unsteady incompressible viscous flows with the Laplace equation modeling inviscid incompressible potential flows. The coupling is done through a domain decomposition procedure with overlapping; with such technique one can take advantage of an operator splitting technique for the time discretization of the Navier-Stokes equations.

Numerical results obtained from finite element approximations are presented showing that the present method provides a matching technique of good quality.

1. Generalities. Synopsis.

The main goal of this paper is to present a computational method for the coupling of two distinct mathematical models describing the same physical phenomenon, namely the flow of an *incompressible viscous fluid*. The basic idea is to replace the Navier-Stokes equations by the potential one in those regions where we can neglect the viscous effects and where the vorticity is small.

Consider for example a flow around an obstacle; we can split the computational domain into two overlapping subdomains:

A first one, containing the obstacle, where the flow is modeled by the Navier-Stokes equations.

A second, that we suppose to be far enough from the obstacle so that the Navier-Stokes equations reduce there to Laplace equation for the velocity potential (assuming of course that in this second region the flow is vorticity free).

Our goal here is to discuss a method for coupling both the Navier-Stokes and Laplace equations for incompressible fluids. We will there-

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fore describe the continous equations, and then using a time discretization of the Navier-Stokes equations by operator splitting, reduce the original problem to a sequence of matching problems for linear models.

Then we will solve the matching problems-which can be seen as linear control problems-by a conjugate gradient algorithm.

The possibilities of such techniques will be illustrated by the results of numerical experiments for two dimensional flows around a cylinder and around a NACA 0012 airfoil.

2. Mathematical Modeling of the Flow Problem.

We consider the unsteady flow of an incompressible viscous fluid around an obstacle B. This flow is modeled by the time dependent Navier-Stokes equations, which here take the following form

$$(2.1)\qquad \frac{\partial \tilde{u}}{\partial t} - \nu\Delta\tilde{u} + (\tilde{u}\cdot\tilde{\nabla})\tilde{u} + \tilde{\nabla} p = \tilde{0} \quad \textit{in} \quad \Omega ,$$

$$(2.2)\qquad \tilde{\nabla}\cdot\tilde{u} = 0 \quad \textit{in} \quad \Omega \ (\textit{incompressibility condition}),$$

$$(2.3)\qquad \tilde{u}(x,0) = \tilde{u}_0\,(x) \quad (\textit{initial condition}),$$

$$(2.4)\qquad \tilde{u} = \tilde{0} \quad \textit{on} \quad \partial B = \Gamma_B \quad (\textit{no-slip condition}),$$

$$(2.5)\qquad \tilde{u} = \tilde{u}_\infty \quad \textit{on} \quad \textit{the external boundary} \quad \Gamma_\infty \ .$$

Here:

(i) $\tilde{u} = \{u_i\}_{i=1}^N$ is the flow *velocity* (N=2,3 *in practice*),

(ii) p is the *pressure*,

(iii) ν is a *viscosity coefficient*,

(iv) $$(\tilde{u}\cdot\tilde{\nabla})\tilde{u} = \left\{ \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} \right\}_{i=1}^{N} .$$

If we assume that the flow is potential in some region of the flow domain Ω , we have

$$(2.6)\qquad \tilde{\nabla}\times\tilde{u} = \tilde{0} \quad ,$$

i.e. there exists a *potential* ϕ such that

$$(2.7)\qquad \tilde{u} = \tilde{\nabla}\phi \quad .$$

Combining (2.7) with the incompressibility condition we obtain

$$(2.8)\qquad \Delta\phi = 0 \ .$$

If we assume that the potential flow region contains Γ_∞ partly or entirely we shall take as boundary condition there

(2.9) $\frac{\partial \phi}{\partial n} = \underset{\sim}{u}_\infty \cdot \underset{\sim}{n}_\infty$,

where $\underset{\sim}{n}_\infty$ denotes the unit outward normal vector at Γ_∞ .

We decompose the computational domain (still denoted by Ω) in two subdomains Ω_1 and Ω_2 such that the flow is governed by (2.1), (2.2) in Ω_2 , and by (2.8) in Ω_1 . The notation is like in Figure 2.1, below

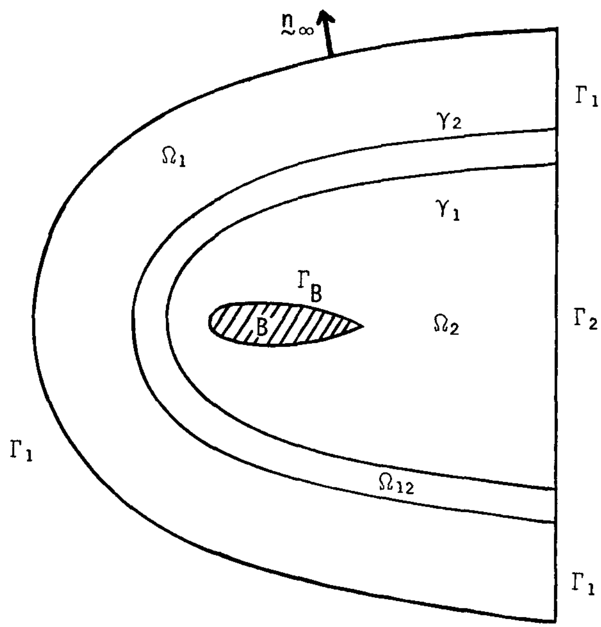


Figure 2.1

where:

(a) $\Omega_{12} = \Omega_1 \cap \Omega_2$,

(b) γ_1 and γ_2 are interfaces between Ω_{12} and Ω_2 , Ω_{12} and Ω_1 , respectively,

(c) $\Gamma_1 = \Gamma_\infty \cap \partial\Omega_1$, $\Gamma_2 = \Gamma_\infty \cap \partial\Omega_2$.

Our goal here is to solve (2.1), (2.2) in Ω_2 coupled to (2.7), (2.8) in Ω_1 . Actually some extra boundary conditions have to be specified to obtain well-posed problems for $\{\underset{\sim}{u},p\}$ and ϕ ; we shall take

(2.10) $\phi = \psi$ *on* γ_1 ,

(2.11) $\underset{\sim}{u} = \underset{\sim}{v}$ *on* γ_2 .

If the (yet unknown) traces ψ and $\underset{\sim}{v}$ are specified we can compute ψ and $\{\underset{\sim}{u},p\}$ (p is in fact known within an arbitrary additive constant) if $\underset{\sim}{v}$ satisfies the following flux condition (direct consequence of the incompressibility):

$$(2.12) \quad \int_{\gamma_2} \underset{\sim}{v}\cdot\underset{\sim}{n}\, d\gamma_2 + \int_{\Gamma_2} \underset{\sim}{u}_\infty \cdot \underset{\sim}{n}\, d\Gamma_2 = 0.$$

To compute ψ and $\underset{\sim}{v}$ and couple the two models, we use a least squares approach in which we minimize over the overlapping region Ω_{12} some distance between $\underset{\sim}{u}$ and $\underset{\sim}{\nabla}\phi$. The minimization problem takes the following formulation

$$(2.13) \begin{cases} \textit{Find } \bar{\psi} \textit{ and } \bar{\underset{\sim}{v}} \textit{ such that} \\ J(\bar{\psi},\bar{\underset{\sim}{v}}) \le J(\psi,\underset{\sim}{v}), \quad \forall \ \{\psi,\underset{\sim}{v}\}, \end{cases}$$

where in (2.13), we have

$$(2.14) \quad J(\psi,\underset{\sim}{v}) = \tfrac{1}{2} \int_{\Omega_{12}} |\underset{\sim}{u}-\underset{\sim}{\nabla}\phi|^2\, dx,$$

and where $\underset{\sim}{u}$ (resp. $\bar{\underset{\sim}{u}}$) is the solution in Ω_2 of the Navier-Stokes equations associated to $\underset{\sim}{v}$ (resp. $\bar{\underset{\sim}{v}}$), with a similar definition for ϕ and $\bar{\phi}$ (we obviously assume that $\underset{\sim}{v}$ and $\bar{\underset{\sim}{v}}$ satisfy (2.12)).

To solve this matching problem which is definitely nonlinear we will take advantage of a time discretization of the Navier-Stokes equations founded on operator splitting; with such technique the time dependent coupling problem is reduced to a sequel of matching problems for linear elliptic equations.

Remark 2.1: Problem (2.13), (2.14) has the structure of an *optimal control problem* in the sense of LIONS [1]; this interpretation is quite interesting since it will suggest applying classical techniques of optimal control to the solution of the above problem, and also of the matching problems obtained at each time step.

3. Time Discretization of the Matching Problem Via an Operator Splitting Time Discretization of the Navier-Stokes Equations.

3.1. *Time Discretization of the Navier-Stokes Equations. Synopsis.*

Following [2] - [4] , we describe here a time discretization which reduces the solution of the unsteady Navier-Stokes equations in Ω_2 , to a sequence of steady Stokes problems and nonlinear elliptic systems.

Let $\Delta t>0$ be a *time discretization step* and with $\Theta \in (0,\tfrac{1}{2})$ define α and β by

$$\alpha = (1-2\Theta)/(1-\Theta) \ , \ \beta = \Theta/(1-\Theta).$$

If z^{s} denotes the approximation of a time dependent variable z at time $s\Delta t$, we approximate the Navier-Stokes equations on Ω_2 by

$$(3.1) \quad \underset{\sim}{u}^0=\underset{\sim}{u}_0 \quad \textit{in } \Omega_2 \ ;$$

then for $n\geq0$, *assuming that* $\underset{\sim}{u}^n$ *is known we compute first*

$\{\underset{\sim}{u}^{n+\theta}, p^{n+\theta}\}$ *solution of*

$$(3.2)\quad \begin{cases} \dfrac{\underset{\sim}{u}^{n+\theta}-\underset{\sim}{u}^{n}}{\theta\Delta t} - \alpha\nu\Delta\underset{\sim}{u}^{n+\theta} + \underset{\sim}{\nabla}p^{n+\theta} = \beta\nu\Delta\underset{\sim}{u}^{n} - (\underset{\sim}{u}^{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n} \quad in \quad \Omega_2 , \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u}^{n+\theta} = 0 \; in \quad \Omega_2, \\ \underset{\sim}{u}^{n+\theta} = \underset{\sim}{0} \quad on \;\; \Gamma_B , \; \underset{\sim}{u}^{n+\theta} = \underset{\sim}{u}_{\infty}^{n+\theta} \;\; on \;\; \Gamma_2, \; \underset{\sim}{u}^{n+\theta} = \underset{\sim}{v}^{n+\theta} \quad on \quad \gamma_2 ; \end{cases}$$

then

$$(3.3)\quad \begin{cases} \dfrac{\underset{\sim}{u}^{n+1-\theta}-\underset{\sim}{u}^{n+\theta}}{(1-2\theta)\Delta t} - \beta\nu\Delta u^{n+1-\theta} + (u^{n+1-\theta}\cdot\nabla)u^{n+1-\theta} = \alpha\nu\Delta u^{n+\theta} - \underset{\sim}{\nabla}p^{n+\theta} \; in \;\; \Omega_2 , \\ \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{0} \quad on \quad \Gamma_B , \; \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{u}_{\infty}^{n+1-\theta} \;\; on \; \Gamma_2, \; \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{v}^{n+1-\theta} \; on \;\; \gamma_2 ; \end{cases}$$

and finally

$$(3.4)\quad \begin{cases} \dfrac{\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{n+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta\underset{\sim}{u}^{n+1} + \underset{\sim}{\nabla}p^{n+1} = \beta\nu\Delta\underset{\sim}{u}^{n+1-\theta} - (\underset{\sim}{u}^{n+1-\theta}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1-\theta} \;\; in \;\; \Omega_2, \\ \nabla\cdot\underset{\sim}{u}^{n+1} = 0 \quad in \quad \Omega_2 , \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{0} \quad on \;\; \Gamma_B, \; \underset{\sim}{u}^{n+1} = \underset{\sim}{u}_{\infty}^{n+1} \;\; on \;\; \Gamma_2, \; \underset{\sim}{u}^{n+1} = \underset{\sim}{v}^{n+1} \quad on \quad \gamma_2 . \end{cases}$$

The basic idea behind the coupling method presented below is to require the optimal matching (defined in Section 2) only for the solutions of the *linear* subproblems (3.2), (3.4), and to "freeze" the interface condition for the *nonlinear* subproblem (3.3). The implementation of this idea is described in Section 3.2, just below.

3.2. Time Discretization for the Coupling of the Navier-Stokes and Potential Equations.

With Δt as in Section 3.1, we generalize scheme (3.1) - (3.4) as follows, in order to solve the problem coupling the Navier-Stokes and potential equations according to the matching criterium of Section 2.

Description of the Algorithm:

$$(3.5)\qquad \underset{\sim}{u}^{0} = \underset{\sim}{u}_{0} \quad given \; in \;\; \Omega_2 .$$

Then for $n\geq 0$, $\underset{\sim}{u}^{n}$ *being known we look for a triple* $\{\phi^{n+\theta}, \underset{\sim}{u}^{n+\theta}, p^{n+\theta}\}$ *satisfying*

$$(3.6)_1 \begin{cases} \dfrac{\underset{\sim}{u}^{n+\theta}-\underset{\sim}{u}^{n}}{\theta\Delta t} - \alpha\nu\Delta\underset{\sim}{u}^{n+\theta}+\underset{\sim}{\nabla}p^{n+\theta}= \beta\nu\Delta\underset{\sim}{u}^{n} - (\underset{\sim}{u}^{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n} & in \ \Omega_2 , \\ \nabla\cdot\underset{\sim}{u}^{n+\theta} = 0 \quad in \ \Omega_2 , \\ \underset{\sim}{u}^{n+\theta} = \underset{\sim}{0} \ on \ \Gamma_B , \ \underset{\sim}{u}^{n+\theta}= \underset{\sim}{u}_{\infty}^{n+\theta} \quad on \ \Gamma_2 , \\ \underset{\sim}{u}^{n+\theta} = \underset{\sim}{v}^{n+\theta} \quad on \ \gamma_2 , \end{cases}$$

$$(3.6)_2 \begin{cases} \Delta\phi^{n+\theta} = 0 \quad on \ \Omega_1, \\ \dfrac{\partial\phi^{n+\theta}}{\partial n} = \underset{\sim}{u}_{\infty}^{n+\theta}\cdot\underset{\sim}{n}_{\infty} \quad on \ \Gamma_1 , \\ \phi^{n+\theta} = \psi^{n+\theta} \quad on \ \gamma_1 , \end{cases}$$

with $\underset{\sim}{v}^{n+\theta}$ *and* $\psi^{n+\theta}$ *chosen such that*

$$(3.6)_3 \begin{cases} \displaystyle\int_{\gamma_2} \underset{\sim}{v}^{n+\theta}\cdot\underset{\sim}{n} \, d\gamma_2 + \int_{\Gamma_2} \underset{\sim}{u}_{\infty}^{n+\theta}\cdot\underset{\sim}{n} \, d\Gamma_2 = 0, \\ and \\ \displaystyle\int_{\Omega_{12}} |\underset{\sim}{u}^{n+\theta}-\underset{\sim}{\nabla}\phi^{n+\theta}|^2 \, dx \quad is \ minimal. \end{cases}$$

Next, we look for $\underset{\sim}{u}^{n+1-\theta}$ *solution of*

$$(3.7)_1 \begin{cases} \dfrac{\underset{\sim}{u}^{n+1-\theta}-\underset{\sim}{u}^{n+\theta}}{(1-2\theta)\ \Delta t} - \beta\nu\Delta\underset{\sim}{u}^{n+1-\theta}+(\underset{\sim}{u}^{n+1-\theta}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1-\theta}=\alpha\nu\Delta\underset{\sim}{u}^{n+\theta}-\underset{\sim}{\nabla}p^{n+\theta} & in \ \Omega_2, \\ \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{0} \ on \ \Gamma_B , \ \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{u}_{\infty}^{n+1-\theta} \\ \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{v}^{n+1-\theta} \quad on \ \gamma_2 , \end{cases}$$

with

$$(3.7)_2 \quad \underset{\sim}{v}^{n+1-\theta} = \underset{\sim}{u}^{n+\theta} .$$

Finally, we compute the triple $\{\phi^{n+1}, \underset{\sim}{u}^{n+1}, p^{n+1}\}$ *solution of a system analogue to* $(3.6)_1$ - $(3.6)_3$ *with* n *and* $n+\theta$ *replaced by* $n+1-\theta$ *and* $n+1$, *respectively*.

The boundary condition $(3.7)_2$ is a variation of the approach introduced in [5] for the solution of the Navier-Stokes equations by domain decomposition methods.

Solution methods for problems like $(3.7)_1$, $(3.7)_2$ are discussed in, e.g., [2] - [4]; we shall therefore concentrate (in Section 4) on the solution of the matching problem $(3.6)_1$ - $(3.6)_3$ which is a problem of a quite new type, belonging however to the class of *optimal control problems* for partial differential equations, in the sense of J. L. LIONS [1].

4. Solution of the Matching Problem.

4.1. Generalities.

The matching problem $(3.6)_1$ - $(3.6)_3$ is a particular case of the following problem:

Find a pair $\{\psi,\underset{\sim}{v}\}$ *and a triple* $\{\underset{\sim}{u},p,\phi\}$ *such that*

$$(4.1)\begin{cases} \alpha\underset{\sim}{u}-\nu\Delta\underset{\sim}{u}+\nabla p = \underset{\sim}{f} & \text{in } \Omega_2, \\ \nabla\cdot\underset{\sim}{u} = 0 & \text{in } \Omega_2, \\ \underset{\sim}{u} = \underset{\sim}{g}_2 & \text{on } \Gamma_B \cup \Gamma_2, \\ \underset{\sim}{u} = \underset{\sim}{v} & \text{on } \gamma_2, \end{cases}$$

$$(4.2)\begin{cases} \Delta\phi = 0 & \text{in } \Omega_1, \\ \dfrac{\partial\phi}{\partial n} = g_1 & \text{on } \Gamma_1, \\ \phi = \psi & \text{on } \gamma_1, \end{cases}$$

$$(4.3)\begin{cases} J(\psi,\underset{\sim}{v}) = \tfrac{1}{2}\displaystyle\int_{\Omega_{12}} (\underset{\sim}{u}-\underset{\sim}{\nabla}\phi)^2\,dx \quad \text{is minimal,} \\ \text{with} \\ \displaystyle\int_{\gamma_2} \underset{\sim}{v}\cdot\underset{\sim}{n}\,d\gamma_2 + \int_{\Gamma_2} \underset{\sim}{g}_2\cdot\underset{\sim}{n}\,d\Gamma_2 = 0, \end{cases}$$

where, in (4.3), $\underset{\sim}{u}$ and ϕ are the solutions obtained from $\underset{\sim}{v}$ and ψ by solving (4.1) and (4.2).

4.2. Variational Formulation of Problem (4.1) - (4.3).

We can formulate problem (4.1) - (4.3) as an *optimal control* problem by

$$(4.4)\qquad \underset{\{\eta,\underset{\sim}{z}\}}{\text{Min}}\ J(\eta,\underset{\sim}{z});\quad \{\eta,\underset{\sim}{z}\}\in W_1\times W_2,$$

where

(4.5) W_1 *is a space of suitable functions defined over* γ_1,

$$(4.6)\qquad W_2 = \{\underset{\sim}{z}\,|\, \int_{\gamma_2}\underset{\sim}{z}\cdot\underset{\sim}{n}\,d\gamma_2 + \int_{\Gamma_2}\underset{\sim}{u}_\infty\cdot\underset{\sim}{n}\,d\Gamma_2 = 0\},$$

and

$$(4.7)\qquad J(\eta,\underset{\sim}{z}) = \tfrac{1}{2}\int_{\Omega_{12}} |\underset{\sim}{u}-\underset{\sim}{\nabla}\phi|^2\,dx,$$

where ϕ and $\underset{\sim}{u}$ are the solutions of

$$(4.8)\quad \begin{cases} \Delta\phi = 0 \quad in \quad \Omega_1 \ , \\ \dfrac{\partial\phi}{\partial n} = g_1 \quad on \quad \Gamma_1 \ , \quad \phi = \eta \quad on \quad \gamma_1 \ , \end{cases}$$

$$(4.9)\quad \begin{cases} \alpha\underset{\sim}{u} - \nu\Delta\underset{\sim}{u} + \underset{\sim}{\nabla}p = \underset{\sim}{f} \quad in \quad \Omega_2 \ , \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \quad in \quad \Omega_2 \ , \\ \underset{\sim}{u} = \underset{\sim}{g}_2 \quad on \quad \Gamma_B \cup \Gamma_2 \ , \quad \underset{\sim}{u} = \underset{\sim}{z} \quad on \quad \gamma_2 \ . \end{cases}$$

The following spaces will be also useful in the sequel:

$$(4.10)\quad V_{10} = \{w \mid w \in H^1(\Omega_1), \quad w = 0 \quad on \quad \gamma_1\} \ ,$$

$$(4.11)\quad V_{20} = (H_0^1(\Omega_2))^N,$$

$$(4.12)\quad V_{1\eta} = \{w \mid w \in H^1(\Omega_1), \quad w = \eta \quad on \quad \gamma_1\} \ ,$$

$$(4.13)\quad V_{2z} = \{\underset{\sim}{w} \mid \underset{\sim}{w} \in (H^1(\Omega_2))^N, \ \underset{\sim}{w} = \underset{\sim}{z} \quad on \quad \gamma_2 \quad \underset{\sim}{w} = \underset{\sim}{g}_2 \quad on \quad \Gamma_B \cup \Gamma_2\} \ ,$$

$$(4.14)\quad Q_2 = L^2(\Omega_2) \ .$$

The state problem (4.8) can then be reformulated as

$$(4.15)\quad \begin{cases} \phi \in V_{1\eta} \ , \\ \displaystyle\int_{\Omega_1} \underset{\sim}{\nabla}\phi\cdot\underset{\sim}{\nabla}w \, dx = \int_{\Gamma_1} g_1 w \, d\Gamma_1 \ , \ \forall \, w \in V_{10} \ . \end{cases}$$

Similarly, the Stokes problem (4.9) can be reformulated as

$$(4.16)_1\quad \begin{cases} \displaystyle\int_{\Omega_2} (\alpha\underset{\sim}{u}\cdot\underset{\sim}{w} + \nu\underset{\sim}{\nabla}\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{w})dx + \int_{\Omega_2} \underset{\sim}{\nabla}p\cdot\underset{\sim}{w} \, dx = \int_{\Omega_2} \underset{\sim}{f}\cdot\underset{\sim}{w} \, dx, \\ \forall \, \underset{\sim}{w} \in V_{20} \ ; \quad \underset{\sim}{u} \in V_{2z} \ , \end{cases}$$

$$(4.16)_2\quad \int_{\Omega_2} (\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) \, q \, dx = 0 \ , \ \forall \, q \in Q_2 \ ; \quad p \in Q_2/\mathbb{R} \ .$$

4.3. Conjugate Gradient Solution of the Matching Problem (4.4).

Define

(4.17) $$W_{20} = \{\underset{\sim}{z} \mid \int_{\gamma_2} \underset{\sim}{z} \cdot \underset{\sim}{n} \, d\Gamma_2 = 0\} \;;$$

problem (4.4) can then be solved by the following *conjugate gradient* method:

(4.18) $\underset{\sim}{z}^0 = \{z_1^0, \underset{\sim}{z}_2^0\} \in W_1 \times W_2$ *is given;*

solve then

(4.19) $$\begin{cases} \underset{\sim}{g}^0 = \{g_1^0, \underset{\sim}{g}_2^0\} \in W_1 \times W_{20} \;;\; \forall \underset{\sim}{w} \in W_1 \times W_{20} \text{ we have} \\ (g_1^0, w_1)_{\gamma_1} + (\underset{\sim}{g}_2^0, \underset{\sim}{w}_2)_{\gamma_2} = (J'(\underset{\sim}{z}^0), \underset{\sim}{w}) \end{cases}$$

and set

(4.20) $\underset{\sim}{w}^0 = \underset{\sim}{g}^0$.

Then for $n \geq 0$ *update* $\underset{\sim}{z}^n, \underset{\sim}{g}^n, \underset{\sim}{w}^n$ *as follows:*

Solve

(4.21) $$\begin{cases} \rho_n \in \mathbb{R} \\ J(\underset{\sim}{z}^n - \rho_n \underset{\sim}{w}^n) \leq J(\underset{\sim}{z}^n - \rho \underset{\sim}{w}^n), \; \forall \; \rho \in \mathbb{R} \end{cases}$$

and set

(4.22) $\underset{\sim}{z}^{n+1} = \underset{\sim}{z}^n - \rho_n \underset{\sim}{w}^n$.

Solve then

(4.23) $$\begin{cases} (g_1^{n+1}, w_1)_{\gamma_1} + (\underset{\sim}{g}_2^{n+1}, \underset{\sim}{w}_2)_{\gamma_2} = (J'(\underset{\sim}{z}^{n+1}), \underset{\sim}{w}) , \\ \forall \underset{\sim}{w} = \{w_1, \underset{\sim}{w}_2\} \in W_1 \times W_{20} , \end{cases}$$

(4.24) $$\lambda_n = \frac{\left(g_1^{n+1}, g_1^{n+1}\right)_{\gamma_1} + \left(\underset{\sim}{g}_2^{n+1}, \underset{\sim}{g}_2^{n+1}\right)_{\gamma_2}}{\left(g_1^n, g_1^n\right)_{\gamma_1} + \left(\underset{\sim}{g}_2^n, \underset{\sim}{g}_2^n\right)_{\gamma_2}} ,$$

(4.25) $\underset{\sim}{w}^{n+1} = \underset{\sim}{g}^{n+1} + \lambda_n \underset{\sim}{w}^n$.

Do $n = n+1$ *and go to* (4.21).

In practice, we have used for $(\cdot,\cdot)_{\gamma_1}$ and $(\cdot,\cdot)_{\gamma_2}$ the $L^2(\gamma_1)$ and $L^2(\gamma_2)$ scalar products (or H^1- scalar products on extensions of the boundary functions).

4.4. Calculation of J' .

Let's introduce some further notation

$$\langle v,v'\rangle_1 = \int_{\Omega_1} \nabla v\cdot\nabla v'\,dx \quad ,$$

$$\langle \underset{\sim}{v},\underset{\sim}{v}'\rangle_2 = \int_{\Omega_2} (\alpha\underset{\sim}{v}\cdot\underset{\sim}{v}' + \nu\nabla\underset{\sim}{v}\cdot\nabla\underset{\sim}{v}')\,dx,$$

$$(v,v')_1 = \int_{\Omega_1} vv'\,dx,$$

$$(\underset{\sim}{v},\underset{\sim}{v}')_2 = \int_{\Omega_2} \underset{\sim}{v}\cdot\underset{\sim}{v}'\,dx \;.$$

We obtain then for J' the following expression

$$(4.26)\begin{cases}(J'(\psi,\underset{\sim}{v}),\ \{\eta,\underset{\sim}{z}\}) = \int_{\Omega_{12}} (\nabla\phi-\underset{\sim}{u})\cdot\nabla\tilde{\eta}\,dx - \int_{\Omega_1} \nabla y\cdot\nabla\tilde{\eta}\,dx \\ + \int_{\Omega_{12}} (\underset{\sim}{u}-\nabla\phi)\cdot\tilde{\underset{\sim}{z}}\,dx - \int_{\Omega_2} (\alpha\underset{\sim}{y}\cdot\tilde{\underset{\sim}{z}} + \nu\nabla\underset{\sim}{y}\cdot\nabla\tilde{\underset{\sim}{z}})\,dx - \int_{\Omega_2} \nabla\Pi\cdot\tilde{\underset{\sim}{z}}\,dx, \\ \forall\ \{\eta,\underset{\sim}{z}\} \in W_1\times W_{20}\ ;\end{cases}$$

in (4.26), ϕ and $\{\underset{\sim}{u},p\}$ are solution of (4.15) and (4.16), respectively. Moreover, $\tilde{\eta}$ and $\tilde{\underset{\sim}{z}}$ are extensions of η and $\underset{\sim}{z}$ vanishing, in practice, outside a neighborhood of γ_1 and γ_2, respectively. Finally, y and $\{\underset{\sim}{y},\Pi\}$ are solutions of the *adjoint equations*

$$(4.27)\begin{cases}\langle y,w\rangle_1 = \int_{\Omega_{12}} (\nabla\phi-\underset{\sim}{u})\cdot\nabla w\,dx,\ \forall\ w\in V_{10}\ , \\ y\in V_{10}\ ,\end{cases}$$

$$(4.28)\begin{cases}\langle \underset{\sim}{y},\underset{\sim}{w}\rangle_2 + (\nabla\Pi,\ \underset{\sim}{w})_2 = \int_{\Omega_{12}} (\underset{\sim}{u}-\nabla\phi)\cdot\underset{\sim}{w}\,dx,\ \forall\ \underset{\sim}{w}\in V_{20}\ , \\ \int_{\Omega_2} \nabla\cdot\underset{\sim}{y}\ q\,dx = 0,\quad \forall\ q\in Q_2\ , \\ \underset{\sim}{y}\in V_{20}\ ,\quad \Pi\in Q_2/\mathbb{R}\ .\end{cases}$$

From the above calculation, we observe that the practical implementation of algorithm (4.18) - (4.25) will require the solution of

(i) 2 Poisson problems for the calculation of the state ϕ^n and the co-state y^n in Ω_1,

(ii) 2 Stokes problems in Ω_2, for the calculation of $\{\underset{\sim}{u}^n,p^n\}$ and $\{\underset{\sim}{y}^n,\Pi^n\}$.

Since the control problem (4.4) is of the *linear quadratic* type, there is no difficulty (in principle) to compute ρ_n exactly in (4.21) (see [6] for further details concerning the implementation of algorithm (4.18) - (4.25) and the calculation of J').

5. Finite Element Approximation.

In practice algorithm (4.18) - (4.25) will have to be implemented through finite element approximations of the various problems involved in the matching process and flow modeling. The page limitation of this paper prevents us to give a precise description of the finite element variants of the techniques discussed in this paper (see [6] for these details). Let's mention however that the potential flow part will be approximated using the finite element space

$$(5.1) \quad V_{1h} = \{w_h | w_h \in C^0(\bar{\Omega}_1),\ w_h|_T \in P_1,\ \forall\ T \in \mathcal{T}_{1h}\}$$

and the Navier-Stokes part via

$$(5.2) \quad V_{2h} = \{\underset{\sim}{v}_h | \underset{\sim}{v}_h \in (\ C^0(\bar{\Omega}_2))^N,\ \underset{\sim}{v}_h|_T \in (P_1)^N,\ \forall\ T \in \mathcal{T}_{2\frac{h}{2}}\}$$

(for the *velocity*) and

$$(5.3) \quad Q_{2h} = \{q_h | q_h \in C^0(\bar{\Omega}_2)\ ,\ q_h|_T \in P_1,\ \forall\ T \in \mathcal{T}_{2h}\ \}$$

(for the *pressure*). In (5.1) - (5.3), $\mathcal{T}_{1h}$ and $\mathcal{T}_{2h}$ are two overlapping triangulations of Ω_1 and Ω_2, respectively and $\mathcal{T}_{2\frac{h}{2}}$ is obtained from $\mathcal{T}_{2h}$ by joining the mid-points in each triangle of $\mathcal{T}_{2h}$ in order to obtain 4 similar sub-triangles; finally, P_1 is the space of those polynominals of degree ≤ 1.

6. Numerical Experiments and Results.

The above methodology has been tested on the two following problems

(i) An incompressible viscous flow around a circular cylinder at Re = 50 (there exists a steady state solution).

(ii) An incompressible viscous flow around a NACA 0012 airfoil at Re = 200, for a 30^0 angle of attack.

We have been comparing the results obtained using the matching method with those obtained via a full Navier-Stokes solution on $\Omega_1 \cup \Omega_2$; the corresponding results are reported on Figures 6.1 to 6.5 for case (i), and 6.6 to 6.9 for case (ii). The details concerning the triangulations are reported in Tables 6.1 and 6.2, below. A natural question arising from those experiments concerns the difference between the results obtaining using the global or matching techniques described in this paper and those obtained via a Navier-Stokes calculation on Ω_2 only (taking this time $\underset{\sim}{u} = \underset{\sim}{u}_\infty$ on $\gamma_2 \cup \Gamma_2$; see Figure 2.1). Indeed the accuracy of the simulation is seriously deteriorated by taking Γ_∞ too close from B, as shown (for the NACA 0012 case) on the color* pictures 6.10 (velocity visualization) and 6.11 (vorticity visualization). The upper left figures concern the global Navier-Stokes solution on $\Omega_1 \cup \Omega_2$, the lower left ones are related to the matching solution and finally the right figure is concerned with the Navier-Stokes solution on Ω_2 only.

* These figures were originally submitted in color.

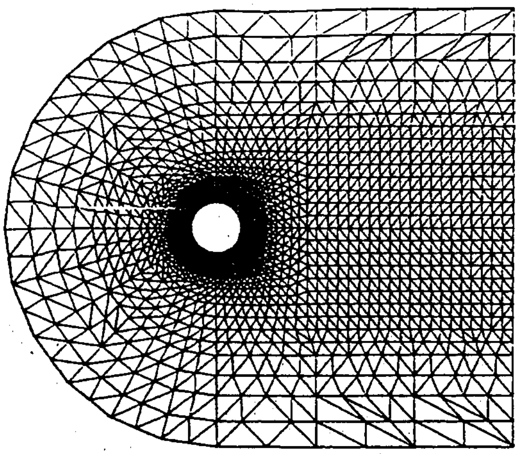
Global Mesh		Viscous Mesh		Potential Mesh	
2588	v.n.	2050	v.n.	1015	nodes
5040	v.e.	2872	v.e.	1800	elements
664	p.n.	541	p.n.		
1260	p.e.	961	p.e.		

v.n. : velocity nodes p.n. : pressure nodes
v.e. : velocity elements p.e. : pressure elements

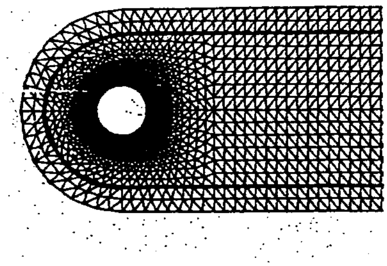
Table 6.1 (Circular cylinder)

Global Mesh		Viscous Mesh		Potential Mesh	
3114	v.n.	2029	v.n.	1015	nodes
6056	v.e.	3884	v.e.	1800	elements
800	p.n.	529	p.n.		
1514	p.e.	971	p.e.		

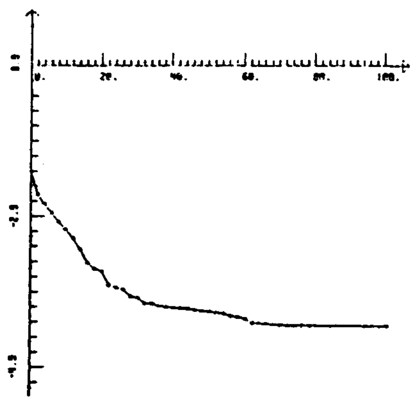
Table 6.2 (NACA 0012)



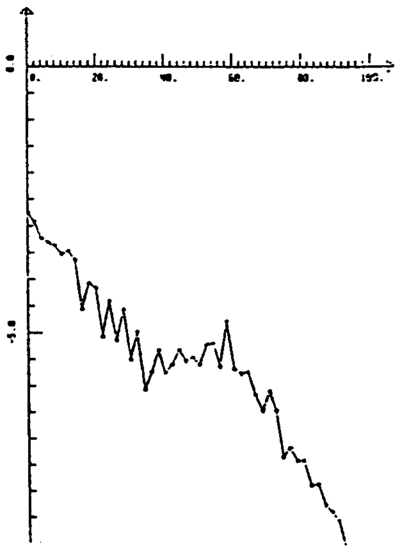
Global mesh
Figure 6.1 (a)



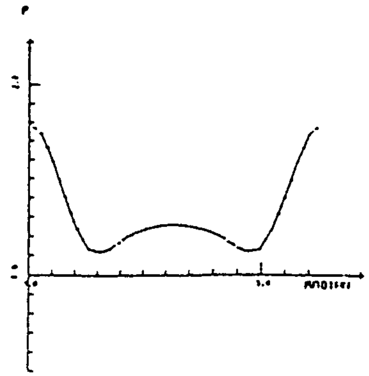
Computational mesh for viscous calculations
Figure 6.1 (b)



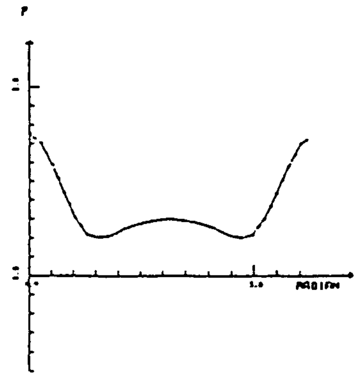
Variation of the cost function
Figure 6.2 (a)



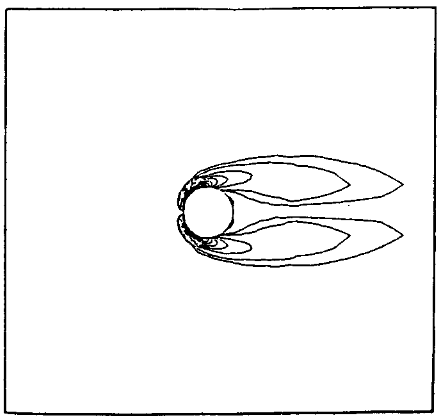
Variation of the gradient norm
Figure 6.2 (b)



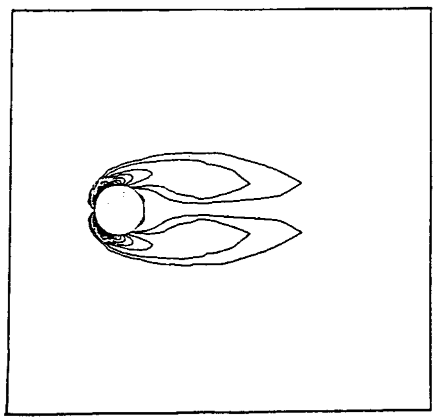
Pressure distribution
(global calculation)
Figure 6.3 (a)



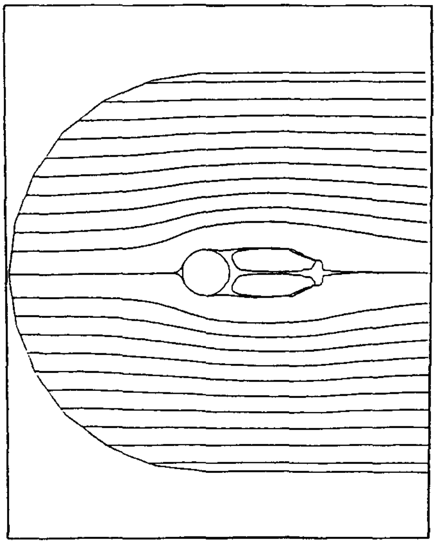
Pressure distribution
(matching calculation)
Figure 6.3 (b)



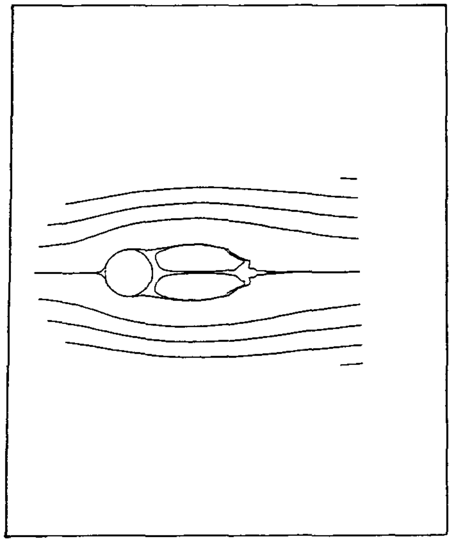
Vorticity contours
(global calculation)
Figure 6.4 (a)



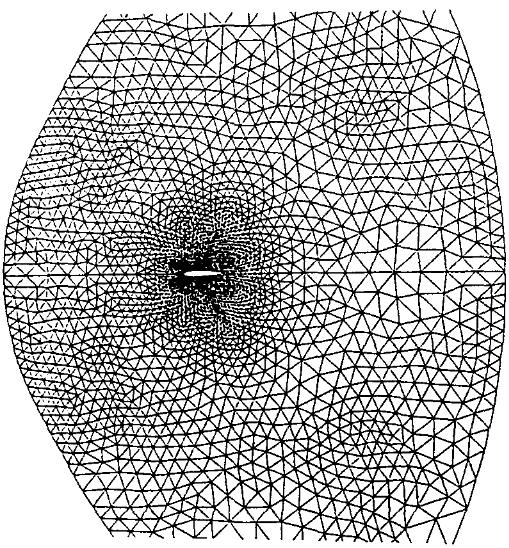
Vorticity contours
(matching calculation)
Figure 6.4 (b)



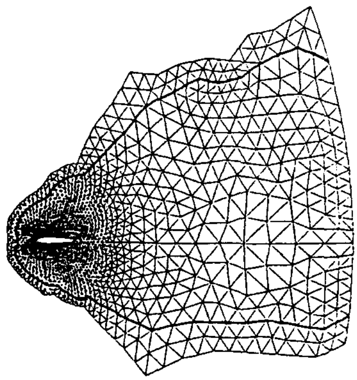
Stream lines
(global solution)
Figure 6.5 (a)



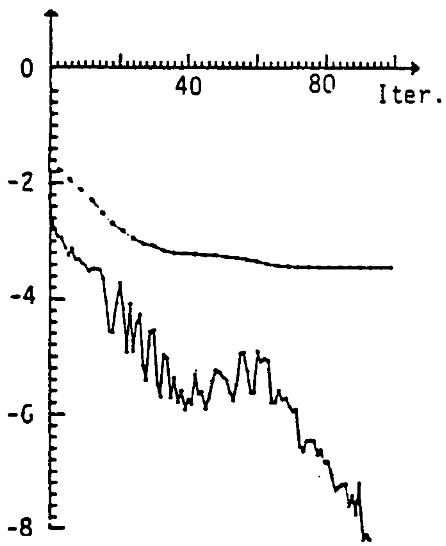
Stream lines
(matching calculation)
Figure 6.5 (b)



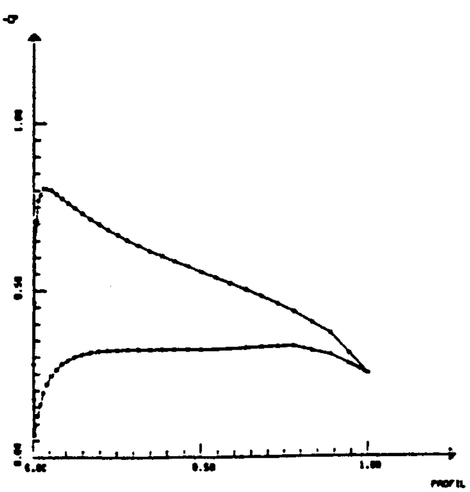
Global mesh
Figure 6.6 (a)



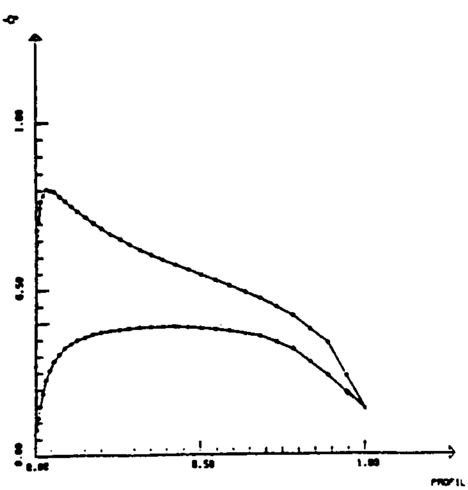
Computational mesh for
viscous calculation
Figure 6.6 (b)



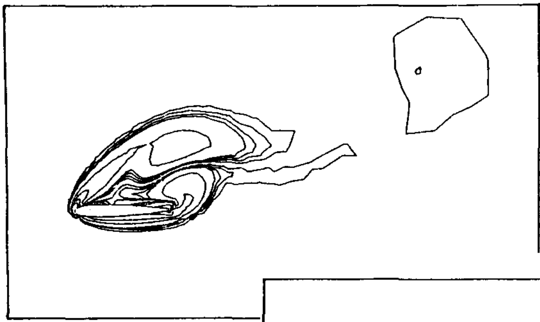
Convergence history
Figure 6.7



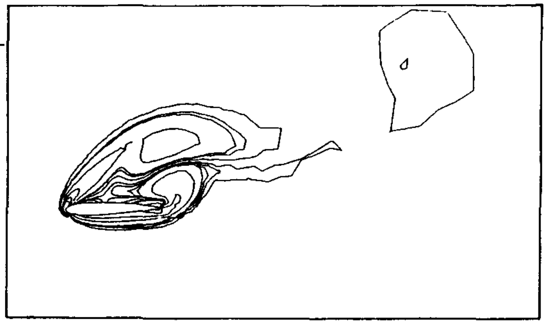
Pressure distribution
(global solution)
Figure 6.7 (a)



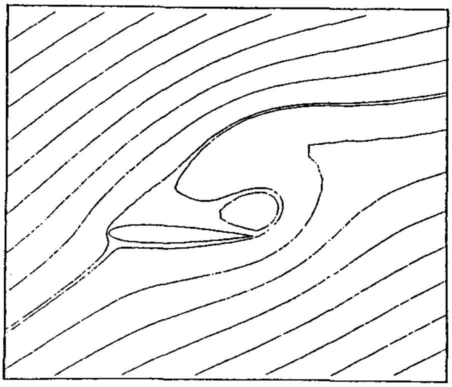
Pressure distribution
(matching calculation)
Figure 6.7 (b)



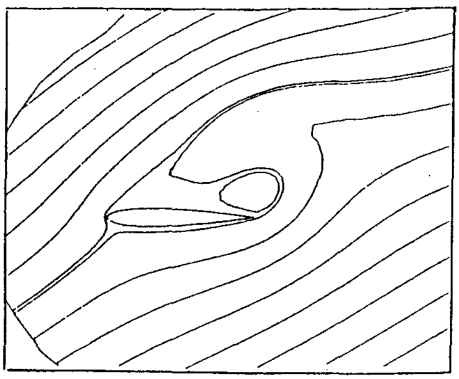
Vorticity contours
(global solution)
Figure 6.8 (a)



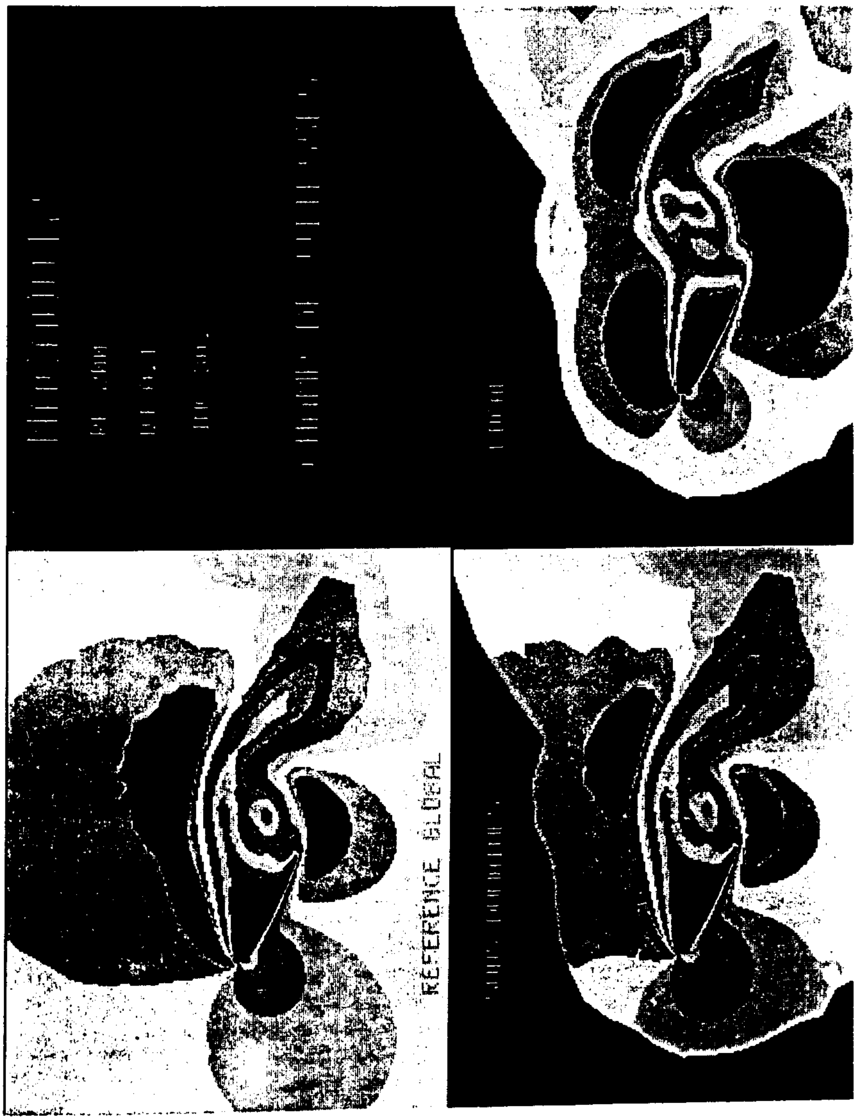
Vorticity contours
(matching calculation)
Figure 6.8 (b)



Streamlines
(global solution)
Figure 6.9 (a)



Streamlines
(matching calculation)
Figure 6.9 (b)



Velocity Visualization
Figure 6.10

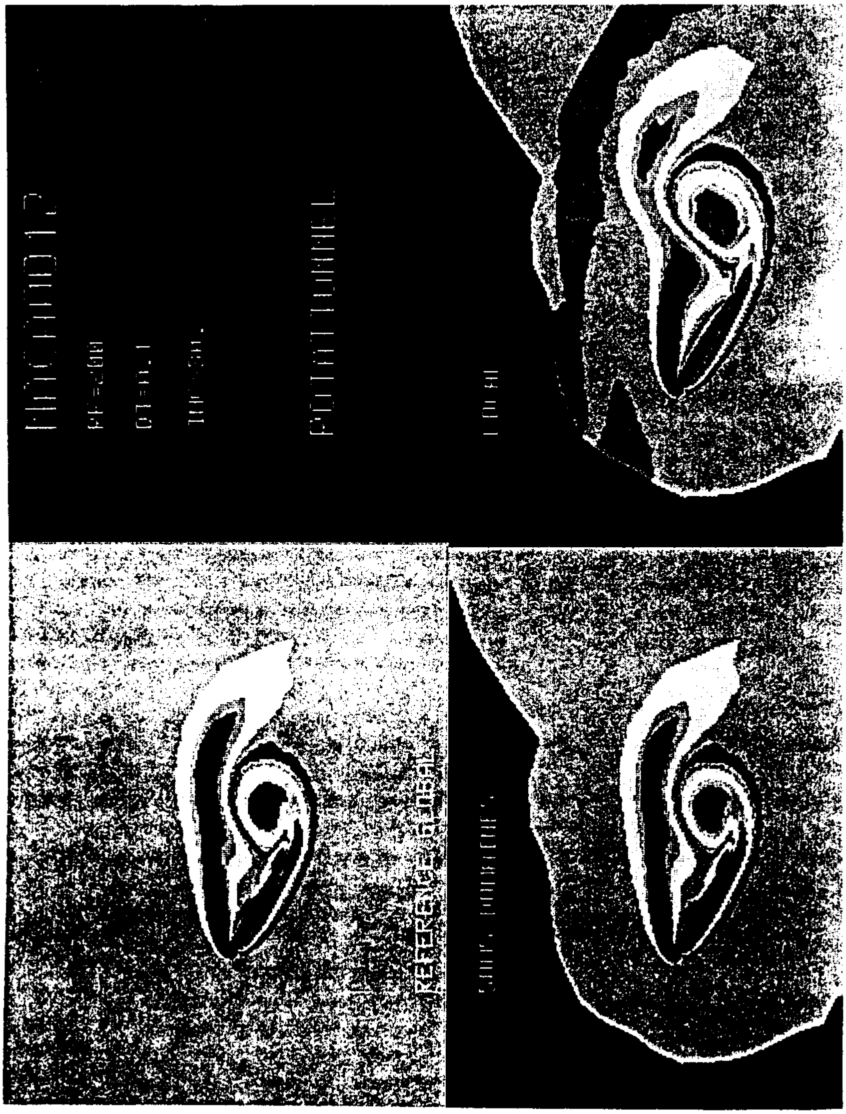


Figure 6.11
Vorticity Visualization

7. Further comments and conclusions.

We have presented here preliminary results showing that the matching method presented here can be applied to couple different mathematical formulations of a given phenomenon.

Actually there is room for many improvements; let's mention some of them:

(i) Use more efficient Stokes solvers, like these introduced by J. Cahouet (see [7]) and also discussed in [8] .

(ii) Improve the preconditioning of the conjugate gradient algorithm (4.18) - (4.25).

We are presently working on these improvements and also to the generalization of the above methods to compressible flow calculations.

Acknowledgments: This work was partly supported by DRET under contract 85/175. We would like to thank Lena Brooks for her careful typing of this paper.

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