On the Schwarz Alternating Method. I

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Introduction.

In [1], H.A. Schwarz proposed an iterative method for the solution of classical boundary value problems for harmonic functions : it consists in solving successively a similar problem in subdomains, going alternatively from one to the other as we recall more precisely below. The convergence of this process was proven by the use of the maximum principle. Since then, this method was studied by various authors including S.L. Sobolev [2], S.G. Michlin [3], M. Prager [4], D. Morgenstern [5], I. Babuska [6], R. Courant and D. Hilbert [7], F.E. Browder [8]... In some of these references the variational interpretation of the method as convenient successive projections was emphasized.

More recently, the interest in such iterative methods was renewed because of the applications to the numerical analysis of boundary value problems. This method was then considered as a method to decompose the original

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problem in a sequence of subproblems or as a domain decomposition method. Many variants, based upon optimal control considerations for instance, were proposed and all these domain decomposition methods have been studied both theoretically and numerically together with their relations with numerical methods for the solution of linear systems. We refer for instance to P.L. Lions [9], R. Glowinski, J. Périaux and Q.V. Dinh [10], [11], R. Glowinski [12], A. Fischler [13], Q.V. Dinh, A. Fischler, R. Glowinski, J. Périaux [14], Q.V. Dinh, J. Périaux, G. Terrasson and R. Glowinski [15], J.M. Trailong and J. Fakleza[16], Q.V. Dinh [17], P.E. Bjorstad and O.B. Widlung [18], P. Lemonnier [19], L. Cambier, W. Ghazzi, J.P. Veillot and H. Viviand [20], P. Anceaux, B. Gay, R. Glowinski, J. Périaux [21], J.P. Benque, J.P. Grégoire, A. Hauguel and M. Maxant [22], Q.V. Dinh, R. Glowinski, B. Mantel, J. Périaux and P. Perrier [23], G.I. Marchuk, Yu.A. Kuznetsov and A.M. Matsokin [24], M. Dryja [25], [26], A.M. Matsokin [27], A.M. Matsokin and S.V. Nepomnyashchikh [28] and their references...

Our goal here is a mathematical study of the Schwarz alternating method where we will emphasize several remarkable properties such as its simplicity, its versality and its good convergence properties for very different classes of equations such as Laplace type equations or Stokes equations and nonlinear variants but also the Hamilton-Jacobi-Bellman equations of optimal stochastic control...

As we will see below, the Fact that the Schwarz method does converge for many different types of problems is due to two reasons : the first one is that it has a variational interpretation and this is the viewpoint that we emphasize in section I. The second one is its interpretation in terms of maximum principle and successive exit times of diffusion processes and this is the viewpoint that we emphasize in section II. Finally, in section III, we present and study various variants of the Schwarz method.

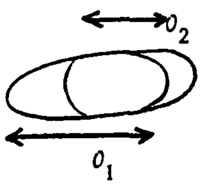
I . Variational interpretation of the Schwarz alternating method.

I.1 Presentation of the method and interpretation.

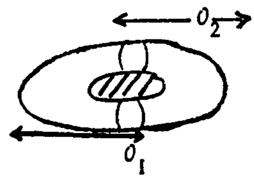
We consider a bounded, open domain $\mathcal{O}$ in $\mathbb{R}^n$ and we assume (to simplify) that $\mathcal{O}$ is smooth and connected. We then decompose $\mathcal{O}$ in two subdomains $\mathcal{O}_1$, $\mathcal{O}_2$ such that

(1) $$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$$

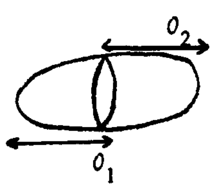
and we denote by $\Gamma = \partial\mathcal{O}$, $\Gamma_1 = \partial\mathcal{O}_1$, $\Gamma_2 = \partial\mathcal{O}_2$, $\gamma_1 = \partial\mathcal{O}_1 \cap \mathcal{O}_2$, $\gamma_2 = \partial\mathcal{O}_2 \cap \mathcal{O}_1$, $\mathcal{O}_{12} = \mathcal{O}_1 \cap \mathcal{O}_2$, $\mathcal{O}_{11} = \mathcal{O}_1 \cap \bar{\mathcal{O}}_2^c$, $\mathcal{O}_{22} = \mathcal{O}_2 \cap \bar{\mathcal{O}}_1^c$. Various decompositions are possible as it can be seen from the following figures



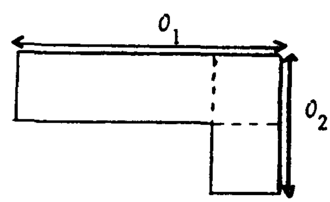
1.a



1.b



2.a



2.b

(even if in case 2.b $\mathcal{O}$ is only Lipschitz). We assume that γ_1, γ_2 are smooth...

Next, suppose that we want to solve the following model problem

(2) $$-\Delta u = f \quad \text{in} \quad \mathcal{O} \quad , \quad u = 0 \quad \text{on} \quad \partial\mathcal{O}$$

where f is a given function in $\mathcal{O}$ say in $L^2(\mathcal{O})$ (or in $H^{-1}(\mathcal{O})$ ...).

The Schwarz alternating procedure consists in solving successively the following problems : let u^o be any initialization say in $H_o^1(\mathcal{O})$, u^{2n+1} $(n \geq 0)$, u^{2n} $(n \geq 1)$ are solutions of (respectively)

$$(3) \qquad -\Delta u^{2n+1} = f \quad \text{in} \quad \mathcal{O}_1 \quad , \quad u^{2n+1} = u^{2n} \quad \text{on} \quad \partial\mathcal{O}_1$$

$$(4) \qquad -\Delta u^{2n} = f \quad \text{in} \quad \mathcal{O}_2 \quad , \quad u^{2n} = u^{2n-1} \quad \text{on} \quad \partial\mathcal{O}_2 \ .$$

where the solutions are taken for instance in $H^1(\mathcal{O}_1)$, $H^1(\mathcal{O}_2)$ respectively. Observe that we may extend u^{2n+1} by u^{2n} on $\overline{\mathcal{O}}_{22}$ and u^{2n} by u^{2n-1} on $\overline{\mathcal{O}}_n$ so that

$$(5) \qquad u^n \in H_o^1(\mathcal{O}) \quad , \quad u^{2n+1}-u^{2n} \in H_o^1(\mathcal{O}_1) \quad , \quad u^{2n+2}-u^{2n+1} \in H_o^1(\mathcal{O}_2)$$

for all $n \geq 0$.

In all that follows, we consider $H_o^1(\mathcal{O}_1)$, $H_o^1(\mathcal{O}_2)$ as closed subspaces of $H_o^1(\mathcal{O})$ by extending their elements to $\mathcal{O}$ by 0 . And we take as scalar product on $H_o^1(\mathcal{O})$ the usual one i.e.

$$(u,v) = \int_{\mathcal{O}} \nabla u \cdot \nabla v \, dx \quad , \quad \forall\, u,v \in H_o^1(\mathcal{O})$$

where ∇ denotes the gradient. With these notations, it is obvious that (3), (4) are equivalent to

$$(6) \qquad (u^{2n+1}-u \, , v_1) = 0 \qquad \forall\, v_1 \in H_o^1(\mathcal{O}_1) \quad , \quad u^{2n+1}-u^{2n} \in H_o^1(\mathcal{O}_1)$$

for all $n \geq 0$,

$$(7) \qquad (u^{2n}-u \, , v_2) = 0 \qquad \forall\, v_2 \in H_o^1(\mathcal{O}_2) \quad , \quad u^{2n}-u^{2n-1} \in H_o^1(\mathcal{O}_2)$$

for all $n \geq 1$.

Now, if we denote by $V_1 = H_o^1(\mathcal{O}_1)$, $V_2 = H_o^1(\mathcal{O}_2)$ we may rewrite (6) and (7) as follows

$$(6') \qquad (u^{2n+1}-u^{2n}, v_1) = (u-u^{2n}, v_1) \qquad \forall\, v_1 \in V_1 \,, \quad u^{2n+1}-u^{2n} \in V_1$$

for all $n \geq 0$,

$$(7') \qquad (u^{2n}-u^{2n-1}, v_2) = (u-u^{2n-1}, v_2) \qquad \forall\, v_2 \in V_2 \,, \quad u^{2n}-u^{2n-1} \in V_2$$

for all $n \geq 1$. And this obviously means

$$(8) \qquad u^{2n+1}-u^{2n} = P_{V_1}(u-u^{2n}) \qquad \text{for all } n \geq 0$$

$$(9) \qquad u^{2n}-u^{2n-1} = P_{V_2}(u-u^{2n-1}) \qquad \text{for all } n \geq 1 \,,$$

or equivalently

$$(10) \qquad u-u^{2n+1} = P_{V_1^\perp}(u-u^{2n}) \qquad \text{for all } n \geq 0$$

$$(11) \qquad u-u^{2n} = P_{V_2^\perp}(u-u^{2n-1}) \qquad \text{for all } n \geq 1$$

where P denotes various orthogonal projections on the subspaces appearing as subscripts.

I.2 Convergence of iterated projections and interpretation.

As we saw in the preceding section, the Schwarz method for the model problem (2) is equivalent to a sequence of projections in an Hilbert space. More precisely, if V_1, V_2 are two closed subspaces of an Hilbert space V we have to investigate the behaviour of sequences $(v_n)_n$ built as follows : $v_o \in V$, $v_{2n+1} = P_{V_1^\perp} v_{2n}$, $v_{2n+2} = P_{V_2^\perp} v_{2n+1}$ for $n \geq 0$. It is well-known that v_n converges (in V) to $P_{V_1^\perp \cap V_2^\perp} v_o$ - we will recall this standard fact below. In addition observing that if we denote by $\tilde{v}_o = v_o - P_{V_1^\perp \cap V_2^\perp} v_o$ then the sequence $\tilde{v}_n$ generated by the above projections starting from $\tilde{v}_o$ instead of v_o is given

$$\tilde{v}_n = v_n - P_{V_1^\perp \cap V_2^\perp} v_o \quad ,$$

we deduce that without loss of generality we may assume $V_1^\perp \cap V_2^\perp = \{0\}$. And we have the

<u>Theorem I.1</u> : Assume that $V_1^\perp \cap V_2^\perp = \{0\}$ or equivalently that $V = \overline{V_1+V_2}$, then v_n converges to 0 . If $V_1+V_2 = V$, then there exists $k \in [0,1[$ such that

$$|P_{V_2^\perp} P_{V_1^\perp}| \leq k \tag{12}$$

therefore

$$|v_{n+1}| \leq k^n |v_o| \qquad \text{for all } n \geq 0 \quad .$$

<u>Proof</u> : We first observe that

$$|v_{n+1}|^2 + |v_{n+1}-v_n|^2 = |v_n|^2 \quad , \qquad \text{for all } n \geq 0 \quad .$$

Hence, $|v_n| \downarrow_n \ell$ for some $\ell \geq 0$ and $v_{n+1}-v_n \to_n 0$. Now, if for some subsequence n_k , v_{n_k} converges weakly to some v we remark that since v_{n_k+1} also converges weakly to v then $v \in V_1^\perp \cap V_2^\perp$ i.e. $v = 0$ and v_n converges weakly to 0 . To conclude, we observe that

$$|v_n|^2 = (v_{2n-1}, v_o) \to_n 0$$

and thus v_n converges strongly to 0 .

Next, if $V_1+V_2 = V$, we deduce (12) from the following

<u>Lemma I.1</u> : There exists a constant $C_o > 0$ such that for all $v \in V$

$$|v| \leq C_o \left(|P_{V_1} v|^2 + |P_{V_2} v|^2\right)^{1/2} \quad . \tag{13}$$

The proof of this lemma is given below. We now conclude the proof of Theorem I.1. To do so, we observe that (13) yields

$$|P_{V_1^\perp} v| \leq C_0 \, |P_{V_2} P_{V_1^\perp} v|$$

hence

$$|P_{V_1^\perp} v|^2 = |P_{V_2^\perp} P_{V_1^\perp} v|^2 + |P_{V_2} P_{V_1^\perp} v|^2$$

$$\geq |P_{V_2^\perp} P_{V_1^\perp} v|^2 + \frac{1}{C_0} |P_{V_1^\perp} v|^2 \qquad \Rightarrow \quad 1 \geq 1/C_0$$

or

(14) $$|P_{V_2^\perp} P_{V_1^\perp} v| \leq (1-1/C_0)^{1/2} \, |P_{V_1^\perp} v| \quad , \quad \forall\, v \in V \ .$$

In particular, this yields (12) and we conclude since we may write $P_{V_2^\perp} P_{V_1^\perp} P_{V_2^\perp}$ as $\left(P_{V_2^\perp} P_{V_1^\perp}\right)\left(P_{V_1^\perp} P_{V_2^\perp}\right)$. ∎

<u>Proof of Lemma I.1</u> : We first observe that by a simple application of the open mapping theorem to $((v_1,v_2) \to v_1+v_2)$ from $V_1 \times V_2$ onto V then there exists $C_0 > 0$ such that

(15) $$\forall\, v \in V \ , \ \exists (v_1,v_2) \in V_1 \times V_2 \ , \quad \left(|v_1|^2+|v_2|^2\right)^{1/2} \leq C_0 |v| \ , \quad v = v_1+v_2 \ .$$

Next, we write

$$|v|^2 = (v,v_1) + (v,v_2) = \left(P_{V_1} v , v_1\right) + \left(P_{V_2} v , v_2\right)$$

$$\leq \left(|P_{V_1} v|^2 + |P_{V_2} v|^2\right)^{1/2} \left(|v_1|^2 + |v_2|^2\right)^{1/2}$$

$$\leq C_0 |v| \left(|P_{V_1} v|^2 + |P_{V_2} v|^2\right)^{1/2} \ .$$ ∎

We may now go back to the Schwarz alternating method for the model problem. In view of the preceding results we have to investigate whether V_1+V_2 is dense in or even equal to V . A simple criterion for the first fact is the following

(16) $$\forall\, \varphi \in \mathcal{D}(\mathcal{O}) \ , \ \exists\, \varphi_1 \in \mathcal{D}(\mathcal{O}_1) \ , \ \exists\, \varphi_2 \in \mathcal{D}(\mathcal{O}_2) \quad , \quad \varphi = \varphi_1 + \varphi_2 \ .$$

Indeed, this implies $\mathcal{D}(\mathcal{O}) \subset \overline{V_1+V_2}$ hence $V = \overline{V_1+V_2}$. Next, we claim that (16) always holds if $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$. Indeed, observe that if K is a compact set in $\mathcal{O}$ then there exists $\varepsilon > 0$ such that $K \subset (\mathcal{O}_1)_\varepsilon \cup (\mathcal{O}_2)_\varepsilon$ where A_ε denotes $\{x \in A \text{ , } \operatorname{dist}(x,A^c) > \varepsilon\}$. Therefore, if $\varphi \in \mathcal{D}(\mathcal{O})$ and $K = \operatorname{Supp} \varphi$, there exists $\psi_1 \in \mathcal{D}(\mathcal{O}_1)$, $\psi_2 \in \mathcal{D}(\mathcal{O}_2)$ such that $0 \leq \psi_i \leq 1$ $(i=1,2)$ and $\psi_i \equiv 1$ on a neighbourhood of $K \cap (\mathcal{O}_i)_\varepsilon$ $(i=1,2)$. Then, we conclude since

$$\varphi = \psi_1(\psi_1+\psi_2)^{-1}\varphi + \psi_2(\psi_1+\psi_2)^{-1}\varphi \quad .$$

Now, if we wish to check $V = V_1+V_2$, the situation is a bit more subtle. It is clearly true if we assume that there exist χ_1,χ_2 smooth on $\overline{\mathcal{O}}$ ($W^{1,\infty}(\mathcal{O})$ is enough in the case of the model problem (2)) vanishing respectively on γ_1,γ_2 such that

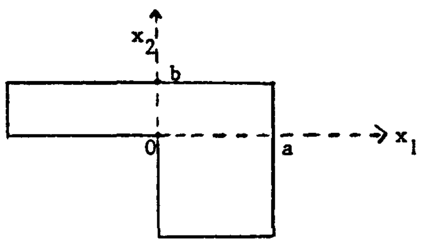
$$\chi_1 + \chi_2 \equiv 1 \qquad \text{on } \mathcal{O} \text{ .} \tag{17}$$

Indeed, if $u \in H^1_o(\mathcal{O})$ then $\chi_1 u \in H^1_o(\mathcal{O}_1)$, $\chi_2 u \in H^1_o(\mathcal{O}_2)$ and our claim is proved.

This assumption is satisfied provided there is some uniform overlapping of $\mathcal{O}_1$ and $\mathcal{O}_2$ as in Figures 1.a or 1.b but it does not hold in the case of Figures 2.a or 2.b. In order to analyze those cases we observe that in those cases we have

$$\left\{\begin{array}{l} \exists\ \chi_1,\chi_2 \in W^{1,\infty}_{loc}(\mathcal{O}) \text{ , } \chi_i = 0 \text{ on } \gamma_i \text{ , } \chi_i \geq 0 \text{ for } i = 1,2 \\ 1 \equiv \chi_1+\chi_2 \text{ on } \mathcal{O} \text{ , } \quad |\nabla\chi_i| \leq C \operatorname{dist}(x,\partial\mathcal{O})^{-1} \text{ a.e. on } \mathcal{O}\ \forall\, i \end{array}\right. \tag{18}$$

for some $C \geq 0$. Indeed, take for instance the case 2.b



then we set $\chi_1(x) = 1$ if $x_1 \leq 0$, $\chi_1(x) = \left(1 - \frac{bx_1}{ax_2}\right)^+$ if $0 \leq x_1 \leq a$ and $0 < x_2 \leq b$, $\chi_1(x) = 0$ if $x_2 \leq 0$ and $\chi_2 \equiv 1-\chi_1$ on $\bar{\mathcal{O}}$.

Observe that χ_1 and χ_2 are Lipschitz continuous on $\bar{\mathcal{O}} - \{0\}$ and if $0 \leq x_1 \leq a$, $0 < x_2 \leq b$

$$\nabla\chi_1(x) = \left(-\frac{b}{ax_2}, \frac{bx_1}{ax_2^2}\right) 1_{(ax_2 > bx_1)} \qquad \text{a.e.}$$

hence

$$|\nabla\chi_1| \leq \left(\frac{b^2}{a^2}+1\right)^{1/2} \frac{1}{x_2}\, 1_{(ax_2 > bx_1)}$$

$$\leq \left(\frac{b^2}{a^2}+1\right)^{1/2} \left(\frac{a^2}{b^2}+1\right)^{1/2} \frac{1}{|x|}$$

and $\frac{1}{|x|} \leq \text{dist}(x,\partial\mathcal{O})^{-1}$ so (18) holds.

Next, we claim that (18) implies that $V = V_1+V_2$. Indeed, arguing as before we only have to check that if $u \in H_o^1(\mathcal{O})$ then $\chi_1 u \in H_o^1(\mathcal{O}_1)$ and this reduces to examining if $\nabla(\chi_1 u) \in L^2$. We then compute

$$\int |\nabla(\chi_1 u)|^2\, dx \leq \int |\chi_1 \nabla u + \nabla\chi_1 u|^2\, dx$$

$$\leq 2\int |\nabla u|^2 + |\nabla\chi_1|^2 u^2\, dx$$

$$\leq 2\int |\nabla u|^2\, dx + 2C\int \frac{u^2}{d^2}\, dx$$

where $d(x) = \text{dist}\,(x,\partial\mathcal{O})$ and by a classical inequality

$$\int |\nabla(\chi_1 u)|^2\, dx \leq C\int |\nabla u|^2\, dx \quad .$$

In conlusion, we have the

Corollary I.1 : If (16) holds, the Schwarz alternating method for the problem (2) converges. In addition if (18) holds, then it converges geometrically.

Remarks : The above arguments also show that the convergence factor k may be estimated by $\left(1 - \frac{1}{C_o}\right)^{1/2}$ where C_o is any constant such that (15) holds. Now, if (18) holds we may choose $v_1 = \chi_1 v$, $v_2 = \chi_2 v$. And we may estimate C_o as follows

$$C^2 = \int |\nabla(\chi_1 u)|^2 + |\nabla(\chi_2 u)|^2 \, dx = \int (\chi_1^2+\chi_2^2) \; |\nabla u|^2 \, dx +$$

$$+ 2 \int \chi_1 u \; \nabla\chi_1 \cdot \nabla u \, dx + 2 \int \chi_2 u \; \nabla\chi_2 \cdot \nabla u \, dx + \int u^2 (|\nabla\chi_1|^2 + |\nabla\chi_2|^2) \, dx$$

and using the relations $\chi_1+\chi_2 = 1$, $\nabla\chi_1 = -\nabla\chi_2$ we find

$$C^2 = \int |\nabla u|^2 \, dx - 2 \int \chi_1\chi_2 |\nabla u|^2 \, dx + 2 \int (\chi_1-\chi_2) u \; \nabla\chi_1 \cdot \nabla u$$
$$+ 2 \int u^2 \; |\nabla\chi_1|^2 \, dx$$

$$= (1+\delta) \int |\nabla u|^2 dx + \left(2+\frac{1}{\delta}\right) \int u^2 |\nabla\chi_1|^2 dx \; , \qquad \text{for all } \delta > 0 \; .$$

Now, if there is a uniform overlapping between $\mathcal{O}_1$ and $\mathcal{O}_2$ i.e. $\chi_1, \chi_2 \in W^{1,\infty}(\mathcal{O})$, we deduce

$$C_o \leq \left(1 + \delta + \left(2+\frac{1}{\delta}\right) L^2 \lambda_1^{-1}\right)^{1/2} \qquad \text{for all } \delta > 0$$

where $L = \|\nabla\chi_1\|_\infty$, λ_1 is the first eigenvalue of $-\Delta$ on $H_o^1(\mathcal{O})$. Therefore, we find in particular

$$C_o \leq 1 + 2 L \lambda_1^{-1/2} \; .$$

If (18) holds we just replace $L^2\lambda_1^{-1}$ by C_1 such that

$$\int u^2 |\nabla\chi_1|^2 dx \leq C_2 \int u^2 d^{-2} dx \leq C_1 \int |\nabla u|^2 dx \; . \qquad \blacksquare$$

We now conclude this section by observing that the above interpretation of the Schwarz alternating method as iterated projections immediately yields similar convergence results for general classes of symmetric variational boundary value problems. In particular, we may consider higher-order problems (biharmonic operator for instance) or more general second-order elliptic problem such as

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + c$$

where $a_{ij} = a_{ji} \in L^\infty(\mathcal{O})$ $(\forall\ 1 \leq i,j \leq n)$, $c \in L^\infty(\mathcal{O})$ and for example we suppose that

(19) $$\exists \nu > 0 \text{ , a.e. } x \in \mathcal{O} \text{ , } \forall \xi \in \mathbb{R}^n \text{ , } \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$$

(20) $$c \geq 0 \quad \text{a.e. in } \mathcal{O} \text{ .}$$

Indeed, for such an operator we just have to endow $H_o^1(\mathcal{O})$ with the scalar product $\langle Au,v\rangle$ i.e.

$$a(u,v) = \int_{\mathcal{O}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + c\,u\,v\,dx$$

and everything we said above adapts immediately.

Similarly, we may replace Dirichlet boundary conditions by various kinds of boundary conditions. The only noticeable modification concerns the case of Neumann boundary conditions. For example, let us consider the following model problem

(21) $$-\Delta u + u = f \quad \text{on } \mathcal{O} \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\mathcal{O}$$

where n denotes the unit outward normal to $\partial\mathcal{O}$. We then set $V = H^1(\mathcal{O})$, $V_1 = \{u \in H^1(\mathcal{O})\ ,\ u = 0 \text{ on } \overline{\mathcal{O}}_{22}\}$, $V_2 = \{u \in H^1(\mathcal{O})\ ,\ u = 0 \text{ on } \mathcal{O}_{11}\}$. The Schwarz alternating method is then defined exactly as in section I.1 and each subproblem in $\mathcal{O}_1$ and $\mathcal{O}_2$ is now a mixed boundary value problem with Neumann conditions on $\partial\mathcal{O}_i \cap \partial\mathcal{O}$ $(i = 1,2)$ and Dirichlet conditions $v_n = v_{n-1}$ on γ_i ... Again, the process converges if $\overline{V_1+V_2} = V$ and the convergence is geometrical if $V_1+V_2 = V$. And if one can show easily (same argument as before) that $V_1+V_2 = V$ in the case when $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap (1.a, 1.b), one can only show that $\overline{V_1+V_2} = V$ in the cases when $\bar{\gamma}_1, \bar{\gamma}_2$ touch on $\partial\mathcal{O}$ (provided everything is smooth...) i.e. in the cases 2.a or 2.b.

Finally, it is obvious that we may replace homogeneous boundary conditions in (2) or (21) by general ones...

I.3 The case of Stokes equation.

We now consider the case of Stokes equation : let u be the solution in $H^1(\mathcal{O};\mathbb{R}^n)$

(22) $$- \Delta u + \nabla p = f \text{ in } \mathcal{O} \quad , \quad u = u^o \text{ on } \Gamma \quad , \text{div } u = 0 \quad \text{in } \mathcal{O}$$

where $u^o \in H^1(\mathcal{O};\mathbb{R}^n)$, $f \in L^2(\mathcal{O};\mathbb{R}^n)$ (for instance) satisfy the compatibility condition

(23) $$\int_\Gamma u^o\cdot n \, dS = 0 \quad .$$

As usual, p denotes the pressure (determined up to a constant). We consider of course the space $V = \{u \in H^1_o(\mathcal{O};\mathbb{R}^n) \ / \ \text{div } u = 0 \text{ in } \mathcal{O}\}$ and we recall that (22) may be written

(24) $$(u,v) = \int_\mathcal{O} f \, v \, dx \quad , \quad \forall v \in V \ ; \ u \in u^o + V$$

where, possibly, we have replaced u^o by $\tilde{u}^o$ such that : $\tilde{u}^o = u^o$ on Γ , $\text{div } \tilde{u}^o = 0$ in $\mathcal{O}$.

We now explain how the Schwarz alternating method applies to this problem. Let $u_o \in u^o + V$, we introduce u_{2n+1} , u_{2n+2} $(n \geq 0)$ solutions of respectively

(25) $$-\Delta u_{2n+1} + \nabla p_{2n+1} = f \text{ in } \mathcal{O}_1 \ , \ u_{2n+1} = u_{2n} \text{ on } \partial\mathcal{O}_1 \ , \quad \text{div } u_{2n+1} = 0 \text{ in } \mathcal{O}_1$$

(26) $$-\Delta u_{2n+2} + \nabla p_{2n+2} = f \text{ in } \mathcal{O}_2 \ , \ u_{2n+2} = u_{2n+1} \text{ on } \partial\mathcal{O}_2 \ , \quad \text{div } u_{2n+2} = 0 \text{ in } \mathcal{O}_2$$

and we extend u_{2n+1} , u_{2n+2} to $\mathcal{O}$ by respectively u_{2n} , u_{2n+1} .

This sequence is well-defined since we have by induction for all $n \geq 1$

$$\operatorname{div} u_n = 0 \qquad \text{a.e. in } \mathcal{O} \quad ;$$

indeed, we have, for example, $\operatorname{div} u_{2n+1} = 0$ a.e. in $\mathcal{O}_1$, $\operatorname{div} u_{2n+1} = \operatorname{div} u_{2n} = 0$ a.e. in $\mathcal{O}_{22}$ and $u_{2n} = u_{2n+1}$ on γ_2 . Now, the results and methods presented in the preceding section apply and we obtain the convergence of this sequence to u as soon as $V = \overline{V_1+V_2}$ where

$$V_i = \{u \in H_o^1(\mathcal{O}_i ; \mathbb{R}^n) \ , \operatorname{div} u = 0 \ \text{ in } \ \mathcal{O}_i\}$$

for $i = 1,2$, where we extend $u \in V_i$ to $\mathcal{O}$ by 0 . Furthermore, the convergence is geometrical if $V = V_1+V_2$. Indeed, we just have to observe that (25)-(26) may then be written : $u-u_{2n+1} = P_{V_1^\perp}(u-u_{2n})$, $u-u_{2n+2} = P_{V_2^\perp}(u-u_{2n+1})$.

Hence, the only question we have to investigate is to check that $V = \overline{V_1+V_2}$ or $V = V_1+V_2$. In fact, we claim that if $H_o^1(\mathcal{O}) = \overline{H_o^1(\mathcal{O}_1) + H_o^1(\mathcal{O}_2)}$ then $V = \overline{V_1+V_2}$, while if $H_o^1(\mathcal{O}) = H_o^1(\mathcal{O}_1) + H_o^1(\mathcal{O}_2)$ then $V = V_1+V_2$. And since we already explained in the preceding sections how to check these conditions, the remaining question will be solved.

To prove our claim, we consider $u \in V$ such that $u = u_1+u_2$ where $u_1 \in H_o^1(\mathcal{O}_1 ;\mathbb{R}^n)$, $u_2 \in H_o^1(\mathcal{O}_2 ;\mathbb{R}^n)$. And we immediately conclude if we show that there then exist $\tilde{u}_1 \in V_1$, $\tilde{u}_2 \in V_2$ such that $u = \tilde{u}_1+\tilde{u}_2$.

Indeed, set $g = \operatorname{div} u_1 \in L^2(\mathcal{O})$ so that

$$\operatorname{div} u_2 = -g \qquad \text{a.e. in } \mathcal{O}$$

and $g = 0$ a.e. in $\mathcal{O} - \mathcal{O}_{12}$.

Next, observe that we have

$$\int_{O_{12}} g\, dx = \int_{O_2} u_1 \cdot \bar{n}\, dS = \int_{O_2} u \cdot \bar{n}\, dS$$

where $\bar{n}$ denotes the unit outward normal to $\gamma_2 = \partial O_2 \cap O$, and in addition denoting by n_1 the unit outward normal to ∂O_{11}

$$\int_{\gamma_2} u \cdot \bar{n}\, dS = -\int_{\partial O_{11}} u \cdot n_1\, dS = -\int_{O_{11}} (\mathrm{div}\, u)\, dx = 0 .$$

Therefore, there exists $\hat{u} \in H_o^1(O_{12})$ such that

$$\mathrm{div}\, \hat{u} = g \quad \text{a.e. in } O_{12}$$

and extending $\hat{u}$ by 0 to O we conclude easily setting

$$\tilde{u}_1 = u_1 - \hat{u} \quad , \quad \tilde{u}_2 = u_2 + \hat{u} \quad .$$

I.4 Extensions to more subdomains.

There are various ways to extend the Schwarz alternating method to geometrical situations where O is split into more subdomains. We will explain these various methods on the model problem (2).

The first one is purely "sequential" : we assume that

(26) $$O = \bigcup_{j=1}^{m} O_j \ , \quad O_j \text{ open set in } \mathbb{R}^n \text{ for } 1 \leq j \leq m$$

and we denote by $V_j = H_o^1(O_j)$. And we consider V_j as a closed subspace of $V = H_o^1(O)$ by extending its elements to O by 0 . Then, we introduce the following sequence : let $u_o \in H_o^1(O)$, for $k \geq 0$ we consider the solutions of

(27) $$-\Delta u_{km+j} = f \text{ in } O_j \quad , \quad u_{km+j} = u_{km+j-1} \text{ on } \partial O_j$$

and we extend u_{km+j} to O by u_{km+j-1} , for $1 \leq j \leq m$.

Or in other words,

(28) $$u-u_{km+j} = P_{V_j^\perp}(u-u_{km+j-1}) \quad , \quad \forall k \geq 0 , \forall 1 \leq j \leq m .$$

And exactly as in section I.2 we show that (26) implies that $\overline{\sum_{j=1}^{m} V_j} = V$ while $\sum_{j=1}^{m} V_j = V$ holds if we assume

(29) $$\begin{cases} \exists \chi_i \in W^{1,\infty}_{loc}(\mathcal{O}) , \ \chi_i = 0 \ \text{on} \ \partial\mathcal{O}_i \cap \mathcal{O} , \ \chi_i \geq 0 , \ \text{for} \ 1 \leq i \leq m \\ \sum_{i=1}^{m} \chi_i \equiv 1 \ \text{on} \ \mathcal{O} , \ |\nabla\chi_i| \leq C \, \mathrm{dist}(x,\partial\mathcal{O})^{-1} \ \text{a.e. on} \ \mathcal{O} \ \text{for} \ 1 \leq i \leq m. \end{cases}$$

And we obtain the

<u>Theorem I.2</u> : The sequence $(u_n)_n$ converges to u in V . In addition if $\sum_{j=1}^{m} V_j = V$, there exists $k \in [0,1[$ such that

$$\|u_n-u\|_V \leq k^n \|u_o-u\|_V \quad , \qquad \text{for all} \ n \geq 0 .$$

<u>Remarks</u> : 1) Exactly as in section I.2, it is possible to estimate k in terms of geometrical quantities (see also the proofs below).

2) The geometrical convergence result is still valid if we replace $(u_n)_n$ by sequences of solutions of subproblems chosen "somewhat randomly". More precisely, if we consider $(u_n)_n$ defined by

$$u_{n+1}-u = P_{V_{j_{n+1}}^\perp}(u_n-u) \qquad \text{for} \ n \geq 0$$

where $j_{n+1} \in \{1,\dots,m\}$, then the above convergence result still holds if we assume

(30) $$\exists M \geq 1 , \exists k_o \geq 1 , \forall k \geq k_o , \quad \{1,\dots,m\} \subset \{j_n \,/\, kM \leq n < (k+1)M\} .$$

Indeed, we just observe that the proof of Theorem I.2 shows that there exists a constant $k \in [0,1[$ such that if $Q_i = P_{V_{\alpha_i}^\perp}$ for some $\alpha_i \in \{1,\dots,m\}$ and for $1 \leq i \leq M$ and if $\{1,\dots,m\} \subset \{\alpha_i \,/\, 1 \leq i \leq M\}$ then

$$\| Q_M \, Q_{M-1} \dots Q_2 \, Q_1 \| \leq k \quad . \qquad \blacksquare$$

Proof of Theorem I.2 : The same method as in Theorem I.1 shows that $v_n = u_n - u$ converges weakly in V to 0, $v_{n+1} - v_n$ converges strongly in V to 0 and $|v_n|^2 \downarrow_n \ell$ for some $\ell \geq 0$.

Next, we argue as in J.B. Baillon and P.L. Lions [29] and we observe that if S denotes $\left(P_{V_m^\perp} \dots P_{V_1^\perp}\right)^p$ for some $p \geq 1$ then for all $w,z \in V$

$$|(w,z) - (Sw,Sz)| \leq \frac{1}{2}|w|^2 + \frac{1}{2}|z|^2 - \frac{1}{2}|Sw|^2 - \frac{1}{2}|Sz|^2 \quad .$$

This yields for all $n \geq 0$, $p \geq 1$, $k \geq 0$

$$\left|\left(v_{nm}, v_{(n+k)m}\right) - \left(v_{(n+p)m}, v_{(n+p+k)m}\right)\right|$$
$$\leq \frac{1}{2}\left(|v_{nm}|^2 + |v_{(n+k)m}|^2 - |v_{(n+p)m}|^2 - |v_{(n+p+k)m}|^2\right)$$

and since $|v_n|^2 \downarrow_n \ell$, this implies that $\left(v_{nm}, v_{(n+k)m}\right)$ converges to some ℓ_k uniformly in k as n goes to $+\infty$. Next, on one hand, recalling that $v_{n+1} - v_n \to_n 0$ we obtain that $\ell_k = \ell$ for all $k \geq 0$ and on the other hand letting k going to $+\infty$ we deduce that $\lim_k \ell_k = 0$. Hence, $\ell = 0$ and $v_n \to_n 0$.

The geometrical convergence result is obtained by showing that there exists $k \in [0,1[$ such that

$$\left\| P_{V_m^\perp} \dots P_{V_1^\perp} \right\| \leq k \quad . \tag{31}$$

To prove this claim, we first observe that Lemma I.1 and its proof may be easily adapted to yield

$$(32)\qquad |v| \leq C_o \left(\sum_{j=1}^{m} |P_{V_j} v|^2\right)^{1/2} , \qquad \forall\, v \in V$$

for some $C_o > 0$. Next, to show (31), we argue by contradiction and we assume there exist $(v_n)_n$ such that

$$1 = |v_n| = \lim_n |P_{V_m^\perp} \cdots P_{V_1^\perp} v_n| .$$

Obviously we also have for all $j \in \{1,\dots,m\}$

$$|P_{V_j^\perp} \cdots P_{V_1^\perp} v_n| \underset{n}{\to} 1$$

therefore for all $j \in \{1,\dots,m-1\}$

$$P_{V_{j+1}} P_{V_j^\perp} \cdots P_{V_1^\perp} v_n \underset{n}{\to} 0 \qquad \text{and} \qquad P_{V_1} v_n \underset{n}{\to} 0 .$$

And if $P_{V_1} v_n, \dots, P_{V_j} v_n \underset{n}{\to} 0$ for some $j \in \{1,\dots,m-1\}$ we deduce

$$P_{V_{j+1}} v_n - P_{V_{j+1}} P_{V_j^\perp} \cdots P_{V_1^\perp} v_n \underset{n}{\to} 0$$

therefore $P_{V_{j+1}} v_n \underset{n}{\to} 0$. This means that

$$P_{V_j} v_n \underset{n}{\to} 0 \qquad \text{for all } 1 \leq j \leq M$$

and we reach a contradiction with (32). ∎

The second extension of the Schwarz alternating method consists in using at each step a possibly new splitting of $\mathcal{O}$ i.e.

$$(33)\qquad \mathcal{O} = \mathcal{O}_1^n \cup \mathcal{O}_2^n , \qquad \mathcal{O}_i^n \text{ open set of } \mathbb{R}^n \quad (i = 1,2) .$$

We then consider $u_o \in H_o^1(\mathcal{O})$ and for $n \geq 0$ the solutions u_{2n+1}, u_{2n+2} of

$$- \Delta u^{2n+i} = f \text{ in } \mathcal{O}_i^n \quad , \quad u^{2n+i} = u^{2n+i-1} \text{ on } \partial\mathcal{O}_i^n \quad (i = 1,2)$$

and we extend u^{2n+i} to $\mathcal{O}$ by u^{2n+i-1} (for $i = 1,2$).

Now, if we assume that (for instance) (18) holds for some constant C independent of n, then this sequence $(u_n)_n$ converges geometrically in V to u. Observe only that there exists $k \in [0,1[$ independent of n such that

$$\| P_{V_1^{n\perp}} P_{V_2^{n\perp}} \| \quad , \quad \| P_{V_2^{n\perp}} P_{V_1^{n\perp}} \| \quad < \quad k$$

where $V_i^n = H_0^1(\mathcal{O}_i^n)$ for $i = 1,2$.

Of course, this extension of the Schwarz method may be combined with the two others described in this section but we will skip such considerations here.

The final extension we wish to consider concerns "parallel" versions of the Schwarz alternating method. We again assume that (26) holds but we assume now that

(34) $$\overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j \cap \overline{\mathcal{O}}_k = \emptyset \quad , \text{ for all distinct } i,j,k \text{ in } \{1,\ldots,m\} .$$

Then, given m initializations $u_0^j \in H_0^1(\mathcal{O})$ ($\forall\, i \leq j \leq m$) we build m sequences inductively as follows : for all $n \geq 0$ and for all $i \in \{1,\ldots,m\}$, u_{n+1}^i is the solution of

(35) $$\begin{cases} - \Delta u_{n+1}^i = f \text{ in } \mathcal{O}_i \quad , \quad u_{n+1}^i = 0 \text{ on } \partial\mathcal{O}_i \cap \partial\mathcal{O} \\ u_{n+1}^i = u_n^j \quad \text{on } \partial\mathcal{O}_i \cap \mathcal{O}_j \quad \text{for all } 1 \leq j \leq m . \end{cases}$$

In the case when $m = 2$, we immediately see that the subsequences $(u_0^2,u_1^1,u_2^2,u_3^1,u_4^2\ldots)$ and $(u_0^1,u_1^2,u_2^1,u_3^2,u_4^1 \ldots)$ are in fact sequences generated by the usual Schwarz alternating method (interchanging $\mathcal{O}_1$ and $\mathcal{O}_2$ in the second case).

As soon as $m \geq 3$, the situation becomes more interesting. And even if, as we will see in section II, each sequence u_n^i converges in $\mathcal{O}_i$ to u, this method does not have always a variational interpretation in terms of iterated projections. A related difficulty is that, using the sequences $(u_n^1)_n$, $(u_n^2)_n$, ..., $(u_n^m)_n$ it is not always possible to define a single-valued function defined on the whole domain $\mathcal{O}$ in a continuous way. In fact, the necessary and sufficient condition for these two difficulties not to happen is that

$$\text{(36)} \quad \begin{cases} \text{For all distinct } i,j,k \in \{1,\dots,m\}, \text{ if } \mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset, \mathcal{O}_i \cap \mathcal{O}_k \neq 0 \\ \text{then} \quad \mathcal{O}_j \cap \mathcal{O}_k = \emptyset \end{cases}$$

Hence, if (36) holds, the results proved above adapt provided we use some of the ideas described below in the particular cases $m = 3$ or $m = 4$.

To be more specific, let us consider the particular cases $m = 3$ and $m = 4$. It will be convenient to use the notation $\mathcal{O}_i \leftrightarrow \mathcal{O}_j$ if $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$. Then, if $m = 3$, up to an irrelevant change of notations, two cases occur.

<u>Case 3.1</u> : $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \leftrightarrow \mathcal{O}_3$

Typical examples are

Figure 3.a

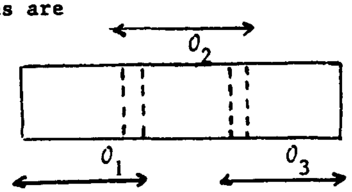
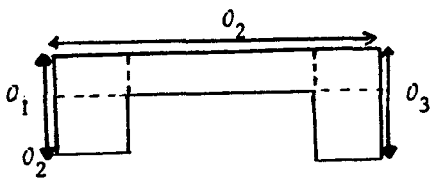


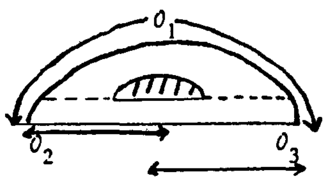
Figure 3.b



<u>Case 3.2</u> : $\mathcal{O}_1 \begin{smallmatrix} \nearrow & \mathcal{O}_2 \\ & \updownarrow \\ \searrow & \mathcal{O}_3 \end{smallmatrix}$

A typical example is

Figure 3.c



Observe that (36) holds in Case 3.1 but does not hold in Case 3.2. Then, the variational interpretation of (35) in case (3.1) may be obtained as follows : we define $\tilde{u}_n$ as follows for $k \geq 0$

$$
(37) \qquad \begin{cases} \tilde{u}_{2k+1} = u_k^1 \text{ on } \bar{\mathcal{O}}_1 , \ = u_{k-1}^2 \text{ on } \mathcal{O}_2 - (\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_3) , \ = u_k^3 \text{ on } \bar{\mathcal{O}}_3 \\ \tilde{u}_{2k+2} = u_k^1 \text{ on } \mathcal{O}_1 - \bar{\mathcal{O}}_2 , \ = u_{k+1}^2 \text{ on } \bar{\mathcal{O}}_2 , \ = u_k^3 \text{ on } \mathcal{O}_3 - \bar{\mathcal{O}}_2 . \end{cases}
$$

(In fact, the other "half" of the sequences $(u_n^1)_n$, $(u_n^2)_n$, $(u_n^3)_n$ may be used similarly to define another sequence of functions on $\mathcal{O}$ with the same properties than $\tilde{u}_n$).

And we remark next that (37) may be interpreted as follows

$$
(38) \qquad \tilde{u}_{2k+2} - u = P_{V_1^\perp \cap V_3^\perp} (\tilde{u}_{2k+1} - u) \quad , \quad \tilde{u}_{2k+1} - u = P_{V_2^\perp} (\tilde{u}_{2k} - u)
$$

(observe that $V_1 \perp V_3$). So, a posteriori, these sequences are in this case somewhat equivalent to sequences generated by a "Schwarz alternating method". And we have of course the same convergence results as before (strong convergence of $\tilde{u}_n$ to u in V and geometric convergence if $V_1 + V_2 + V_3 = V$ as it is the case in Figures 3.a and 3.b).

In case 3.2, the situation is not as simple and in order to have a variational argument to prove convergence (we will see in section II that the process does converge via maximum principle...) we have to modify (35). We give an example of such a modification below which, however, is not parallel : u_n^i is defined for $n \geq 2$ and for $i = 1,2,3$ as the solution of

$$
\begin{cases} -\Delta u_n^1 = f \quad \text{in} \quad \mathcal{O}_1 \quad , \quad u_n^1 = 0 \quad \text{on} \quad \partial\mathcal{O}_1 \cap \partial\mathcal{O} \\ u_n^1 = u_{n-1}^2 \quad \text{on} \quad \partial\mathcal{O}_1 \cap \mathcal{O}_2 \quad , \quad u_n^1 = u_{n-1}^3 \quad \text{on} \quad \partial\mathcal{O}_1 \cap \mathcal{O}_3 \end{cases}
$$

$$
\begin{cases} -\Delta u_n^2 = f \quad \text{in} \quad \mathcal{O}_2 \quad , \quad u_n^2 = 0 \quad \text{on} \quad \partial\mathcal{O}_2 \cap \partial\mathcal{O} \\ u_n^2 = u_{n-1}^1 \quad \text{on} \quad \partial\mathcal{O}_2 \cap \mathcal{O}_1 \quad , \quad u_n^2 = u_{n-1}^3 \quad \text{on} \quad \partial\mathcal{O}_2 \cap \mathcal{O}_3 \end{cases}
$$

$$\begin{cases} -\Delta u_n^3 = f \quad \text{in} \quad \mathcal{O}_3 \quad , \quad u_n^3 = 0 \quad \text{on} \quad \partial\mathcal{O}_3 \cap \partial\mathcal{O} \\ u_n^3 = u_{n-1}^1 \quad \text{on} \quad \partial\mathcal{O}_3 \cap \mathcal{O}_1 \quad , \quad u_n^3 = u_n^2 \quad \text{on} \quad \partial\mathcal{O}_3 \cap \mathcal{O}_2 \quad . \end{cases}$$

We may then define a sequence $(\tilde{u}_n)_n$ as follows

$$\begin{cases} \tilde{u}_{3k} = u_{k-1}^1 \quad \text{on} \quad \bar{\mathcal{O}}_1 - \bar{\mathcal{O}}_2 \ , \ = u_k^2 \quad \text{on} \quad \bar{\mathcal{O}}_2 \ , \ = u_{k-2}^3 \quad \text{on} \quad \bar{\mathcal{O}}_3 - (\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_2) \\ \tilde{u}_{3k+1} = u_{k-1}^1 \quad \text{on} \quad \bar{\mathcal{O}}_1 - (\bar{\mathcal{O}}_2 \cup \bar{\mathcal{O}}_3) \ , \ = u_k^2 \quad \text{on} \quad \bar{\mathcal{O}}_2 - \bar{\mathcal{O}}_3 \ , \ = u_k^3 \quad \text{on} \quad \bar{\mathcal{O}}_3 \\ \tilde{u}_{3k+2} = u_{k+1}^1 \quad \text{on} \quad \bar{\mathcal{O}}_1 \ , \ = u_k^2 \quad \text{on} \quad \bar{\mathcal{O}}_2 - (\bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_3) \ , \ = u_k^3 \quad \text{on} \quad \bar{\mathcal{O}}_3 - \bar{\mathcal{O}}_1 \ . \end{cases}$$

And we observe that

$$\begin{cases} \tilde{u}_{3k+2} - u = P_{V_1^\perp} (\tilde{u}_{3k+1} - u) \quad , \quad \tilde{u}_{3k+1} - u = P_{V_3^\perp} (\tilde{u}_{3k} - u) \\ \tilde{u}_{3k} - u = P_{V_2^\perp} (\tilde{u}_{3k-1} - u) \end{cases}$$

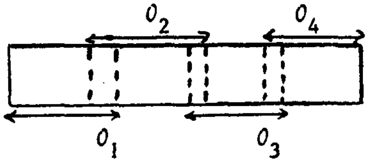
So we see that this modification of (35) is in some sense equivalent to the sequential algorithm (27) and thus presents the same convergence properties.

Now, in the case $m = 4$, up to irrelevant changes of notations, six cases may occur.

<u>Case 4.1</u> : $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \leftrightarrow \mathcal{O}_3 \leftrightarrow \mathcal{O}_4$.

A typical example is

Figure 4.a

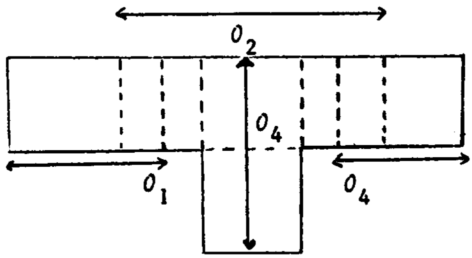


<u>Case 4.2</u> : $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \leftrightarrow \mathcal{O}_4$, $\mathcal{O}_2 \updownarrow \mathcal{O}_3$

$$\begin{matrix} \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \leftrightarrow \mathcal{O}_4 \\ \updownarrow \\ \mathcal{O}_3 \end{matrix}$$

A typical example is

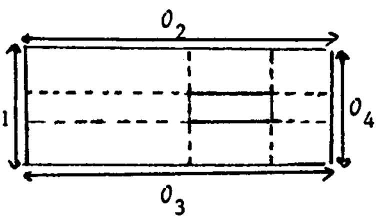
Figure 4.b



<u>Case 4.3</u> : $O_1 \nearrow O_2 \searrow O_4$, $O_1 \searrow O_3 \nearrow O_4$

A typical example is

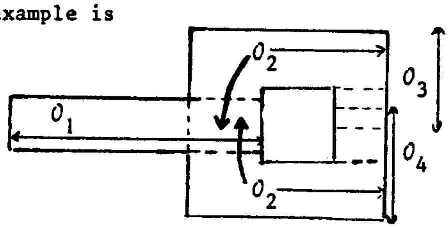
Figure 4.c



<u>Case 4.4</u> : $O_1 \leftrightarrow O_2 \nearrow O_3$, $O_2 \searrow O_4$, $O_3 \updownarrow O_4$

A typical example is

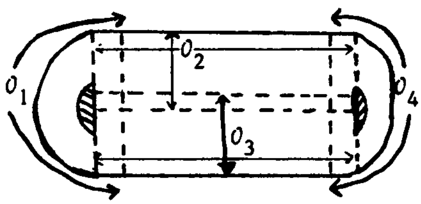
Figure 4.d



<u>Case 4.5</u> : $O_1 \nearrow O_2 \searrow O_4$, $O_2 \updownarrow O_3$, $O_1 \searrow O_3 \nearrow O_4$

A typical example is

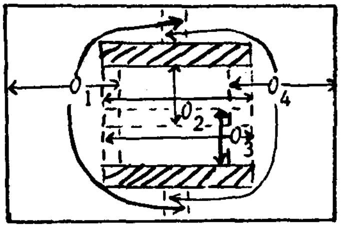
Figure 4.e



Case 4.6 : $\mathcal{O}_1 \nearrow \mathcal{O}_2 \searrow \mathcal{O}_3$, $\mathcal{O}_1 \searrow \mathcal{O}_4 \nearrow \mathcal{O}_3$

A typical example is

Figure 4.f



It is quite clear that (36) holds in Cases 4.1, 4.2 and 4.3 but does not hold in Cases 4.4, 4.5 and 4.6. In these latter cases, however we can modify (35) along the lines of the modification of Case 3.2 proposed above to obtain an algorithm consisting of iterated projections and yielding the possibility of defining globally approximated solutions on the whole domain $\mathcal{O}$.

We now explain the variational interpretation of the parallel algorithm in Cases 4.1, 4.2 and 4.3. In Case 4.1, we introduce for $k \geqslant 1$

$$(39)\quad \begin{cases} \tilde{u}_{2k} = u^1_{k-1} \text{ in } \mathcal{O}_1 - \mathcal{O}_2 , = u^2_k \text{ in } \mathcal{O}_2 , = u^3_{k-1} \text{ in } \mathcal{O}_3 - (\mathcal{O}_2 \cup \mathcal{O}_4), \\ \qquad = u^4_k \text{ in } \mathcal{O}_4 \\ \tilde{u}_{2k+1} = u^1_{k+1} \text{ in } \mathcal{O}_1 , = u^2_k \text{ in } \mathcal{O}_2 - (\mathcal{O}_1 \cup \mathcal{O}_3) , = u^3_{k+1} \text{ in } \mathcal{O}_3 , \\ \qquad = u^4_k \text{ in } \mathcal{O}_4 - \mathcal{O}_3 \end{cases}$$

so that

$$\tilde{u}_{2k+1} - u = P_{V_1^\perp \cap V_3^\perp}(\tilde{u}_{2k} - u) \quad , \quad \tilde{u}_{2k} - u = P_{V_2^\perp \cap V_4^\perp}(\tilde{u}_{2k-1} - u)$$

and the convergence properties are easily analysed.

In Case 4.2, we introduce for $k \geqslant 1$

$$(40)\quad \begin{cases} \tilde{u}_{2k} = u^1_{k-1} \text{ in } \mathcal{O}_1 - \mathcal{O}_2 , = u^2_k \text{ in } \mathcal{O}_2 , = u^3_{k-1} \text{ in } \mathcal{O}_3 - \mathcal{O}_2 , \\ \qquad = u^4_{k-1} \text{ in } \mathcal{O}_4 - \mathcal{O}_2 \\ \tilde{u}_{2k+1} = u^1_{k+1} \text{ in } \mathcal{O}_1 , = u^2_k \text{ in } \mathcal{O}_2 - (\mathcal{O}_1 \cup \mathcal{O}_3 \cup \mathcal{O}_4) , = u^3_{k+1} \text{ in } \mathcal{O}_3, \\ \qquad = u^4_{k+1} \text{ in } \mathcal{O}_4 \end{cases}$$

so that

$$\tilde{u}_{2k+1}-u = P_{V_1^\perp \cap V_3^\perp \cap V_4^\perp}(\tilde{u}_{2k}-u) \quad , \quad \tilde{u}_{2k}-u = P_{V_2^\perp}(\tilde{u}_{2k-1}-u)$$

and the convergence properties are easily analysed.

In Case 4.3, we introduce for $k \geq 1$

$$(41) \quad \begin{cases} \tilde{u}_{2k} = u^1_{k-1} \text{ in } \mathcal{O}_1-(\mathcal{O}_2 \cup \mathcal{O}_3) \ , \ = u^2_k \text{ in } \mathcal{O}_2 \ , \ = u^3_k \text{ in } \mathcal{O}_3 \ , \\ \qquad = u^4_{k-1} \text{ in } \mathcal{O}_4-(\mathcal{O}_2 \cup \mathcal{O}_3) \\ \tilde{u}_{2k+1} = u^1_{k+1} \text{ in } \mathcal{O}_1 \ , \ = u^2_k \text{ in } \mathcal{O}_2-(\mathcal{O}_1 \cup \mathcal{O}_4), \ = u^3_k \text{ in } \mathcal{O}_3-(\mathcal{O}_1 \cup \mathcal{O}_4), \\ \qquad = u^4_{k+1} \text{ in } \mathcal{O}_4 \end{cases}$$

so that

$$\tilde{u}_{2k+1}-u = P_{V_1^\perp \cap V_4^\perp}(\tilde{u}_{2k}-u) \quad , \quad \tilde{u}_{2k}-u = P_{V_2^\perp \cap V_3^\perp}(\tilde{u}_{2k-1}-u)$$

and the convergence properties are easily analysed.

I.5 Nonlinear monotone problems.

First of all, we begin with some general abstract minimization considerations. Indeed, observe that the model problem consists in minimizing over $H^1_o(\mathcal{O})$ the functional

$$(42) \qquad J(u) = \int_{\mathcal{O}} \frac{1}{2} |\nabla u|^2 - fu \, dx \quad .$$

Then, the Schwarz alternating method is equivalent to : $u_o \in V$, $J(u_o) < \infty$ and

$$(43) \quad \begin{cases} u_{2n+1} - u_{2n} & \text{is the minimum over } V_1 \text{ of } J(u_{2n}+\cdot) \\ u_{2n+2} - u_{2n+1} & \text{is the minimum over } V_2 \text{ of } J(u_{2n+1}+\cdot) \end{cases}$$

for all $n \geq 0$. In the form (43), it is now quite clear that the Schwarz sequence $(u_n)_{n \geq 0}$ is related to classical minimization methods over product spaces (see for instance J. Céa [30]).

We now give a brief study of the algorithm (43) assuming that V is an Hilbert space (we identify $V^{\star}$ with V), V_1, V_2 are two closed subspaces such that $V = V_1 + V_2$ and J is a proper, lower semi-continuous, coercive convex function from V into $\mathbb{R} \cup \{+\infty\}$. It is possible to give many convergence results for the method (43) and the one which follows is only the simplest we could think of. Let us also emphasize the obvious fact that such convergence results have immediate applications to convergence results for the Schwarz alternating method when applied to general classes of nonlinear monotone partial differential equations (or at least those having a variational structure). We will assume that for all $R < \infty$, denoting by $K_R = \{u \in V \,/\, J(u) < R\}$, $J \in C^1(K_R)$ and

$$(44) \qquad \exists\, \alpha_R > 0 \,,\ \forall\, v,u \in K_R \,, \quad J(v) - J(u) - (J'(u), v-u) \geq \alpha_R |v-u|^2$$

$$(45) \qquad J' \text{ is uniformly continuous on } K_R \,.$$

Then we have the

<u>Theorem I.3</u> : Under the above assumptions, the sequence $(u_n)_n$ defined by (43) converges in V to the minimum u of J. If in addition, J' is Lipschitz in a neighborhood of u then $(u_n)_n$ converges geometrically to u.

<u>Proof :</u> Obviously, $J(u_n) \downarrow_n \ell$ for some $\ell \in \mathbb{R}$ and thus u_n is bounded in V. Furthermore, (43) may be written as

$$(46) \qquad \begin{cases} (J'(u_{2n+1}), v_1) = 0 & \forall\, v_1 \in V_1 \,,\ u_{2n+1} - u_{2n} \in V_1 \\ (J'(u_{2n+2}), v_2) = 0 & \forall\, v_2 \in V_2 \,,\ u_{2n+2} - u_{2n+1} \in V_2 \end{cases}$$

for all $n \geq 0$. We choose $R > J(u_0)$, and we denote by $\alpha = \alpha_R$, ω the modulus of continuity of J' over K_R.

Newt, if we apply (44) with $v = u_{2n}$, $u = u_{2n+1}$ and with $v = u_{2n+1}$, $u = u_{2n+2}$ and if we combine the resulting inequalities with (46), we deduce

$$(47) \qquad J(u_n) - J(u_{n+1}) \geq \alpha|u_n-u_{n+1}|^2 \quad , \quad \text{for all} \quad n \geq 0 .$$

Hence, in particular, $u_n-u_{n+1} \underset{n}{\to} 0$ in V.

Now, we recall from the proof of Lemma I.2 that for each $v \in V$ there exist $v_1 \in V_1$, $v_2 \in V_2$ such that

$$v = v_1+v_2 \quad , \quad \max(|v_1|,|v_2|) \leq C_o|v|$$

for some $C_o > 0$ independent of v. Combining this with (46) we deduce for all $n \geq 0$

$$|(J'(u_n),v)| \leq C_o |J'(u_{n+1})-J'(u_n)| \, |v| \quad , \quad \forall v \in V$$

i.e. using (45)

$$(48) \qquad |J'(u_n)| \leq C_o \, \omega(|u_{n+1}-u_n|) \quad , \quad \forall n \geq 0 .$$

Using again (44), we deduce, denoting by u the minimum of J

$$|u_n-u|^2 \leq \frac{1}{\alpha} (J'(u_n)-J'(u) , u_n-u) = \frac{1}{\alpha} (J'(u_n) , u_n-u)$$

hence

$$(49) \qquad |u_n-u| \leq \frac{C_o}{\alpha} \, \omega(|u_{n+1}-u_n|)$$

and the convergence is proved.

If J' is Lipschitz in a neighborhood of u then for n large enough we deduce from (49)

$$|u_n-u|^2 \leq C_1 |u_{n+1}-u_n|^2$$

for some $C_1 > 0$ independent of n. And going back to (47) we obtain

$$(50) \qquad [J(u_n)-J(u)] - [J(u_{n+1})-J(u)] \geq \frac{\alpha}{C_1} |u_n-u|^2 .$$

It is then easy to conclude since J' being Lipschitz near u we have

$$J(u_n) - J(u) \leq C_2 \, |u_n-u|^2$$

while on the other hand by (44)

$$J(u_n) - J(u) \geq \alpha |u_n-u|^2 \quad . \qquad \blacksquare$$

We next give another convergence result for functionals J of the type

$$J(u) = \frac{1}{2} |u|^2 - L(u) + \Pi_K(u) \quad , \quad \forall \, u \in V \tag{51}$$

where $\Pi_K(u) = 0$ if $u \in K$, $\Pi_K(u) = +\infty$ if $u \notin K$, $L \in V^*$ and K is closed, convex, nonempty set. Recall that a typical choice for applications to partial differential equations is $V = H_o^1(\mathcal{O})$ in which case the minimum u of J over V solves the following variational inequality

$$\int_{\mathcal{O}} (\nabla u, \nabla(v-u)) \, dx \geq \int_{\mathcal{O}} f(v-u) \, dx \quad , \quad \forall \, v \in K \, , \, u \in K \tag{52}$$

where $f \in L^2(\mathcal{O})$ for instance $(H^{-1}(\mathcal{O})$ in general). Typical examples of convex sets K are

$$K_1 = \{v \in H_o^1(\mathcal{O}) \, / \, v \geq \phi \text{ a.e. in } \mathcal{O}\} \tag{53}$$

$$K_2 = \{v \in H_o^1(\mathcal{O}) \, / \, \phi_1 \leq v \leq \phi_2 \text{ a.e. in } \mathcal{O}\} \tag{54}$$

where $\phi, \phi_1, \phi_2 \in H^1(\mathcal{O})$ for instance and $\phi, \phi_1 \leq 0$ on $\partial\mathcal{O}$ while $\phi_2 \geq 0$ on $\partial\mathcal{O}$.

Then, on such variational inequalities, the Schwarz alternating method yields the following sequence : let $u_o \in K$, u_{2n+1} and u_{2n+2} are determined respectively for $n \geq 0$ by the solutions of the following variational inequalities

$$\left\{ \begin{array}{lll} (u_{2n+1}, v-u_{2n+1}) \geq L(v-u_{2n+1}) & , & \forall \, v \in K \cap (u_{2n}+V_1) \, , \\ & & u_{2n+1} \in K \cap (u_{2n}+V_1) \; ; \\ (u_{2n+2}, v-u_{2n+2}) \geq L(v-u_{2n+2}) & , & \forall \, v \in K \cap (u_{2n+1}+V_2) \, , \\ & & u_{2n+2} \in K \cap (u_{2n+1}+V_2) \quad . \end{array} \right. \tag{55}$$

In the case when $V = H_o^1(\mathcal{O})$ and $V_1 = H_o^1(\mathcal{O}_1)$, $V_2 = H_o^1(\mathcal{O}_2)$, (55) become

$$(56)\quad \begin{cases} \int_{\mathcal{O}_1} (\nabla u_{2n+1},\nabla(v-u_{2n+1})) - f(v-u_{2n+1})\,dx \geq 0 \quad , \quad \forall\, v \in K \, , \\ \qquad\qquad v \equiv u_{2n} \text{ on } \mathcal{O}_1^c \, , \; u_{2n+1} \in K \, , \; u_{2n+1} \equiv u_{2n} \text{ on } \mathcal{O}_1^c \, , \\ \int_{\mathcal{O}_2} (\nabla u_{2n+2},\nabla(v-u_{2n+2})) - f(v-u_{2n+2})\,dx \geq 0 \quad , \quad \forall\, v \in K \, , \\ \qquad\qquad v \equiv u_{2n+1} \text{ on } \mathcal{O}_2^c \, , \; u_{2n+2} \in K \, , \; u_{2n+2} \equiv u_{2n+1} \text{ on } \mathcal{O}_2^c \, . \end{cases}$$

For general K, V, V_1, V_2 (even such that $V = V_1 + V_2$) the algorithm does not necessarily converge to the minimum u of J i.e. to the solution of the variational inequality

$$(57)\qquad (u,v-u) \geq L(v-u) \quad , \quad \forall\, v \in K \, , \; u \in K \quad .$$

Indeed, assume that there exist $u_o \in K$, V_1, V_2 closed subspaces such that $V_1 + V_2 = V$, $K \cap (u_o+V_1) = \{u_o\}$, $K \cap (u_o+V_2) = \{u_o\}$, then $u_n = u_o$ for all $n \geq 0$ and if $u \neq u_o$ the method does not converge.

<u>Example</u> : $V = \mathbb{R}^2$, $V_i = \{(x_1,x_2) \in \mathbb{R}^2 \,/\, x_i = 0\}$ for $i = 1,2$, $K = \{(x_1,x_2) \in \mathbb{R}^2 \,/\, x_1=x_2\}$, $u_o = 0$.

However, if $V = H_o^1(\mathcal{O})$, $V_1 = H_o^1(\mathcal{O}_1)$, $V_2 = H_o^1(\mathcal{O}_2)$ as in sections I.1-I.2, then we prove below that the method (55) or equivalently (56) converges to u provided (18) holds and K has the following structure

$$(58)\qquad K = \{v \in H_o^1(\mathcal{O}) \,/\, v(x) \in C(x) \text{ a.e. in } \mathcal{O}\}$$

where for each x in $\mathcal{O}$, $C(x)$ is aclosed (nonempty) interval in $\mathbb{R}$.

Observe that K_1,K_2 given respectively by (53),(54) satisfy the above property. As we will see below, the fact that we obtain in this case the convergence to u is based upon the following property

$$(59)\qquad \forall\, v,w \in K \, , \qquad \chi_1 v + \chi_2 w \in K \quad .$$

The abstract version of this property may be written as follows

$$(60)\quad \begin{cases} \exists T : V \times V \to V, \quad T(v,w) \text{ is uniformly continuous in } v \text{ bounded}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall w \in V. \\ T(v,w) + T(w,v) = v+w, \quad \forall v,w \in K \\ \forall v,w \in K, \quad T(v,w) \in K \cap (w+V_1) \cap (v+V_2). \end{cases}$$

Then, we have the

<u>Theorem I.4</u> : Assume that $V_1 + V_2 = V$ and that (60) holds. Then, the sequence $(u_n)_n$ generated by (55) converges to the solution u of (57).

<u>Remark</u> : As we will see in section II, it is possible to prove the geometric convergence in the case of elliptic variational inequalities with K given (for instance) by (58).

<u>Proof of Theorem I.4</u> : Exactly as in the proof of Theorem I.3, we see that $J(u_n) \downarrow_n$, u_n is bounded and inserting $v = u_{2n}$ or $v = u_{2n+1}$ in (55) we deduce also easily

$$J(u_{n+1}) + \frac{1}{2} |u_n - u_{n+1}|^2 \leq J(u_n) \qquad \text{for all } n \geq 0$$

and thus $u_{n+1} - u_n \underset{n}{\to} 0$ in V.

Next, we take in (55) $v = T(u,u_{2n})$ and $v = T(u_{2n+1},u)$ and because of (60) we find

$$(u_{2n+1}, T(u,u_{2n}) - u_{2n+1}) \geq L(T(u,u_{2n}) - u_{2n+1})$$

$$(u_{2n+2}, T(u_{2n+1},u) - u_{2n+2}) \geq L(T(u_{2n+1},u) - u_{2n+1})$$

and summing we deduce easily using again (60)

$$(u_{2n+1}, u + u_{2n} - u_{2n+1} - u_{2n+2}) \geq L(u + u_{2n} - u_{2n+1}) +$$
$$+ (u_{2n+1}, T(u_{2n+1},u) - T(u_{2n},u)) + (u_{2n+2} - u_{2n+1}, T(u_{2n+1},u) - u_{2n+2}).$$

Now, by the uniform continuity of $T(\cdot,u)$ on bounded sets we obtain

$$(u_{2n+1}, u-u_{2n+1}) - L(u-u_{2n+1}) \geq \varepsilon_{2n+1} \underset{n}{\to} 0$$

arguing similarly for u_{2n}, we deduce for all $n \geq 0$

$$(u_n, u-u_n) - L(u-u_n) \geq \varepsilon_n \underset{n}{\to} 0 \quad .$$

While on the other hand (57) yields for all $n \geq 0$

$$(u, u_n-u) - L(u_n-u) \geq 0 \quad ,$$

hence $|u_n-u|^2 \leq \varepsilon_n$, and we conclude. ■

<u>Remark</u> : We want to conclude this section by a few observations on a different extension of the Schwarz alternating method to variational inequalities. We first rewrite the usual "Schwarz sequence" as follows : set $w_o = u_o$, and then

(61) $$v_n = P_{V_1}(u-w_n) \quad , \quad w_{n+1} = P_{V_2}(u-v_n) \qquad \text{for } n \geq 0 \ .$$

We then claim that $v_n+w_n = u_{2n+1}$, $v_n+w_{n+1} = u_{2n+2}$ where $(u_n)_n$ is the sequence generated by the Schwarz method $(u_{2n+1}-u_{2n} = P_{V_1}(u-u_{2n})$, $u_{2n+2}-u_{2n+1} = P_{V_2}(u-u_{2n+1}))$. Indeed observe that

$$(v_o+w_o)-u_o = (v_o+w_o)-w_o = v_o = P_{V_1}(u-w_o) = P_{V_1}(u-u_o)$$

and if our claim is proved up to $n-1$ then

$$v_n+w_n = P_{V_1}(u-w_n-v_{n-1}) + w_n + v_{n-1} = P_{V_1}(u-u_{2n}) + u_{2n}$$

i.e. $v_n+w_n = u_{2n+1}$, and

$$v_n+w_{n+1} = P_{V_2}(u-v_n-w_n) + v_n + w_n = P_{V_2}(u-u_{2n+1}) + u_{2n+1}$$

i.e. $v_n+w_{n+1} = u_{2n+2}$, proving thus our claim.

Next, we observe that if $V = H^1_o(\mathcal{O})$, $V_1 = H^1_o(\mathcal{O}_1)$, $V_2 = H^1_o(\mathcal{O}_2)$, (18) holds and K is given by (53) with $\phi \equiv 0$, then $K = K_1 + K_2$ where $K_1 = K \cap V_1$, $K_2 = K \cap V_2$. Indeed, one just observe that if $\phi \in K$, $\phi = (\chi_1 \phi) + (\chi_2 \phi)$ and $\chi_i \phi \in K_i$ $(i = 1,2)$ while if $\phi_i \in K_i$ $(i = 1,2)$, $\phi = \phi_1 + \phi_2 \in K$.

And the above considerations lead to the following algorithm

$$(62) \qquad v_n = P_{K_1}(\bar{u}-w_n) \quad , \quad w_{n+1} = P_{K_2}(\bar{u}-v_n) \qquad \text{for } n \geq 0$$

with w_o arbitrary in K_2 and $\bar{u}$ is a given element of V , where P_K denotes the orthogonal projection onto K .

Let us give now some convergence results concerning the method (62) : for general convex closed sets K_1, K_2 in an Hilbert space V , denoting by $K = \overline{K_1 + K_2}$, we can show $v_n + w_{n+1}$, $v_n + w_n$ converge to $u = P_K(\bar{u})$. Furthermore, if $K = K_1 + K_2$, then $(v_n)_n$ and $(w_n)_n$ converge weakly to some $v \in K_1$, $w \in K_2$ such that $u = v+w$. Indeed, we have for all $n \geq 0$

$$(v_n + w_n - \bar{u} , k_1 - v_n) \geq 0 \qquad \forall k_1 \in K_1 \quad ,$$

$$(v_n + w_{n+1} - \bar{u} , k_2 - w_{n+1}) \geq 0 \qquad \forall k_2 \in K_2$$

then choosing $k_1 = v_{n-1}$, $k_2 = w_n$ we deduce for all $n \geq 0$

$$\frac{1}{2}|v_{n+1} + w_{n+1} - \bar{u}|^2 + \frac{1}{2}|v_{n+1} - v_n|^2 \leq \frac{1}{2}|v_n + w_{n+1} - \bar{u}|^2$$

$$\frac{1}{2}|v_n + w_{n+1} - \bar{u}|^2 + \frac{1}{2}|w_n - w_{n+1}|^2 \leq \frac{1}{2}|v_n + w_n - \bar{u}|^2$$

and thus $v_n + w_n$, $v_n + w_{n+1}$ are bounded and $w_{n+1} - w_n \underset{n}{\to} 0$, $v_{n+1} - v_n \underset{n}{\to} 0$. Hence,

$$(v_n + w_n - \bar{u} , k_1 + k_2 - v_n - w_n) \geq (w_n - w_{n+1} , k_2 - w_{n+1}) + \\ - |w_n - w_{n+1}| \, |v_n + w_n - \bar{u}| \qquad \forall k_1, k_2$$

while

$$(u - \bar{u} , v_n + w_n - u) \geq 0 \quad .$$

Therefore, we find for all $n \geq 0$, $k_1 \in K_1$, $k_2 \in K_2$

$$|v_n+w_n-u|^2 \leq C\,|u-(k_1+k_2)| + C\,|w_n-w_{n+1}| + (w_{n+1}-k_2\,,\,w_n-w_{n+1})$$

for some constant $C \geq 0$ independent of n .

Let n' be a subsequence such that $|v_{n'}+w_{n'}-u| \underset{n'}{\to} \overline{\lim_n}\, |v_n+w_n-u|$. Then, if $(|w_{n'}|)_{n'}$ is bounded, we deduce easily letting n' go to ∞ in the above inequality

$$\overline{\lim_n}\, |v_n+w_n-u|^2 \leq C\,|u-(k_1+k_2)| \quad , \quad \forall\, k_1 \in K_1 \;,\; \forall\, k_2 \in K_2$$

and since $K = \overline{K_1 + K_2}$, we deduce : $v_n+w_n \underset{n}{\to} u$.

On the other hand, if $(|w_{n'}|)_{n'}$ is unbounded, we can find a new subsequence n_k such that

$$|w_{n_k+1}| \leq |w_{n_k}|$$

and we deduce

$$|v_{n_k}+w_{n_k}-u|^2 \leq C\,|u-(k_1+k_2)| + C\,|w_{n_k}-w_{n_k+1}| + |k_2|\,|w_{n_k}-w_{n_k+1}| \,.$$

Letting k go to ∞ , and using the fact that $K = \overline{K_1 + K_2}$ we deduce

$$\overline{\lim_n}\, |v_n+w_n-u| = \lim_{n'}\, |v_{n'}+w_{n'}-u| = \lim_k\, |v_{n_k}+w_{n_k}-u| = 0 \;.$$

Therefore in all cases $v_n+w_n \underset{n}{\to} u$. One proves similarly that $v_n+w_{n+1} \underset{n}{\to} u$.

If $K = K_1 + K_2$, we observe that $u = k_1+k_2$ with $k_i \in K_i$ $(i = 1,2)$ and we remark that

$$k_1 = P_{K_1}(\bar{u}-k_2) \qquad , \qquad k_2 = P_{K_2}(\bar{u}-k_1) \;.$$

In other words, we see there exists a fixed point of the contra ctions

$$T = P_{K_1}\big(\bar{u} - P_{K_2}(\bar{u}-\cdot)\big) \quad , \quad S = P_{K_2}\big(\bar{u} - P_{K_1}(\bar{u}-\cdot)\big)$$

and since $v_{n+1} = Tv_n$, $w_{n+1} = Sw_n$ for all $n \geq 0$, we immediately deduce that v_n and w_n are bounded.

In addition, if k_1 is a fixed point of T then $k_2 = P_{K_2}(\bar{u}-k_1)$ is a fixed point of S and $k_1+k_2 = u$. Furthermore, if for some subsequence n' , $v_{n'}$ converge weakly to some v then $w_{n'}$ converge weakly to $u-v = w$ and one checks easily that $v = Tv$, $w = Sw$. Since finally $|v_n-k_1| \downarrow_n$, $|w_n-k_2| \downarrow_n$ for any fixed points respectively of T ans S , we deduce that v_n and w_n converge weakly using Opial's lemma [31].

We conclude these observations by remarking that if $K = \overline{K_1+K_2}$ but $K \neq K_1+K_2$ then in general $(v_n)_n$ and $(w_n)_n$ are not bounded (take for instance $V = \mathbb{R}^2$, $K_1 = \left\{(x_1,x_2) \in \mathbb{R}^2 \ / \ x_1 > 0 \ , \ x_2 \geq \frac{1}{x_1}\right\}$, $K_2 = \{(x_1,0) \ / \ x_1 \in \mathbb{R}\}$, $u = \bar{u} = 0$ then for all w_0 , $|v_n| \underset{n}{\to} \infty$, $|w_n| \underset{n}{\to} \infty$). We wish also to remark that even if $V = H_0^1(\mathcal{O})$, $K = \{\phi \geq 0 \ , \ \phi \in H_0^1(\mathcal{O})\}$, $K_1 = K \cap V_1$, $K_2 = K \cap V_2$ in general the sequence generated by (62) is distinct from the one generated by the Scharz alternating method.

I.6 Evolution problems.

A model evolution problem is of course the heat equation : let $T > 0$, we now want to approximate the solution of

$$(63) \qquad \frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad \mathcal{O} \times (0,T) \quad , \quad u|_{\partial\mathcal{O} \times (0,T)} = 0$$

with the initial condition

$$(64) \qquad u|_{t=0} = u_0 \qquad \text{in } \mathcal{O}$$

where for instance $u_0 \in H_0^1(\mathcal{O})$, $f \in L^2(\mathcal{O} \times (0,T)) \cap L^\infty(0,T\,;H^{-1}(\mathcal{O}))$. Then, as it is well-known, there exists a unique solution in (say) such that $u \in L^2(0,T\,;H^2 \cap H_0^1)$

$$\frac{\partial u}{\partial t} \in L^2(\mathcal{O} \times (0,T)) \qquad (\text{and thus} \quad u \in C([0,T]\,;\, H_0^1)) \quad .$$

The above problem is of course a model problem for the following class of evolution problems

$$(65) \qquad u_t + Au = f \quad , \quad u|_{t=0} = u_o$$

where A is a bounded, self-adjoint, coercive operator from V into V^* and $V \hookrightarrow H \hookrightarrow V^*$ with V,H Hilbert spaces... We could treat as well such abstract problems.

The first (and naive) way to apply Schwarz decomposition idea is to consider the following sequence of problems : take for instance $u^o(x,t) = u^o(x)$ for all $(x,t) \in \mathcal{O} \times (0,T)$ and solve for $n \geq 0$

$$(66) \qquad \frac{\partial u^{2n+1}}{\partial t} - \Delta u^{2n+1} = f \ \text{in} \ \mathcal{O}_1 x(0,T) \quad , \quad u^{2n+1} = u^{2n} \ \text{on} \ \partial\mathcal{O}_1 x(0,T)$$

$$(67) \qquad \frac{\partial u^{2n+2}}{\partial t} - \Delta u^{2n+2} = f \ \text{in} \ \mathcal{O}_2 x(0,T) \quad , \quad u^{2n+2} = u^{2n+1} \ \text{on} \ \partial\mathcal{O}_2 x(0,T)$$

with of course the initial conditions : $u^{2n+2}|_{t=0} = u^{2n+1}|_{t=0} = u_o$ in $\mathcal{O}$; and one extends u^{2n+1}, u^{2n+2} to $\mathcal{O}$ by respectively u^{2n}, u^{2n+1} . This simple adaptation of the Schwarz alternating method does not seem to converge for variational reasons : it does converge but we will prove this convergence in section II by maximum principle arguments.

Then, since for any practical purpose one is obliged to discretize the various evolution equations (63),(66),(67), it is obviously tempting to combine the Schwarz procedure with the iterations of schemes corresponding to time discretisation. We will explain these possibilities by combining the Schwarz procedure and the simplest implicit scheme. To this end, for each $N \geq 1$, we set $\delta = \frac{T}{N}$ and we consider $t_k = k\delta$ for $0 \leq k \leq N$. We are going to build (hopefully) an approximation of the solution u of (63) : to this end, let $f_k = \frac{1}{\delta} \int_{t_{k-1}}^{t_k} f(s)\, ds$ for $k \geq 1$ and we wish to define in $\mathcal{O}_1 , \mathcal{O}_2$

approximations of u on (t_{k-1},t_k) that we denote by u_k,v_k. The main question concerns of course the approximation of $\frac{\partial u}{\partial t}$: since we are dealing with implicit schemes, we want to replace $\frac{\partial u}{\partial t}$ by $\frac{1}{\delta}\{u_k-\tilde{u}_{k-1}\}$ and $\frac{\partial v}{\partial t}$ by $\frac{1}{\delta}\{v_k-\tilde{v}_{k-1}\}$ for some $\tilde{u}_{k-1},\tilde{v}_{k-1}$ to be determined. It is quite obvious that various choices are possible : $\tilde{u}_{k-1}=u_{k-1}$, $\tilde{v}_{k-1}=v_{k-1}$; or $\tilde{u}_{k-1}=v_k$, $\tilde{v}_{k-1}=u_{k-1}$ ... We also have to determine the boundary conditions for u_k and v_k like for instance : $u_k=v_k$ on $\partial\mathcal{O}_1$, $v_k=u_{k-1}$ on $\partial\mathcal{O}_2$; or

$$u_k=v_{k-1} \text{ on } \partial\mathcal{O}_1 \quad , \quad v_k=u_{k-1} \text{ on } \partial\mathcal{O}_2$$

(and of course in all cases we extend the functions to $\mathcal{O}$ as usual by the corresponding functions appearing in the boundary conditions). Of course, in all cases, we take u_o as initializations of $u_k,v_k,\tilde{u}_k,\tilde{v}_k$ (for $k=0$). At this stage, we wish to observe that if we choose the simpleminded analogue of the Schwarz method i.e. $\tilde{u}_{k-1}=v_k$, $\tilde{v}_{k-1}=u_{k-1}$ and $u_k=v_k$ on $\partial\mathcal{O}_1$, $v_k=u_{k-1}$ on $\partial\mathcal{O}_2$ i.e. if we solve first at step k

$$\frac{1}{\delta}v_k-\Delta v_k=f_k+\frac{1}{\delta}u_{k-1} \text{ in } \mathcal{O}_2 \quad , \quad v_k=u_{k-1} \text{ on } \partial\mathcal{O}_2 \tag{68}$$

and then

$$\frac{1}{\delta}u_k-\Delta u_k=f_k+\frac{1}{\delta}v_k \text{ in } \mathcal{O}_1 \quad , \quad u_k=v_k \text{ on } \partial\mathcal{O}_1 \tag{69}$$

and if we set $u^N(\cdot,t)=u_{k-1}(\cdot)$ if $t_{k-1}\leq t<t_k$, $v^N(\cdot,t)=v_{k-1}(\cdot)$ if $t_{k-1}\leq t<t_k$ then by the same arguments than the ones we give below we can show that u^N,v^N converge to the solution of

$$\chi\frac{\partial u}{\partial t}-\Delta u=f \text{ in } \mathcal{O}\times(0,T) \quad , \quad u|_{\partial\mathcal{O}\times(0,T)}=0 \quad , \quad u|_{t=0}=u_o \text{ in } \mathcal{O} \tag{70}$$

with $\chi(x)\equiv 1$ in $\mathcal{O}_{11}\cup\mathcal{O}_{22}$, $\equiv\frac{1}{2}$ in $\mathcal{O}_{12}$! This apparent mistake is due to the fact that roughly speaking there are twice more iterations in $\mathcal{O}_{12}$ than in $\mathcal{O}_{11}\cup\mathcal{O}_{22}$. Clearly, this observation shows that some care is necessary. To simplifiy the presentation, we will study only four different methods and it will be clear that many variants are possible. Let us now describe these four methods.

Method 1 : At step k , we solve

(71) $$\frac{1}{\delta} u_k - \Delta u_k = f_k + \frac{1}{\delta} u_{k-1} \quad \text{in } \mathcal{O}_1 \;, \quad u_k = v_{k-1} \quad \text{on } \partial\mathcal{O}_1 \;,$$

(72) $$\frac{1}{\delta} v_k - \Delta v_k = f_k + \frac{1}{\delta} v_{k-1} \quad \text{in } \mathcal{O}_2 \;, \quad v_k = u_{k-1} \quad \text{on } \partial\mathcal{O}_2 \;.$$

Method 2 : At step k , we solve first

(73) $$\frac{1}{\delta} v_k - \Delta v_k = f_k + \frac{1}{\delta} v_{k-1} \quad \text{in } \mathcal{O}_2 \;, \quad v_k = u_{k-1} \quad \text{on } \partial\mathcal{O}_2$$

and then

(74) $$\frac{1}{\delta} u_k - \Delta u_k = f_k + \frac{1}{\delta} u_{k-1} \quad \text{in } \mathcal{O}_1 \;, \quad u_k = v_k \quad \text{on } \partial\mathcal{O}_1 \;.$$

Method 3 : At step k , we solve first

(75) $$\frac{1}{\delta} \chi_2 v_k - \Delta v_k = f_k + \frac{1}{\delta} \chi_2 u_{k-1} \quad \text{in } \mathcal{O}_2 \;, \quad v_k = u_{k-1} \quad \text{on } \partial\mathcal{O}_2$$

and then

(76) $$\frac{1}{\delta} \chi_1 u_k - \Delta u_k = f_k + \frac{1}{\delta} \chi_1 v_k \quad \text{in } \mathcal{O}_1 \;, \quad u_k = v_k \quad \text{on } \partial\mathcal{O}_1$$

where $\chi_1 \equiv 1$ in $\mathcal{O}_{11}$, $\equiv 2$ in $\mathcal{O}_{12}$; $\chi_2 \equiv 1$ in $\mathcal{O}_{22}$, $\equiv 2$ in $\mathcal{O}_{12}$.

Method 4 : At step k , we solve

(77) $$\frac{1}{\delta} \zeta_1 u_k - \Delta u_k = f_k + \frac{1}{\delta} \zeta_1 v_{k-1} \quad \text{in } \mathcal{O}_1 \;, \quad u_k = v_{k-1} \quad \text{on } \partial\mathcal{O}_1$$

(78) $$\frac{1}{\delta} \zeta_2 v_k - \Delta v_k = f_k + \frac{1}{\delta} \zeta_2 u_{k-1} \quad \text{in } \mathcal{O}_2 \;, \quad v_k = u_{k-1} \quad \text{on } \partial\mathcal{O}_2 \;,$$

with $\zeta_1 \equiv \frac{1}{2}$ in $\mathcal{O}_{11}$, $\equiv 1$ in $\mathcal{O}_{12}$; $\zeta_2 \equiv \frac{1}{2}$ in $\mathcal{O}_{22}$, $\equiv 1$ in $\mathcal{O}_{12}$.

Then, in these four examples, we define u^N and v^N as above. Furthermore, in the case of Method 1 we set

$$w^N(\cdot,t) = \frac{1}{\delta} (u_k+v_k) - \frac{1}{\delta} (u_{k-1}+v_{k-1}) \qquad \text{for } t_{k-1} < t < t_k$$

while in the cases of Methods 2,3,4 we define w_1^N , w_2^N on the same interval by respectively : (Method 2) $w_1^N = \frac{1}{\delta}(u_k - u_{k-1})$, $w_2^N = \frac{1}{\delta}(v_k - v_{k-1})$; (Method 3) $w_1^N = \frac{1}{\delta}\chi_1(u_k - v_k)$, $w_2^N = \frac{1}{\delta}\chi_2(v_k - u_{k-1})$; (Method 4) $w_1^N = \frac{1}{\delta}\zeta_1(u_k - v_{k-1})$, $w_2^N = \frac{1}{\delta}\zeta_2(v_k - u_{k-1})$.

And we have the

<u>Theorem I.5</u> : As N goes to $+\infty$, for the four above methods, u^N, v^N converge in $L^\infty(0,T;H_o^1)$ to u . In addition, the following convergences hold in $L^2(0,T;L^2)$: (Method 1) $w^N \underset{N}{\rightharpoonup} \frac{\partial u}{\partial t}$; (Method 2) $w_1^N, w_2^N \underset{N}{\rightharpoonup} \frac{\partial u}{\partial t}$; (Methods 3,4) $w_1^N \underset{N}{\rightharpoonup} 1_{\mathcal{O}_1} \frac{\partial u}{\partial t}$, $w_2^N \underset{N}{\rightharpoonup} 1_{\mathcal{O}_2} \frac{\partial u}{\partial t}$.

<u>Sketch of proof</u> : We first explain how to obtain a priori estimates. In Method 1, we multiply (71) by $(u_k - v_{k-1})$ and (72) by $(v_k - u_{k-1})$ and we integrate by parts. We then obtain

$$(79)\quad \begin{cases} \frac{1}{\delta}\left[\frac{1}{2}|u_k - v_k|^2 - \frac{1}{2}|u_{k-1} - v_{k-1}|^2 + \frac{1}{2}|(u_k + v_k) - (u_{k-1} + v_{k-1})|^2\right] + \\ + \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|v_k\|^2 - \frac{1}{2}\|u_{k-1}\|^2 - \frac{1}{2}\|v_{k-1}\|^2 + \frac{1}{2}\|u_k - v_{k-1}\|^2 \\ + \frac{1}{2}\|v_k - u_{k-1}\|^2 \quad = \quad (f_k , (u_k + v_k) - (u_{k-1} + v_{k-1})) \end{cases}$$

where we denote by (φ,ψ) the scalar product in $L^2(\mathcal{O})$, by $|\varphi|$ the L^2 norm and by $\|\varphi\|$ the H_o^1 norm $(= |\nabla\varphi|)$. And summing from $k = 1$ to any $n \geq 1$ we find by Cauchy-Schwarz inequality

$$\frac{1}{2\delta}|u_n - v_n|^2 + \frac{\delta}{4}\sum_{k=1}^n \left|\frac{1}{\delta}\{(u_k + v_k) - (u_{k-1} + v_{k-1})\}\right|^2 + \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|v_n\|^2$$
$$+ \frac{1}{2}\sum_{k=1}^n \left(\|u_k - v_{k-1}\|^2 + \|v_k - u_{k-1}\|^2\right) \leq \|u_o\|^2 + \delta\sum_{k=1}^n |f_k|^2$$

hence u^N, v^N are bounded in $L^\infty(0,T;H_o^1)$, w^N is bounded in $L^2(0,T;L^2)$.

In the case of Method 2, we multiply (73) by $(v_k - u_{k-1})$ and (74) by $(u_k - v_k)$ and we find

$$
(80)\qquad \begin{cases} \frac{1}{2\delta}\,|v_k-v_{k-1}|^2 - \frac{1}{2\delta}\,|u_{k-1}-v_{k-1}|^2 + \frac{1}{2\delta}\,|v_k-u_{k-1}|^2 + \frac{1}{2}\,\|v_k\|^2 + \\ \quad -\frac{1}{2}\,\|u_{k-1}\|^2 + \frac{1}{2}\,\|v_k-u_{k-1}\|^2 = (f_k\,,\,v_k-u_{k-1}) \quad ; \\ \frac{1}{2\delta}\,|u_k-u_{k-1}|^2 - \frac{1}{2\delta}|v_k-u_{k-1}|^2 + \frac{1}{2\delta}|u_k-v_k|^2 + \frac{1}{2}\,\|u_k\|^2 - \frac{1}{2}\,\|v_k\|^2 + \\ \quad + \frac{1}{2}\,\|u_k-v_k\|^2 = (f_k,u_k-v_k) \quad . \end{cases}
$$

And, summing and using the fact that $f \in L^\infty(0,T\,;H^{-1})$, we deduce easily that u^N, v^N are bounded in $L^\infty(0,T\,;H_o^1)$, w_1^N, w_2^N are bounded in $L^2(0,T\,;L^2)$.

In the case of Method 3, we multiply (75) by (v_k-u_{k-1}) and (76) by (u_k-v_k) and we find

$$
(81)\qquad \begin{cases} \frac{1}{\delta}\,|\chi_2^{1/2}(v_k-u_{k-1})|^2 + \frac{1}{2}\,\|v_k\|^2 - \frac{1}{2}\,\|u_{k-1}\|^2 + \frac{1}{2}\,\|v_k-u_{k-1}\|^2 = (f_k,v_k-u_{k-1}) \\ \frac{1}{\delta}\,|\chi_1^{1/2}\,(u_k-v_k)|^2 + \frac{1}{2}\,\|u_k\|^2 - \frac{1}{2}\,\|v_k\|^2 + \frac{1}{2}\,\|u_k-v_k\|^2 = (f_k\,,\,u_k-v_k) \end{cases}
$$

And summing, we deduce that u^N, v^N are bounded in $L^\infty(0,T\,;H_o^1)$, w_1^N, w_2^N are bounded in $L^2(0,T\,;L^2)$.

Finally, in the case of Method 4, we multiply (77) by (u_k-v_{k-1}) and (78) by (v_k-u_{k-1}) and we find

$$
(82)\qquad \begin{cases} \frac{1}{\delta}\,|\zeta_1^{1/2}(u_k-v_{k-1})|^2 + \frac{1}{2}\,\|u_k\|^2 - \frac{1}{2}\,\|v_{k-1}\|^2 + \frac{1}{2}\,\|u_k-v_{k-1}\|^2 = (f_k,u_k-v_{k-1}) \\ \frac{1}{\delta}\,|\zeta_2^{1/2}(v_k-u_{k-1})|^2 + \frac{1}{2}\,\|v_k\|^2 - \frac{1}{2}\,\|u_{k-1}\|^2 + \frac{1}{2}\,\|v_k-u_{k-1}\|^2 = (f_k,v_k-u_{k-1}) \end{cases}
$$

And summing, we deduce that u^N, v^N are bounded in $L^\infty(0,T\,;H_o^1)$, w_1^N, w_2^N are bounded in $L^2(0,T\,;L^2)$.

The rest of the proof is a standard exercise in evolution problems : one first shows that the weak convergence to the solution holds and then that the convergence is strong because of the above equalities... ∎

It is of course possible to obtain some more precise convergence estimates but we will skip here these technical considerations.

It is also possible to consider different evolution problems such as for instance linear wave equations

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in} \quad \mathcal{O} \times (0,T) \quad , \quad u|_{\partial\mathcal{O} \times (0,T)} = 0 \tag{83}$$

with initial conditions

$$u|_{t=0} = u_o \quad , \quad \frac{\partial u}{\partial t}\Big|_{t=0} = u_1 \quad \text{in} \quad \mathcal{O} \quad . \tag{84}$$

Then, it is possible to combine the Schwarz alternating scheme with time discretizations along the above lines. And again some care is needed in the approximation of $\frac{\partial^2 u}{\partial t^2}$. The analogues of the above methods and results still hold in this case.

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