Poincaré–Steklov's Operators and Domain Decomposition Methods in Finite Dimensional Spaces

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ABSTRACT.

In Section 1, the boundary value problems are formulated in terms of bilinear functionals. The finite element method is then used in Section 2 to approximate these problems. Next, in Section 3 finite dimensional variants of Poincaré-Steklov operators are introduced and their properties are discussed. In Section 4 to 6 iterative domain decomposition methods founded on the properties of the discrete Poincaré-Streklov operators are presented. Finally, in Section 7 algorithms for solving some specific elliptic problems are described and their rate of convergence is analyzed there.

INTRODUCTION

At the present moment, domain decomposition methods have been developed in various directions [1-22]. One of them is based on a theory of special operators - Poincaré-Steklov's operators - and it has been investigated in [7,9,16,17,19,20], where the basic procedure to construct these domain decomposition algorithms has been presented. In this paper we propose and investigate domain decomposition methods for abstract boundary value problems in terms of bilinear forms. Our investigations

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are based again on the use of Poincaré-Steklov's operators. But unlike the above papers our research will be focussing on applications to finite dimensional approximations of the problems.

To approximate the problems we use finite element techniques. For simplicity we will consider only piecewise linear basis functions. Moreover we will study problems corresponding to Dirichlet problems for second order partial differentional equations. The investigations will be carried out in spaces of real functions of two independent variables. But it is easy to see that most of our results are true for problems in which the above restrictions are omitted.

Let us describe the contents of this paper. In Section 1 we introduce functional spaces and formulate the problems in abstract form. In Section 2 we approximate the problems using the finite element method. The Poincaré-Steklov's operators in finite dimensional spaces are introduced and their properties are investigated in Section 3. Then using known iterative processes (minimal residuals method, splitting methods and optimal linear iterative process) we construct some domain decomposition algorithms for equations in terms of Poincaré-Steklov's operators and describe the various steps of their implementation (Sections 4-6). In Section 7 we consider domain decomposition algorithms applied to two concrete elliptic problems and estimate their rates of convergence.

1. NOTATION. FUNCTIONAL SPACES. FORMULATION OF THE PROBLEMS.

Let D be a bounded open set of $\mathbf{R}^2$ with a Lipschitz continuous boundary ∂D and $\bar{D} = D \cup \partial D$, $x = (x_1 , x_2) \in \bar{D}$. Decompose D into two subsets D_1 , D_2 with Lipschitz boundaries ∂D_1 , ∂D_2 respectively; D_1 , D_2 are adjacent along the set γ . Let $\Gamma_i = \partial D_i \setminus \gamma$, mes $(\Gamma_i) \geq 0$, mes $(\gamma) > 0$, $i = 1 , 2$. We denote by P_X the trace operator: $P_X u \equiv u|_X$ ($X = \partial D , \partial D_i ,..., \gamma$; $u = u(x)$).

In the paper we will use the following (real) Hilbert spaces of vector valued functions $u(x) = (u_1(x) ,..., u_N(x))$:

$$\mathbf{L}_2(D):\ (u,v)_{\mathbf{L}_2(D)} = \sum_{i=1}^{N}(u_i\,,\,v_i)_{L_2(D)}\ ,\quad \|u\|_{\mathbf{L}_2(D)} = (u,u)^{1/2}_{\mathbf{L}_2(D)}\ ;$$

$$\mathbf{W}^1_2(D):\ (u,v)_{\mathbf{W}^1_2(D)} = \sum_{i=1}^{N}(u_i\,,\,v_i)_{W^1_2(D)}\ ,\ \|u\|_{\mathbf{W}^1_2(D)} = (u,u)^{1/2}_{\mathbf{W}^1_2(D)}\ ;$$

$$\mathbf{W}^{1/2}_2(\partial D):(u,v)_{\mathbf{W}^{1/2}_2(\partial D)} = \sum_{i=1}^{N}(u_i\,,\,v_i)_{W^{1/2}_2(\partial D)}\ ,\ \|u\|_{\mathbf{W}^{1/2}_2(\partial D)} = (u,u)^{1/2}_{\mathbf{W}^{1/2}_2(\partial D)}\ ;$$

$$\overset{0}{\mathbf{W}}{}^1_2(D) = \{u : u\in\mathbf{W}^1_2(D)\ ,\ P_{\partial D}u \equiv u|_{\partial D} = 0\}\ .$$

The definitions of spaces $\mathbf{L}_2(D_i)$, $\mathbf{W}^1_2(D_i)$, $\mathbf{W}^{1/2}_2(\partial D_i)$, $\overset{0}{\mathbf{W}}{}^1_2(D_i)$ $(i = 1,2)$ are the same as for $\mathbf{L}_2(D)$, $\mathbf{W}^1_2(D)$, $\mathbf{W}^{1/2}_2(\partial D)$, $\overset{0}{\mathbf{W}}{}^1_2(D)$ (with the formal substitution of D with D_i) ;

$$\mathbf{W}^1_{2,0(\Gamma_i)}(D_i) = \left\{u : u\in\mathbf{W}^1_2(D_i)\ ,\ \ u|_{\Gamma_i} = 0\right\}\ ,\ \|u\|_{\mathbf{W}^1_{2,0(\Gamma_i)}(D_i)} = \|u\|_{\mathbf{W}^1_2(D_i)}\ .$$

We introduce the closed subspace $\mathbf{W}^{1/2}_{2,0(\Gamma_i)}(\gamma)$ of $\mathbf{W}^{1/2}_2(\gamma)$, which consists of the restrictions to γ of the traces of the functions $u\in\mathbf{W}^1_{2,0(\Gamma_i)}(D_i)$. Let's $\mathbf{L}_2(D)$ be identified with $\mathbf{L}^*_2(D)$ and let's denote by $(\mathbf{W}^1_2(D))^* \equiv \mathbf{W}^{-1}_2(D)$ the dual space of $\mathbf{W}^1_2(D)$, i.e.

$$\mathbf{W}^{-1}_2(D) \equiv (\mathbf{W}^1_2(D))^* = \left\{u : \|u\|_{\mathbf{W}^{-1}_2(D)} = \sup_{\substack{v\in\mathbf{W}^1_2(D)\\ v\neq 0}} \frac{|(u,v)_{\mathbf{L}_2(D)}|}{\|v\|_{\mathbf{W}^1_2(D)}}\right\}\ .$$

In the same manner we introduce the other dual space $\overset{0}{\mathbf{W}}{}^{-1}_2(D) \equiv (\overset{0}{\mathbf{W}}{}^1_2(D))^*,\ldots,$ $\mathbf{W}^{-1/2}_{2,0(\Gamma_i)}(\gamma) \equiv (\mathbf{W}^{1/2}_{2,0(\Gamma_i)}(\gamma))^*$. In this paper we suppose that the spaces $\mathbf{L}_2(D_i)$, $\mathbf{L}_2(\Gamma_i)$, $\mathbf{L}_2(\Gamma)$, $\mathbf{L}_2(\gamma)$ coincide with their dual spaces. Let $a_D(u,v)$ be a bilinear form over $\mathbf{W}^1_2(D)\times\mathbf{W}^1_2(D)$, for which we have

$$a_D(u,v) = \sum_{i=1}^{2} a_{D_i}(u,v) , \tag{1.1}$$

where $a_{D_i}(u,v)$ is a bilinear form over $\mathbf{W}_2^1(D_i)\times\mathbf{W}_2^1(D_i)$. We suppose that the bilinear form $a_{D_i}(u,v)$ $(i=1,2)$ is $\mathbf{W}_2^1$ - bounded and $\mathbf{W}^1_{2,0(\Gamma_i)}$ - elliptic, i.e. there exist constants C_1 , $C_2>0$ such that

$$\left.\begin{aligned} |a_{D_i}(u,v)| &\le C_1 \|u\|_{\mathbf{W}_2^1(D_i)}\cdot\|v\|_{\mathbf{W}_2^1(D_i)} , && \forall\ u,v\in \mathbf{W}_2^1(D_i) , \\ C_2\|u\|^2_{\mathbf{W}_2^1(D_i)} &\le a_{D_i}(u,u) , && \forall\ u\in\mathbf{W}^1_{2,0(\Gamma_i)}(D_i) . \end{aligned}\right\} \tag{1.2}$$

Now, we consider the following problem: *Given* $(f,\ \varphi_{(\Gamma)})\in\overset{0}{\mathbf{W}}{}_2^{-1}(D)\times\mathbf{W}_2^{1/2}(\partial D)$ *find* $\varphi\in\mathbf{W}_2^1(D)$ *such that*

$$\left.\begin{aligned} &a_D(\varphi,v) = (f,v)_{\mathbf{L}_2(D)} , \qquad \forall v\in\overset{0}{\mathbf{W}}{}_2^1(D) , \\ &\varphi|_{\partial D} = \varphi_{(\Gamma)} . \end{aligned}\right\} \tag{1.3}$$

It is easy to see that under the assumptions (1.2), problem (1.3) has a unique solution $\varphi\in\mathbf{W}_2^1(D)$ which satisfies also

$$\|\varphi\|_{\mathbf{W}_2^1(D)} \le C\ [\ \|f\|_{\overset{0}{\mathbf{W}}{}_2^{-1}(D)} + \|\varphi_{(\Gamma)}\|_{\mathbf{W}_2^{1/2}(\partial D)}\ ] . \tag{1.4}$$

In the following we will formulate algorithms for solving problem (1.3). The first step of these algorithms consists in constructing finite-dimensional approximation of problem (1.3).

2. FINITE-DIMENSIONAL APPROXIMATION OF PROBLEM (1.3).

To approximate problem (2.3) we use a finite element method. To simplify the procedure for constructing basis functions let D_1 , D_2 be polygons of $\mathbf{R}^2$. Set on $\bar{D}_i = D_i \cup \partial D_i$ $(i=1,2)$ a triangular mesh. We suppose that the meshes on D_1 , D_2 have common nodes on γ and that the corner points of the boundary ∂D_i $(i=1,2)$ coincide with some mesh points of $\bar{D}_i$. As a result we have a triangulation of $\bar{D}$.

Now to each node $x_j = (x_{1,j}, x_{2,j})$ of the mesh on $\bar{D} = D \cup \partial D$ we associate a function $\omega_j(x)$ which is a linear in x on each triangle and satisfies $\omega_j(x_j)=1$, $\omega_j(x_i)=0$, $i \neq j$. Let $\mathcal{N}$ be the number of these functions. We denote by $W_2^{1,h}(D) \subset (W_2^1(D) \cap C(\bar{D}))$ the space generated by $\{\omega_j\}_{j=1}^{\mathcal{N}}$ and let $W_2^{1,h}(D_i)$ be the set of the restrictions to D_i of the functions $u^h \in W_2^{1,h}(D)$. It is obvious that $W_2^1(D_i) \subset (W_2^1(D_i) \cap C(\bar{D}_i))$. Next, let's $W_2^{1/2,h}(\partial D_i)$, $W_2^{1/2,h}(\Gamma_i)$, $W_2^{1/2,h}(\partial D_i)$ be the restrictions of $W_2^{1,h}(D_i)$ to γ , Γ_i , ∂D_i respectively. Let us observe that when we restrict $W_2^{1,h}(D_1)$, $W_2^{1,h}(D_2)$ to γ we have the same subspace $W_2^{1/2,h}(\gamma) \subset W_2^{1/2}(\gamma)$.

We introduce also the following subspaces

$$W^{1,h}_{2,0(\Gamma_i)}(D_i) = \{u^h : u^h \in W_2^{1,h}(D_i) \ , \ u^h|_{\Gamma_i} = 0\} \ ;$$

$$W^{1/2,h}_{2,0}(\gamma) \equiv W^{1/2,h}_{2,0(\Gamma_i)}(\gamma) = \{w^h : w^h = P_\gamma u^h \ , \ u^h \in W^{1,h}_{2,0(\Gamma_i)}(D_i)\} \ .$$

The basis functions in $W^{1/2,h}_{2,0(\Gamma_i)}(\gamma)$ are denoted by $\{\omega_j^{(\gamma)}(x)\}_{j=1}^{M}$ (where $\omega_j^{(\gamma)}(x) = P_\gamma \omega_j$ if $x_j \in \gamma$ and $x_j \notin \partial D$).
With the finite-dimensional spaces $W_2^{1,h}(D)$,..., $W_{2,0}^{1/2,h}(\gamma)$ we construct in the usual manner the corresponding spaces of vector-functions $\mathbf{W}_2^{1,h}(D)$,..., $\mathbf{W}_{2,0}^{1/2,h}(\gamma)$.
Finally, in the following we will suppose that

$$\varphi_{\Gamma,i} \equiv \varphi_{(\Gamma)}|_{\Gamma_i} \in \mathbf{W}_2^{1/2,h}(\Gamma_i)$$

(i.e. $\varphi_{\Gamma,i}$ is a piecewise linear function over Γ_i, $i=1,2$). (Observe that we impose the last restriction only to simplify the anlysis in this paper). Now, with all the above assumptions we will approximate problem (2.3).

We construct the approximate solution of the problem in the form

$$\varphi^h = \sum_{i=1}^{\mathcal{N}} a_i \omega_i(x) , \quad a_i = (a_{1,i} ,\dots, a_{\mathcal{N},i})^T , \tag{2.1}$$

where the coefficients $\{a_i\}_{i=1}^{\mathcal{N}}$ are determined by the following system

$$a_D(\varphi^h, \omega_{kj}) = (f,\omega_{kj})_{\mathbf{L}_2(D)}, \quad k=1 ,\dots, N , \quad j=1,2,\dots, \mathcal{N}' \tag{2.2}$$

$$\varphi^h|_{\Gamma_i} = \varphi_{\Gamma,i} , \ i=1,2 , \tag{2.3}$$

where $\omega_{kj}(x)=e_k\omega_j(x)$, $e_k=(0,\dots, 0,\underset{k}{1}, 0)^T$, $\mathcal{N}'$ is the number of internal nodes of the mesh on D, and the index j in (2.2) takes the values corresponding to these nodes. Conditions (1.2) guarantee the existence and the uniqueness of the solution φ^h of (2.2), (2.3). The above process for constructing the solution φ^h has been fully investigated in the scientific literature and the rate of convergence of φ^h to φ has been estimated.

We transform now problem (2.2), (2.3). Let us introduce the function

$$\varphi_{(\gamma)} \in \mathbf{W}_2^{1/2,h}(\gamma) , \quad \text{for which we have}$$

$$\varphi^h_{\Gamma,i} = \{\varphi_{(\gamma)}, x\in\gamma; \varphi_{\Gamma,i} , x\in\Gamma_i\} \in \mathbf{W}_2^{1/2,h}(\partial D_i) .$$

Consider the two auxiliary problems

$$a_{D_i}(\varphi_i^{(o),h}, \omega_{kj}) = (f, \omega_{kj})_{L_2(D_i)}, \quad k=1,\dots,N, \quad j=1,\dots, \mathcal{N}_i, \tag{2.4}$$

$$\varphi_i^{(o),h} = \varphi_{\Gamma,i}, \quad x \in \Gamma_i, \tag{2.5}$$

$$\varphi_i^{(o),h} = \varphi_{(\gamma)}, \quad x \in \gamma, \quad i=1,2 \qquad (\varphi_i^{(o),h} \in \mathbf{W}_2^{1,h}(D_i)), \tag{2.6}$$

where the index j in (2.4) takes the values corresponding to the internal nodes in D_i and let $\mathcal{N}_i$ be the number of these nodes. Each of the problems (2.4) - (2.6) has a unique solution.

Let us consider an arbitrary function $\psi^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D) = \{\psi^h \in \mathbf{W}_2^{1,h}(D), \psi^h|_{\partial D} = 0\}$ and represent it in the form:

$$\psi^h = \psi_1^h + \psi_2^h + \psi_\gamma^h, \tag{2.7}$$

where $\psi_i^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D_i) = \{\psi_i^h \in \mathbf{W}_2^{1,h}(D), \psi_i^h|_{D \setminus D_i} = 0\}$. Here $\psi_\gamma^h = \psi^h - \psi_1^h - \psi_2^h$ is a function of the subspace $\overset{0}{\mathbf{W}}{}_2^{1,h}(\gamma) = (\overset{0}{\mathbf{W}}{}_2^{1,h}(D) - \overset{0}{\mathbf{W}}{}_2^{1,h}(D_1) - \overset{0}{\mathbf{W}}{}_2^{1,h}(D_2)) \subset$

$\subset \overset{0}{\mathbf{W}}{}_2^{1,h}(D)$. Then problem (2.2), (2.3) may be rewritten as follows:

$$\left.\begin{aligned} &a_D(\varphi^h, \psi^h) = (f, \psi^h)_{L_2(D)}, \quad \forall \psi^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D), \\ &\varphi^h|_{\Gamma_i} = \varphi_{\Gamma,i}, \quad i=1,2, \end{aligned}\right\} \tag{2.8}$$

or

$$\left.\begin{aligned}&\sum_{i=1}^{2} a_{D_i}(\varphi^h, \psi_i^h) + a_D(\varphi^h, \psi_\gamma^h) = \sum_{i=1}^{2} (f, \psi_i^h)_{\mathbf{L}_2(D_i)} + (f, \psi_\gamma^h)_{\mathbf{L}_2(D)} , \\ &\varphi|_{\Gamma_i} = \varphi_{\Gamma,i} \quad , \quad i=1,2.\end{aligned}\right\} \quad (2.9)$$

Now, (2.4) - (2.6) implies

$$\left.\begin{aligned}&a_{D_i} (\varphi_i^{(o),h} , \psi_i^h) = (f, \psi_i^h)_{\mathbf{L}_2(D_i)} , \ \forall \psi_i^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D_i) , \\ &\varphi_i^{(o),h}|_{\Gamma_i} = \varphi_{\Gamma,i} , \\ &\varphi_i^{(o),h}|_{(\gamma)} = \varphi_{(\gamma)} \quad , \quad i=1,2,\end{aligned}\right\} \quad (2.10)$$

and as a consequence of (2.9) and (2.10) for the functions

$$\left.\begin{aligned}&U_i^h = \varphi_i^h - \varphi_i^{(o),h} \quad , \quad i=1,2 \\ &(\varphi_i^h \equiv \varphi^h \quad \text{on} \quad D_i)\end{aligned}\right\} \quad (2.11)$$

we obtain the following equations

$$\left.\begin{aligned}&\sum_{i=1}^{2} a_{D_i} (U_i^h , \psi_i^h + \psi_\gamma^h) = g(\psi_\gamma^h) , \\ &U_i^h|_{\Gamma_i} = 0 , \\ &U_i^h|_\gamma = \underset{\sim}{\varphi}^h - \varphi_{(\gamma)} , \quad i=1,2,\end{aligned}\right\} \quad (2.12)$$

where

$$\widetilde{\varphi}^h \equiv \varphi_1^h|_\gamma - \varphi_2^\gamma|_\gamma ,$$

$$g(\psi_\gamma^h) = (f,\psi_\gamma^h)_{\mathbf{L}_2(D)} - \sum_{i=1}^{2} a_{D_i}(\varphi_i^{(o),h} , \psi_\gamma^h) ,$$

$$(\psi_i^h+\psi_\gamma^h) \in \mathbf{W}^{1,h}_{2,0(\Gamma_i)} (D_i) = \{u^h: u^h \in \mathbf{W}_2^{1,h}(D_i) , u^h|_{\Gamma_i} = 0\} .$$

If in (2.12) we take $\psi^h_\gamma = 0$, $\psi^h_1 = 0$ or $\psi^h_\gamma = 0$, $\psi^h_2 = 0$ then we obtain

$$\left.\begin{aligned} & a_{D_i}(U^h_i , \psi^h_i) = 0 , \quad \forall\ \psi^h_i \in \overset{0}{W}{}^{1,h}_2(D_i) , \\ & U^h_i|_{\Gamma_i} = 0 , \\ & U^h_i|_\gamma = \widetilde{\varphi}^h - \varphi_{(\gamma)} , \quad i=1,2 . \end{aligned}\right\} \qquad (2.13)$$

On the other hand if we know the function

$$\widetilde{U}^h \equiv \widetilde{\varphi}^h - \varphi_{(\gamma)} ,$$

then we can construct U^h_i over D_i according to (2.13) and as a result we have the solution φ^h_i of the problem (2.2) - (2.3) :

$$\varphi^h_i = U^h_i + \varphi^{(o),h}_i , \quad i=1,2 .$$

Therefore the iterative methods discussed in the next sections of this paper are intended mainly for calculating the functions $\{U^h_i\}$.

The formulation of these methods will be given in terms of the Poincaré-Steklov's operators introduced in the following section.

3. POINCARÉ-STEKLOV'S OPERATORS IN FINITE DIMENSION

Consider the following problem: *Given* $g^h \in \mathbf{W}^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)$, *find* $V^h_1 \in$ $\in \mathbf{W}^{1,h}_{2,o(\Gamma_1)}(D_1)$ *such that*

$$a_{D_1}(V^h_1 , \psi^h) = \int_\gamma g^h \psi^h d\Gamma , \ \forall\ \psi^h \in \mathbf{W}^{1,h}_{2,o(\Gamma_1)}(D_1) . \qquad (3.1)$$

Here $\mathbf{W}^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)$ is the dual space of $\mathbf{W}^{1/2,h}_{2,o(\Gamma_1)}$. We equip $\mathbf{W}^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)$ with the norm

$$\|g^h\|_{W^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)} = \sup_{w^h \in W^{1/2,h}_{2,o(\Gamma_1)}(\gamma)} \frac{|(g^h, w^h)_{L_2(\gamma)}|}{\|w^h\|_{W^{1/2,h}_{2,o(\Gamma_1)}(\gamma)}} ,$$

where

$$\|w^h\|_{W^{1/2,h}_{2,o(\Gamma_1)}(\gamma)} = \inf_{\substack{\varphi \in W^1_{2,o(\Gamma_1)}(D_1) \\ \varphi|_\gamma = w^h}} \|\varphi^h\|_{W^{1/2}_{2,o(\Gamma_1)}(\partial D_1)} .$$

Under the assumptions (1.2) problem (3.2) has a unique solution

$V_1^h \in W^{1,h}_{2,o(\Gamma_1)}(D_1)$ and

$$\| V_1^h\|_{W^1_2(D_1)} \leq C \|g^h\|_{W^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)} , \tag{3.2}$$

where the constant C does not depend of V_1^h, g^h and h. The function V_1^h has the unique trace $V_1^h|_{\partial D_1} = P_{\partial D_1} V_1^h \in W^{1/2}_{2,0}$ and

$$\|V_1^h|_{W^{1/2}_{2,o(\Gamma_1)}(\partial D_1)} \leq \widetilde{C}\|V_1^h\|_{W^1_2(D_1)} \leq C \|g^h\|_{W^{-1/2,h}_{2,o(\Gamma_1)}(\gamma)} \tag{3.3}$$

with the two constants $\widetilde{C}$, $C>0$ independent of V_1^h, g^h and h. We have therefore

$$\|\widetilde{V}_1^h\|_{W^{1/2,h}_{2,o(\Gamma_1)}(\gamma)} \leq \|V_1^h\|_{W^{1/2}_{2,o(\Gamma_1)}(\partial D_1)} \leq C\|g^h\|_{W^{-1/2}_{2,o(\Gamma_1)}(\gamma)} , \tag{3.4}$$

where $\widetilde{V}_1^h \equiv V_1^h|_\gamma$ is the restriction to γ of the trace $V_1^h|_{\partial D_1}$. The mapping $g^h \to \widetilde{V}_1^h$ defines the Poincaré-Steklov's operator S_1^h, i.e.

$$S_1^h g^h = \widetilde{V}_1^h \ , \qquad S_1^h : \mathbf{W}^{-1/2,h}_{2,0(\Gamma_1)}(\gamma) \to \mathbf{W}^{1/2,h}_{2,0(\Gamma_1)}(\gamma) \ .$$

We have then:

<u>**Theorem 3.1:**</u> *The following statements hold:*

(*i*) S_1^h *is bounded and the norm* $\|S_1^h\|$ *may be estimated by a constant independent of* h .

(ii) *If the form* $a_{D_1}(V^h, W^h)$ *is symmetric then the operator* S_1^h *is also symmetric.*

(iii) S_1^h *is a compact operator in* $\mathbf{L}_2(\gamma)$.

(iv) S_1^h *is a positive operator in* $\mathbf{L}_2(\gamma)$.

(v) *There exists an inverse operator* $(S_1^h)^{-1} : \mathbf{W}^{1/2,h}_{2,0(\Gamma_1)} \to \mathbf{W}^{-1/2,h}_{2,0(\Gamma_1)}(\gamma)$.

<u>**Proof:**</u>

(i) As a consequence of (3.4) we have

$$\|S_1^h\|_{\mathbf{W}^{-1/2,h}_{2,0(\Gamma_1)}(\gamma) \to \mathbf{W}^{1/2,h}_{2,0(\Gamma_1)}(\gamma)} \leq C = \text{const.} < \infty \ .$$

(ii) It follows from (3.1) that

$$\int_\gamma S_1^h g^h \cdot g^h d\Gamma = \int_\gamma g^h \cdot V_1^h d\Gamma = a_{D_1}(V_1^h \ , \ V_1^h) \geq$$

$$\geq C\|V_1^h\|^2_{\mathbf{W}^1_2(D_1)} \geq C\|V_1^h\|^2_{\mathbf{W}^{1/2}_2(\partial D_1)} \geq C \ \|\widetilde{V}_1^h\|^2_{\mathbf{L}_2(\gamma)} \geq 0 \ .$$

If $\int_\gamma S_1^h g^h \cdot g^h d\Gamma = 0$ then $\widetilde{V}_1^h = 0$ and from (3.1) we have

$$a_{D_1}(V_1^h \ , \ \psi^h) = 0 \ , \quad \forall \ \psi^h \in \overset{0}{\mathbf{W}}{}^{1,h}_2 \ (D_1) \ ,$$

$$a_{D_1}(V_1^h \ , \ V_1^h) = 0 \ ,$$

$$\int_\gamma g^h \psi^h d\Gamma = 0 \quad , \quad \forall \quad \psi^h \in \mathbf{W}^{1/2,h}_{2,0(\Gamma_1)}(\gamma) \quad ,$$

$$g^h = 0 \ .$$

Therefore the operator S_1^h is positive and the inverse operator $(S_1^h)^{-1}$ exists.

(iii) Let V_1^h , $\bar{V}_1^h$ be two solutions of problem (3.1) corresponding to some functions $\mathbf{g}^h$, $\bar{\mathbf{g}}^h$. Then if the form $a_{D_1}(V^h, w^h)$ is symmetric we have

$$\int_\gamma (S_1^h)^{-1}\, \widetilde{V}^h . \widetilde{\widetilde{V}}^h d\Gamma = a_{D_1}(V_1^h, \bar{V}_1^h) = a_{D_1}(\bar{V}_1^h, V_1^h)$$

$$= \int_\gamma (S^h)^{-1}\, \widetilde{\widetilde{V}}^h . \widetilde{V}^h d\Gamma = \int_\gamma \widetilde{V}^h . (S_1^h)^{-1}\, \widetilde{\widetilde{V}}^h d\Gamma \,,$$

i.e. the operators $(S_1^h)^{-1}$, S_1^h are also symmetric.

(iv) Using the following relations

$$\|S_1^h \mathbf{g}^h\|^2_{\mathbf{W}^{1/2}_{2,0(\Gamma_1)}(\gamma)} = \|V_1^h\|_{\mathbf{W}^{1/2}_{2,0(\Gamma_1)}(\gamma)} \le C\|V_1^h\|_{\mathbf{W}_2^1(D_1)} \le$$

$$\le C\|\mathbf{g}^h\|^2_{\mathbf{W}^{-1/2,h}_{2,0(\Gamma_1)}(\gamma)} \le C\|\mathbf{g}^h\|^2_{\mathbf{L}_2(\gamma)}$$

and the property of a compactness of the imbedding operator of $\mathbf{W}^{1/2}_{2,0(\Gamma_1)}(\gamma)$ into $\mathbf{L}_2(\gamma)$ we have the compactness of the operator S_1^h , acting in $\mathbf{L}_2(\gamma)$. Observe that the compactness of S_1^h is true with an arbitrary number of mesh nodes and it does not depend on mesh parameters. □

<u>**Lemma 3.1:**</u> *The operator* $(S_1^h)^{-1}$ *is positive definite in* $\mathbf{L}_2(\gamma)$, *i.e.*

$$((S^h)^{-1}\, \widetilde{V}_1^h \,,\, \widetilde{V}_1^h)_{\mathbf{L}_2(\gamma)} \ge C\, \|\widetilde{V}_1^h\|^2_{\mathbf{L}_2(\gamma)} \tag{3.5}$$

with the constant $C>0$ *independent of* $\widetilde{V}_1^h$ *and of* h .

<u>**Proof:**</u> Let $\widetilde{V}_1^h$ be a function of $\mathbf{W}^{1/2,h}_{2,0(\Gamma_1)}(\gamma)$. Set $\mathbf{g}^h \equiv (S^h)^{-1}\, \widetilde{V}_1^h$. Then from (3.1) we have

$$((S_1^h)^{-1}\widetilde{V}_1^h \, , \, \widetilde{V}_1^h)_{\mathbf{L}_2(\gamma)} = (g^h \, , \, \widetilde{V}_1^h)_{\mathbf{L}_2(\gamma)} = a_{D_1}(V_1^h \, , \, V_1^h) \geq$$

$$\geq C \cdot \|\widetilde{V}_1^h\|^2_{\mathbf{W}^{1,2}_{2,0(\Gamma_1)}(\partial D_1)} \geq C_1 \|\widetilde{V}_1^h\|^2_{\mathbf{L}_2(\partial D_1)} \geq C_2 \|\widetilde{V}_1^h\|^2_{\mathbf{L}_2(\gamma)} \quad ,$$

where the constants C_i , $C>0$ do not depend of $\widetilde{V}_1^h$ and h (they are determined only from imbedding theorems and the relations (1.2)) . □

<u>**Lemma 3.2:**</u> *Let* (e^h, λ^h) *be a solution of the eignvalue problem* $S_1^h e^h = \lambda^h e^h$. *Then the function* V^h *defined by*

$$a_{D_1}(V^h, \psi^h) = 0 \quad , \quad \forall \quad \psi^h \in \overset{0}{\mathbf{W}}{}_2^{1,h} \, (D_1) \tag{3.6$_1$}$$

$$V^h|_\gamma = e^h \tag{3.6$_2$}$$

satisfies the equaton

$$a_{D_1}(V^h \, , \, \psi^h) \equiv \frac{1}{\lambda^h} \int_\gamma V^h \psi^h d\Gamma \, , \quad \forall \ \psi^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_1)}(D_1) \quad . \tag{3.7}$$

<u>Proof:</u> Set $g^h \equiv (S_1^h)^{-1} e^h$ and consider (3.1) . Then we have

$$a_{D_1}(V^h \, , \, \psi^h) = \int_\gamma g^h \psi^h d\Gamma = \int_\gamma (S_1^h)^{-1} e^h \psi^h d\Gamma = \frac{1}{\lambda^h} \int_\gamma e^h \psi^h d\Gamma.$$

But the function $S_1^h g^h = \widetilde{V}^h$ is equal to the eigenfunction e^h . Therefore the equations (3.6) , (3.7) are valid. □

<u>**Lemma 3.3:**</u> *Assume that the bilinear form* $a_{D_1}(V^h \, , \, w^h)$ *is symmetric and the eignefunctions* $\{e_k^h\}$ *of* S_1^h *are normalized as* $\|e_k^h\|_{\mathbf{L}_2(\gamma)} = 1$.
Then the solution of the following problem

$$\left.\begin{aligned} & a_{D_1}(V^h, \psi^h) = 0, \quad V^h \in \mathbf{W}^1_{2,0(\Gamma_1)}(D_1), \quad \forall\ \psi^h \in \overset{0}{\mathbf{W}}{}^{1,h}_2(D_1), \\ & V^h|_\gamma = w^h \in \mathbf{W}^{1/2,h}_{2,0(\Gamma_1)}(\gamma) \end{aligned}\right\} \quad (3.8)$$

has the form

$$V^h = \sum_k w^h_k V^h_k, \quad w^h_k = (w^h, e^h_k)_{\mathbf{L}_2(\gamma)}, \qquad (3.9)$$

where $\{V^h_k\}$ *are the solutions of* (3.6) *with* $e^h = e^h_k$, $k=1,2,\dots$.

(To prove lemma 3.3 it is enough to show the expansions (3.9) satisfies (3.8).)

By analogy with the above consideration we can consider the problem

$$a_{D_2}(V^h_2, \psi^h) = \int_\gamma g^h \psi^h d\Gamma, \quad \forall\ \psi^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_2)}(D_2) \qquad (3.10)$$

and introduce the operator S^h_2, which has properties, analogous to those of the operator S^h_1.

Now let us use the operators $\{S^h_i\}$ to rewrite the problem (2.12) in equivalent forms.

By definitions of S^h_i, $i=1,2$, we can represent the bilinear forms

$$a_{D_i}(U^h_i, \psi^h_i + \psi^h_\gamma) \text{ in (2.12) as}$$

$$a_{D_i}(U^h_i, \psi^h_i + \psi^h_\gamma) = \int_\gamma ((S^h_i)^{-1}\widetilde{U}^h_i)\cdot(\psi^h_i + \psi^h_\gamma)\, d\Gamma = \int_\gamma ((S^h_i)^{-1}\widetilde{U}^h)\, \psi^h_\gamma d\Gamma,$$

where $U^h_i|_\gamma = \widetilde{\varphi}^h - \varphi_{(\gamma)} \equiv \widetilde{U}^h$. Therfore the first equivalent form of the problem (2.12) is the following one:

$$\int_\gamma \mathcal{A}^h \widetilde{U}^h \cdot \psi_\gamma^h d\Gamma = g(\psi_\gamma^h) , \quad \mathcal{A}^h = \sum_{i=1}^{2} (S_i^h)^{-1} . \tag{3.11}$$

Represent $\widetilde{U}^h$, ψ_γ^h by the expansions

$$\left.\begin{aligned} \widetilde{U}^h &= \sum_{k=1}^{N} \sum_{i=1}^{M} b_{ki}\omega_{ki}^{(\gamma)} (x) , \quad \omega_{ki}^{(\gamma)} = e_k \omega_i^{(\gamma)} , \\ \psi_\gamma^h &= \sum_{k=1}^{N} \sum_{i=1}^{M} C_{ki}\omega_{ki}^{(\gamma)}(x) , \end{aligned}\right\} \tag{3.12}$$

and substitute them into (3.11). Then taking into account the arbitrariness of the coefficients $\{C_{ki}\}$ we have the following system of linear algebraic equations

$$\hat{\mathcal{A}}^h \vec{b} = \vec{g} , \tag{3.13}$$

where

$$\vec{b} = (b_1, \ldots , b_N)^T , \quad b_k = (b_{k1}, \ldots, b_{kM})^T ,$$

$$\vec{g} = (g_1, \ldots , g_N)^T , \quad g_k = (g_{k1}, \ldots, g_{kM})^T ,$$

$$g_{ki} = g(\omega_{k_i}^{(\gamma)}) = (f , \omega_{k_i}^{(\gamma)})_{L_2(D)} - \sum_{i=1}^{2} a_{D_j}(\varphi_j^{(0),h} , \omega_{k_i}^{(\gamma)}) ,$$

$$\hat{\mathcal{A}}^h = (\mathcal{A}^h_{ki,k'i'}) , \quad \mathcal{A}^h_{ki,k'i'} = \int_\gamma \mathcal{A}^h \omega_{k'i'}^{(\gamma)} \cdot \omega_{ki}^{(\gamma)} d\Gamma .$$

Therefore if φ^h is the solution of the problem (2.2) , (2.3) then the vector $\vec{b}$ satisfies the equaton (3.13) . On the contrary if $\vec{b}$ is the solution of (3.13) then the function $\varphi_i^h = U_i^h + \varphi_i^{(o),h}$, where U_i^h is the solution of (2.13) with $U_i^h|_\gamma = \underset{\sim}{U}^h \equiv \sum_{k,i} b_{ki}\omega_{ki}^{(\gamma)}(x)|_\gamma$, coincides on D_i with the function φ_h.

As a consequence of the properties of operators $(S_i^h)^{-1}$, i=1,2, we have the following

Lemma 3.4: *The matrix* $\hat{\mathcal{A}}^h$ *is positive definite in* $\mathbf{R}^{N\text{-}M}$ *and it has the form* $\hat{\mathcal{A}}^h = \hat{\mathcal{A}}_1^h + \hat{\mathcal{A}}_2^h$, *where the matrix* $\hat{\mathcal{A}}_l^h$ $(l=1,2)$ *is also positive definite in* $\mathbf{R}^{N\text{-}M}$ *and its elements are as follows:*

$$\mathcal{A}_{l,ki,k'i'}^h = \int_\gamma ((S_l^h)^{-1}\omega_{k'i'}^{(\gamma)})\cdot\omega_{ki}^{(\gamma)}\, d\Gamma\ ,\quad l=1,2,\ \ k,k'=1,\ldots,N,\ i,i'=1,\ldots,M.$$

To formulate the next equivalent form of (3.11) we introduce the orthogonal projection operator over the span of $\{\omega_{kj}^{(\gamma)}\}$ defined by

$$P_\omega u = \sum_{k=1}^{N}\sum_{j=1}^{M}\left[\sum_{k'=1}^{N}\sum_{j'=1}^{M}(M_\omega^{-1})_{kj,k'j'}\ (u,\omega_{k'j'}^{(\gamma)})_{\mathbf{L}_2(\gamma)}\right]\omega_{kj}^{(\gamma)}(x)\ , \tag{3.14}$$

where $M_\omega = (M_{kj,k'j'})$ is the *Gramm matrix* with the elements $M_{kj,k'j'} = (\omega_{kj}^{(\gamma)}, \omega_{k'j'}^{(\gamma)})_{\mathbf{L}_2(\gamma)}$. Now, using operator P_ω we can rewrite equation (3.11) as follows

$$P_\omega\mathcal{A}^h\widetilde{U}^h = \bar{F}^h\ , \tag{3.15}$$

where

$$F^h = \sum_{k=1}^{N}\sum_{j=1}^{M}\left[\sum_{k'=1}^{N}\sum_{j'=1}^{M}(M_\omega^{-1})_{kj,k'j'}\ g(\omega_{k'j'}^{(\gamma)})\right]\omega_{kj}^{(\gamma)} \in \mathbf{W}_{2,0(\Gamma_i)}^{1/2,h}(\gamma)\ .$$

Observe that equation (3.15) is valid for any point $x \in \gamma$. Besides we can consider operator $P_\omega \mathcal{A}^h$ as an operator on $\mathbf{L}_2(\gamma)$ with $D(P_\omega \mathcal{A}^h) = \mathbf{W}^{1/2,h}_{2,0(\Gamma_i)}(\gamma)$ as domain of definition. Some properties of this operator follows from the properties of $\{S_i^h\}$ and from

$$(P_\omega \mathcal{A}^h \widetilde{U}^h , \widetilde{V}^h)_{\mathbf{L}_2(\gamma)} = (\mathcal{A}^h \widetilde{U}^h , P_\omega \widetilde{V}^h)_{\mathbf{L}_2(\gamma)} = (\mathcal{A}^h \widetilde{U}^h , \widetilde{V}^h)_{\mathbf{L}_2(\gamma)} \cdot \quad (3.16)$$

In particular, using (3.16) it is easy to show that the operators $P_\omega (S_i^h)^{-1}$, $P_\omega \mathcal{A}^h$ are positive definite and if the bilinear forms $\{a_{D_i}(V_i^h , W_i^h)\}$ are symmetric then the operators are also symmetric.

So we have obtained some equivalent forms of the equation (3.11). In the next sections we will use them to formulate the iterative algorithms. In conclusion of this section observe that in the following we will frequently calculate the values $((S_k^h)^{-1}\widetilde{U}_k^h , \psi_k^h)_{\mathbf{L}_2(\gamma)}$ with $\widetilde{U}_k^h \in \mathbf{W}^{1/2,h}_{2,0(\Gamma_k)}$, $\psi_k^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_k)}(D_k)$ given. To find them, it is enough to solve the problem

$$\left.\begin{aligned} & a_{D_k}(U_k^h , \bar{\psi}_k^h) = 0 , \quad \forall \ \bar{\psi}_k^h \in \overset{0}{\mathbf{W}}{}_2^{1,h} (D_k) , \\ & U_k^h|_\gamma = \widetilde{U}_k^h , \qquad U_k^h|_{\Gamma_i} = 0 \quad (U_k^h \in \mathbf{W}_2^{1,h} (D_k)) \end{aligned}\right\} \quad (3.17)$$

and use

$$a_{D_k}(U_k^h, \psi_k^h) = \int_\gamma (S_k^h)^{-1}\widetilde{U}_k^h \cdot \psi_k^h d\Gamma , \quad \psi_k^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_k)}(D_k) . \quad (3.18)$$

If the values

$$\alpha_{l,kj} = a_{D_l}(U_l^h , \omega_{kj}^{(\gamma)}) \quad (3.19)$$

have been also calculated then constructing the function $P_\omega (S_l^h)^{-1}\widetilde{U}^h$ may be done by

$$P_\omega(S_l^h)^{-1}\widetilde{U}^h = \sum_{k=1}^{N}\sum_{j=1}^{M}\left[\sum_{k'=1}^{N}\sum_{j'=1}^{M}(M_\omega^{-1})_{kj,k'j'}\,\alpha_{lk'j'}\right]\omega_{kj}^{(\gamma)}\,. \tag{3.20}$$

From the other hand let the function $g^h = (S_k^h)^{-1}\widetilde{U}^h$ be known and suppose that we have to compute the function $\widetilde{U}^h \in W^{1/2,h}_{2,0(\Gamma_k)}(\gamma)$. To find it, we may solve the problem

$$a_{D_k}(U_k^h, \psi_k^h) = \int_\gamma g^h\psi_k^h d\Gamma\,, \quad \forall\ \psi_k^h \in W^{1,h}_{2,0(\Gamma_k)}(D_k) \tag{3.21}$$

and then set

$$\widetilde{U}^h = U_k^h|_\gamma\,. \tag{3.22}$$

The above methods for computing $(S_k^h)^{-1}\widetilde{U}^h$ or $\widetilde{U}^h$ will be used frequently in the following.

4. DOMAIN DECOMPOSITION ALGORITHM BASED ON THE MINIMAL RESIDUALS METHOD.

The first domain decomposition algorithm will be formulated as a solution methods for system (3.13) . Let us use the minimal residuals method to solve this system:

$$\left.\begin{aligned}
&\vec{\xi} = \hat{\mathcal{A}}^h\vec{b} - \vec{g}\,,\\
&\tau_j = (\hat{\mathcal{A}}^h\vec{\xi}\,, \vec{\xi})_2/\|\hat{\mathcal{A}}^h\vec{\xi}^j\|_2^2\,,\\
&\vec{b}^{j+1} = \vec{b}^j - \tau_j\vec{\xi}^j\,,\quad j = 0,1,\dots,
\end{aligned}\right\} \tag{4.1}$$

where $\vec{\xi}^0 = -\vec{g}$ and $\vec{b}^0 \equiv \vec{0}$, $(\vec{a}\,, \vec{b})_2 \equiv \sum_k a_k b_k$, or

$$\left.\begin{aligned}
&\vec{p}^j = \hat{\mathcal{A}}^h\vec{\xi}^j\,,\\
&\tau_j = (\vec{p}^j, \vec{\xi}^j)_2/\|\vec{p}^j\|_2^2\,,\\
&\vec{\xi}^{j+1} = \vec{\xi}^j - \tau_j\,\vec{p}^j\,,\quad j=0,1,2,\dots\,.
\end{aligned}\right\} \tag{4.2}$$

Here we have

$$\vec{b}^{j+1} = \vec{b}^{j} - \tau_j \vec{\xi}^{j} \qquad \text{with} \qquad \vec{\xi}^{0} = -\vec{g}\,, \quad \vec{b}^{0} = \vec{0}\;. \tag{4.3}$$

Suppose that at a step J of the iterative procedure the vector $\vec{b}^{J}$ approximates $\vec{b}$ within a needed accuracy. Then we set

$$\widetilde{U}^{J} = \sum_{k+1}^{N} \sum_{j=1}^{M} b^{J}_{kj}\, \omega^{(\gamma)}_{kj}(x)\;, \quad x \in \gamma \tag{4.4}$$

and solve the problems

$$\left.\begin{aligned} &a_{D_l}(U^{J}_{l}, \omega_{kj}) = 0, \quad k=1,\dots,N\,,\ j=1,\dots,\mathcal{N}_l\,, \quad \omega_{kj} \in \overset{0}{\mathbf{W}}{}^{1,h}_{2}(D_l)\,, \\ &U^{J}_{l}|_{\Gamma_l} = 0\,, \qquad U^{J}_{l}|_{\gamma} = \widetilde{U}^{J}\,, \end{aligned}\right\} \tag{4.5}$$

where $U^{J}_{l} \in \mathbf{W}^{1,h}_{2,0(\Gamma_l)}(D_l)$, $l = 1,2$. Now the approximate solution of initial problem on D_l *is defined by*

$$\varphi^{J}_{l} = U^{J}_{l} + \varphi^{(0),h}_{l}\;, \quad l = 1,2. \tag{4.6}$$

Consider the steps of the realisation of the corresponding algorithm.

<u>Step 0.</u> We solve the auxiliary problems

$$\left.\begin{aligned} &a_{D_l}(\varphi^{(0),h}_{l}, \psi^{h}_{l}) = (f, \psi^{h}_{l})_{\mathbf{L}_2(D_l)}\;, \qquad \forall\, \psi^{h}_{l} \in \mathbf{W}^{1,h}_{2}(\overset{0}{D}_l)\,, \\ &\varphi^{(0),h}_{l}|_{\Gamma_l} = \varphi_{\Gamma,l}\,, \quad \varphi^{(0),h}_{l}|_{\gamma} = \varphi_{(\gamma)}\;, \quad l=1,2, \end{aligned}\right\} \tag{4.7}$$

and calculate the vector $\vec{\xi}^{0} = (\xi^{0}_{1},\dots,\xi^{0}_{N})^{T}$, $\xi_k = (\xi_{k1},\dots,\xi_{kM})^{T}$ with the components

$$\xi^{0}_{ki} = \sum_{l=1}^{2} a_{D_l}(\varphi^{(0),h}_{l}, \omega^{(\gamma)}_{ki}) - (f, \omega^{(\gamma)}_{ki})_{\mathbf{L}_2(D)}\;. \tag{4.8}$$

(So, we know the vector $\vec{\xi}^{j}$ with $j=0$.)

Step 1. Solve the problems

$$\left.\begin{aligned} &a_{D_l}(U_l^h, \psi_l^h) = 0, \; U_l^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_l)}(D_l) , \quad \forall \; \psi_l^h \in \overset{0}{\mathbf{W}}{}^{1,h}_2 (D_l) , \\ &U_l^h|_\gamma = \xi^{h,j} , \; l=1,2, \end{aligned}\right\} \quad (4.9)$$

and compute the vector $\vec{p}^j = \hat{\mathcal{A}}^h \vec{\xi}^j$ with the components

$$p_{kl}^j = (\hat{\mathcal{A}}^h \vec{\xi}^j)_{kl} = \int_\gamma \sum_{l=1}^{2} (S_l^h)^{-1} \xi^{h,j} \omega_{kl}^{(\gamma)} d\Gamma = \sum_{l=1}^{2} a_{D_l}(U_l^h , \omega_{kl}^{(\gamma)}) ,$$

where

$$\xi^{h,j} = \sum_{k=1}^{N} \sum_{l=1}^{M} \xi_{kl}^j \; \omega_{kl}^{(\gamma)} (x) , \quad x \in \gamma .$$

Step 2. The quantities τ_j , $\vec{\xi}^{j+2}$, $\vec{b}^{j+1}$ are computed by (4.2) , (4.3) .

After this step the calculations are repeated again back to the beginning of Step. 1.

Remark 4.1: The above algorithm is valid for another decomposition of D , for example, for the decomposition, represented on Figure 7.2. It is easy to see that in this case Steps 0 to 2 are sets of independent subproblems. Therefore these subproblems may be solved in parallel on several computers. It is a very important property of our domain decomposition methods. But if we have a single computer then the algorithm may be realized too. To do this we must solve the subproblems sequentially. □

Consider the convergence of the above algorithm. Because the matrix $\hat{\mathcal{A}}^h$ is positive definite then by known results of the theory of minimal residual methods ([23] , [24]) we have

$$\|\vec{b}^J - \vec{b}\|_2 \to 0 , \quad \|\hat{\mathcal{A}}^h \vec{b}^J - \vec{g}\|_2 \to 0 , \quad J \to \infty . \qquad (4.11)$$

If the functions $\{\omega_{kj}^{(\gamma)}\}$ are normalized and, with $C,\widetilde{C} > 0$,

$$C \; \|\vec{b}\|_2 \leq \| \sum_{k=1}^{N} \sum_{l=1}^{M} b_{kl} \, \omega_{kl}^{(\gamma)} \|_{\mathbf{L}_2(\gamma)} \leq \widetilde{C} \|\vec{b}\|_2 , \qquad (4.12)$$

then from (4.11) we obtain

$$\|P_\omega \mathcal{A}^h \widetilde{U}^J - F^h\|_{\mathbf{L}_2(\gamma)} \to 0 , \quad \|\widetilde{U}^J - \widetilde{U}^h\|_{\mathbf{L}_2(\gamma)} \to 0 , \ J \to \infty , \tag{4.13}$$

where $\widetilde{U}^J = \sum_{k=1}^{N} \sum_{t=1}^{M} b_{kt}^J \, \omega_{kt}^{(\gamma)}(x)$. Now for the solutions of (4.5) we have

$$\sum_{l=1}^{2} a_{D_l} (U_l^J - U_l, U_l^J - U_l) = (\mathcal{A}^h (\widetilde{U}^J - \widetilde{U}^h) , \widetilde{U}^J - \widetilde{U}^h)_{\mathbf{L}_2(\gamma)}$$

$$= (P_\omega \mathcal{A}^h \widetilde{U}^J - F^h , \widetilde{U}^J - \widetilde{U}^h)_{\mathbf{L}_2(\gamma)} \to 0 , \quad J \to \infty .$$

Therefore,

$$\sum_{l=1}^{2} \|U_l^J - U_l\|_{\mathbf{W}_2^1(D_l)} \to 0 , \quad \sum_{l=1}^{2} \|\varphi_l^J - \varphi_l^h\|_{\mathbf{W}_2^1(D_l)} \to 0 , \ J \to \infty , \tag{4.14}$$

where the functions φ_l^J , $l=1,2,$ are defined by (4.6).

<u>Remark 4.2:</u> If bounds for the eigenvalues of $\hat{\mathcal{A}}^h$ are known we may in addition to (4.14) estimate the rate of the convergence of the above algorithm. □

5. DOMAIN DECOMPOSITION ALGORITHMS BASED ON A SPLITTING METHOD.

A wide class of domain decomposition methods may be formulated if to solve (3.13) , (3.15) we apply a suitable splitting method. In this section we consider only one of them. But it is easy to see that the following considerations may be reformulated for other splitting methods.

Since the operator $P_\omega \mathcal{A}^h$ in (3.15) is the sum of two positive definite operators $P_\omega \mathcal{A}_1^h$, $P_\omega \mathcal{A}_2^h$ then to solve (3.15) we may apply the following iterative procedure ([23] , p. 206) :

$$(E+\tau B_1)(E+\tau B_2)(\widetilde{U}^{j+1} - \widetilde{U}^j) = -2\tau (B\widetilde{U}^j - F^h) , \ j=0,1,\dots , \tag{5.1}$$

where $B_l \equiv P_\omega \mathcal{A}_l^h$, $\tau =$ const >0 , $B=B_1+B_2$, E is the identity operator. As long as the operators B_1 , B_2 are positive definite then process (5.1) converges and

$$\|(E + \tau B_2)(\widetilde{U}^J - \widetilde{U}^h)\|_{\mathbf{L}_2(\gamma)} \to 0 , \quad J \to \infty . \tag{5.2}$$

Let's consider now the steps of the implementation of the iterative procedure (5.2). First of all let's observe that it may be rewritten as follows as a system of equtions in $\mathbf{L}_2(\gamma)$:

$$\begin{aligned} &\xi^{j+1/4} = B\,\widetilde{U}^j - F^h , \\ &(E + \tau B_1)\,\xi^{j+1/2} = -2\tau\,\xi^{j+1/4} , \\ &(E + \tau B_2)\,\xi^{j+3/4} = \xi^{j+1/2} , \\ &\widetilde{U}^{j+1} = \widetilde{U}^j + \xi^{j+3/4} , \quad j=0,1,2,..., \end{aligned} \tag{5.3}$$

or as the linear algebraic equations

$$\left.\begin{aligned} &(\xi^{j+1/4}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = (\sum_{l=1}^{2} (S_l^h)^{-1}\widetilde{U}^j, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} - (F^h, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} , \\ &(\xi^{j+1/2}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} + \tau((S_1^h)^{-1}\xi^{j+1/2}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = -2\tau(\xi^{j+1/4}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} , \\ &(\xi^{j+3/4}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} + \tau((S_2^h)^{-1}\xi^{j+3/4}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = (\xi^{j+1/2}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} , \\ &(\widetilde{U}^{j+1} - \widetilde{U}^j - \xi^{j+3/4}, \omega_{kl}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = 0 , \quad j=0,1,2 \dots . \end{aligned}\right\} \tag{5.4}$$

Therefore taking into account (3.17) , (3.18) , we conclude that the realization of the first equation of (5.4) consists in solving the problems

$$\left.\begin{aligned} &a_{D_l}(U_l^j , \psi_l^h) = 0 , \; U_l^j \in \mathbf{W}^{1,h}_{2,0(\Gamma_l)}(D_l) , \quad \forall\, \psi_l^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D_l) , \\ &U_l^j|_\gamma = \widetilde{U}^j , \quad l=1,2, \end{aligned}\right\} \tag{5.5}$$

and in computing the piecewise linear function. $\xi^{j+1/4}$ by

$$(\xi^{j+1/4}, \omega_{kt}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = \sum_{l=1}^{2} a_{D_l}(U_l^j , \omega_{kt}^{(\gamma)}) - (F^h, \omega_{kt}^{(\gamma)})_{\mathbf{L}_2(\gamma)}$$

or by the equations

$$\left.\begin{aligned} &(\xi^{j+1/4}, \omega_{kt}^{(\gamma)})_{\mathbf{L}_2(\gamma)} = \sum_{l=1}^{2} a_{D_l}(U_l + \varphi_l^{(0),h}, \omega_{kt}^{(\gamma)}) - (F^h, \omega_{kt}^{(\gamma)})_{\mathbf{L}_2(D)} , \\ &k=1,\dots,N , \quad j=1,\dots, M. \end{aligned}\right\} \quad (5.6)$$

The step corresponding to the second equation of (5.4) consists in the solution of

$$\left.\begin{aligned} &\tau\, a_{D_1}(U_1^{j+1/2}, \psi^h)+(U_1^{j+1/2}, \psi^h)_{\mathbf{L}_2(\gamma)} = -\, 2\tau(\xi^{j+1/4}, \psi^h)_{\mathbf{L}_2(\gamma)} , \\ &U_1^{j+2/2}, \psi^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_1)}(D_1) \end{aligned}\right\} \quad (5.7)$$

and in setting

$$\xi^{j+1/2} = U^{j+1/2}|_\gamma . \qquad (5.8)$$

Indeed if we know $U_1^{j+1/2}$ and if the function $\xi^{j+1/2}$ has been determined by (5.8) , then with $\psi^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D_1)$ in (5.7) we have

$$a_{D_1}(U_1^{j+1/2}, \psi^h) = 0 , \quad \forall \quad \psi^h \in \overset{0}{\mathbf{W}}{}_2^{1,h} (D_1) ,$$

$$U_1^{j+1/2}|_\gamma = \xi^{j+1/2} , \quad U_1^{j+1/2} \in \overset{0}{\mathbf{W}}{}^{1,h}_{2,0(\Gamma_1)}(D_1) .$$

Now let ψ^h be $\omega_{kt}(x)$, where $\omega_{kt}|_\gamma = \omega_{kt}^{(\gamma)}$, and let us remember the definition of S_1^h . Hence we conclude that equation (5.7) coincides with the second equation of (5.4).

Taking into account the already formulated propositions we can write down the realisation of the third equation of (5.4) as follows

$$\left.\begin{aligned} &\tau a_{D_2}(U_2^{j+3/4}, \psi^h) + (U_2^{j+3/4}, \psi^h)_{L_2(\gamma)} = (\xi^{j+1/2}, \psi^h)_{L_2(\gamma)}, \\ &U_2^{j+3/4}, \psi^h \in W^{1,h}_{2,0(\Gamma_2)}(D_2), \end{aligned}\right\} \quad (5.9)$$

$$\xi^{j+3/4} \equiv U_2^{j+3/4}|_\gamma . \qquad (5.10)$$

Then, we set

$$\widetilde{U}^{j+1} = \widetilde{U}^j + \xi^{j+3/4} \equiv \widetilde{U}^j + U_2^{j+3/4}|_\gamma . \qquad (5.11)$$

Thus the solution of the initial problem consists in the implementation of the following steps.

<u>**Step 0:**</u> *Solve the auxiliary problems* (4.7).

<u>**Step 1:**</u> *Compute the solutions of* (5.5) *and the function* $\xi^{j+1/4}$ *according to* (5.6).

<u>**Step 2:**</u> *Solve equation* (5.7) *and compute* $\xi^{j+1/2} \equiv U^{j+1/2}|_\gamma$.

<u>**Step 3:**</u> *From* (5.9) *define* $U_2^{j+3/4}$ *and compute* $\xi^{j+3/4} \equiv U^{j+3/4}|_\gamma$. *and* $\widetilde{U}^{j+1} = \widetilde{U}^j + \xi^{j+3/4}$.

After this step the calculations are repeated again beginning from Step 1. If the process has been finished at step J then with the computed piecewise linear function $\widetilde{U}^J$, solve the problems (4.5) and define the approximate solution of the problem by (4.6).

Back to the question of convergence, it follows from (5.2)

$$\left.\begin{array}{l}
\|\widetilde{U}^J - \widetilde{U}^h\|^2_{\mathbf{L}_2(\gamma)} + 2\tau\, ((S_2^h)^{-1} (\widetilde{U}^J - \widetilde{U}^h) , \widetilde{U}^J - \widetilde{U}^h)_{\mathbf{L}_2(\gamma)} \\
\quad + \tau^2 \, \|P_\omega (S_2^h)^{-1} (\widetilde{U}^J - \widetilde{U}^h)\|^2_{\mathbf{L}_2(\gamma)} \to 0 \, , \, J \to \infty \, , \\
\|\widetilde{U}^J - \widetilde{U}^h\|^2_{\mathbf{L}_2(\gamma)} + 2\tau \, \|U_2^J - U_2\|^2_{\mathbf{W}_2^1(D_2)} \\
\quad + \tau^2 \, \|P_\omega (S_2^h)^{-1} \, (\widetilde{U}^J - \widetilde{U}^h) \, \|^2_{\mathbf{L}_2(\gamma)} \to 0 \, , \, J \to \infty \, , \\
\|U_2^J - U_2\|^2_{\mathbf{W}_2^{1/2}(\partial D_2)} \to 0 \, , \quad \|\widetilde{U}^J - \widetilde{U}^h\|^2_{\mathbf{W}^{1/2}_{2,0(\Gamma_2)}(\gamma)} \to 0, \, J \to \infty .
\end{array}\right\} \quad (5.12)$$

To prove the convergence of U_1^J to U_1 on D_1 let us formulate the following

Lemma 5.1: *If* U_l *is the solution of problem*

$$a_{D_l}(U_l \, , \psi^h) = 0 \, , \quad U_l|_\gamma = \widetilde{U} \, , \, U_l \in \mathbf{W}^{1,h}_{2,0(\Gamma_l)}(D_l) \, , \quad \psi^h \in \overset{0}{\mathbf{W}}{}^{1,h}_2(D_l) \, , \qquad (5.13)$$

then

$$\| (S_l^h)^{-1} \widetilde{U} \|_{\mathbf{W}^{-1/2,h}_{2,0(\Gamma_l)}(\gamma)} \le C a^{1/2}_{D_l} (U_l \, , U_l) \, , \qquad (5.14)$$

with the constant C *independent of* U_l .

Proof: Let the components of the vector-function V be the solutions of the problems

$$\left.\begin{array}{l}
\inf \, \|\nabla v\|^2_{\mathbf{L}_2(D_l)} \, , \\
v \in \mathbf{W}_2^1(D_l) \, , \quad v|_{\Gamma_l} = 0 \, , \quad v|_\gamma = \widetilde{v} \in \mathbf{W}_2^{1/2}(\partial D_l)
\end{array}\right\}$$

Denote by $v^h \in \mathbf{W}^{1,h}_{2,0}(\Gamma_l)$ the piecewise linear approximation of v by those functions $\{\omega_{kj}\}$ such that $\|v - v^h\|_{\mathbf{W}^1_2(D_l)} \to 0$, $h \to 0$ (h is the maximal edge of the mesh tringles). Then

$$\|v - v^h\|_{\mathbf{W}^{1/2}_2(\partial D_l)} \to 0 \ , \ h \to 0 \ .$$

Let h be small enough; then for the solution of (5.13) we have

$$\left|\int_\gamma (S^h_l)^{-1}\widetilde{U}\cdot v^h d\Gamma\right| = |a_{D_l}(U_l \ , \ v^h)| \ \le$$

$$\le C \, a^{1/2}_{D_l}(U_l \ , \ U_l) \ \|v^h\|_{\mathbf{W}^1_2(D_l)} \le C a^{1/2}_{D_l}(U_l \ , \ U_l) \ \|v\|_{\mathbf{W}^1_2(D_l)} \le$$

$$\le C \, a^{1/2}_{D_l}(U_l \ , \ U_l) \ \|v\|_{\mathbf{W}^{1/2}_2(\partial D_l)} \le C \, a^{1/2}_{D_l}(U_l \ , \ U_l) \ \|v^h\|_{\mathbf{W}^{1/2}_2(\partial D_l)} \ .$$

Therefore (5.14) is valid. □

Due to (5.14) we conclude that:

$$|a_{D_1}(U^J_1 - U_1 \ , \ U^J_1 - U_1)| = \left|\int_\gamma (S^h_1)^{-1}(\widetilde{U}^J_1 - \widetilde{U}^h_1) \cdot (\widetilde{U}^J - \widetilde{U}^h) \, d\Gamma\right| \le$$

$$\le C \, a^{1/2}_{D_l}(U^J_1 - U_1 \ , \ U^J_1 - U_1) \ \|U^J_1 - U^h_1\|_{\mathbf{W}^{1/2}_2(\partial D_1)} .$$

However the norm $\|U^J_1 - U^h_1\|_{\mathbf{W}^{1/2}_2(\partial D_1)}$ is equivalent to the norms

$\|\widetilde{U}^J - \widetilde{U}^h\|_{\mathbf{W}^{1/2}_2(\gamma)}$, $\|U^J_2 - U^h_2\|_{\mathbf{W}^{1/2}_2(\partial D_2)}$ (cf. [19]) therefore

$$\|U_1^J - U_1^h\|_{W_2^{1/2}(\partial D_1)} \le C\|U_1^J - U_1^h\|_{W_2^1(D_1)} \le$$

$$\le C\|U_2^J - U_2^h\|_{W_2^{1/2}(\partial D_2)} \to 0 \ , \quad J \to \infty \ .$$

So we have the following result:

$$\sum_{l=1}^{2} \|U_l^J - U_l\|_{W_2^1(D_l)} \to 0 \ , \ \sum_{l=1}^{2} \|\varphi_l^J - \varphi_l^h\|_{W_2^1(D_l)} \to 0 \ , \ J \to \infty \ . \tag{5.15}$$

<u>Remark 5.1:</u> From the proof of Lemma 5.1 it follows that

$$\|U_l\|_{W_2^1(D_l)} \le C\|\widetilde{U}\|_{W_2^{1/2}(\partial D_l)} \qquad (l=1,2) \ . \tag{5.16}$$

This inequality is the finite dimensional analogue of the corresponding inequality for the solution of the Dirichlet problem for elliptic equations of the second order.

6. DOMAIN DECOMPOSITION ALGORITHM BASED ON OPTIMAL LINEAR ITERATIVE PROCEDURE.

In previous sections we constructed iterative processes by applying gradient methods and aplitting methods. Here we will use to this aim a method of another class of iterative procedures: the optimal linear procedure. In this section we will assume that the bilinear forms $a_{D_l}(u,v)$, $l=1,2$, are symmetric.

Consider again equation (3.25) , which is equivalent to an equation of the form

$$C^h \widetilde{V}^h = G^h \ , \tag{6.1}$$

where

$$C^h = (P_\omega (S_1^h)^{-1})^{-1/2} \mathcal{A}^h \, (P_\omega (S_1^h)^{-1})^{-1/2},$$

$$\widetilde{V}^h = (P_\omega (S_1^h)^{-1})^{1/2} \, \widetilde{U}^h \ , \quad G^h = (P_\omega (S_1^h)^{-1})^{-1/2} \, F^h \ .$$

Suppose that the eigenvalues of C^h belong to the interval $[\widetilde{m}, \widetilde{M}]$,

with $0 < \widetilde{m} < \widetilde{M} < \infty$, $\widetilde{m}$, $\widetilde{M}$=const. If we set

$$\beta = \frac{2}{\widetilde{m}+\widetilde{M}}, \tag{6.2}$$

then the linear optimal iterative procedure to solve (6.1) has the form

$$\widetilde{V}^{j+1} = \widetilde{V}^{j} - \beta\,(C^h\widetilde{V}^{j} - G^h)\ ,\ j=0,1,2,\ \ldots, \tag{6.3}$$

and the rate of convergence is estimated by the formula:

$$\|\widetilde{V}^{j} - \widetilde{V}^{h}\|_{\mathbf{L}_2(\gamma)} \le \frac{\widetilde{m}+\widetilde{M}}{2\widetilde{m}}\,|\theta|^{-j}\|\,C^h\widetilde{V}^{0} - G^h\|_{\mathbf{L}_2(\gamma)}\ ,$$

where $\theta = (\widetilde{M}+\widetilde{m})/(\widetilde{M}-\widetilde{m}) > 1$.

For the equation (3.15) the process (6.3) is formulated as

$$\widetilde{U}^{j+1} = \widetilde{U}^{j} - \beta(P_\omega(S_1^h)^{-1})^{-1}(P_\omega\mathcal{A}^h\widetilde{U}^{j} - F^h)\ ,\quad j=0,1,2,\ \ldots, \tag{6.4}$$

with the following rate of convergence

$$((S_1^h)^{-1}(\widetilde{U}^{j}-\widetilde{U}^{h})\ ,\ \widetilde{U}^{j}-\widetilde{U}^{h})^{1/2}_{\mathbf{L}_2(\gamma)} \le \frac{\widetilde{M}+\widetilde{m}}{2\widetilde{m}}\,|\theta|^{-j}\|\,C^h\widetilde{V}^{0} - G^h\|_{\mathbf{L}_2(\gamma)}\ . \tag{6.5}$$

The realization of (6.3) may be carried out by solving the equations:

$$\left.\begin{aligned} &\xi^j = P_\omega\mathcal{A}^h\widetilde{U}^h - F^h\ ,\\ &P_\omega(S_1^h)^{-1}\xi^{j+1/2} = \xi^j\ ,\\ &\widetilde{U}^{j+1} = \widetilde{U}^{j} - \beta\,\xi^{j+1/2}\ ,\quad j=0,1,2,\ \ldots\ . \end{aligned}\right\} \tag{6.6}$$

Taking into account the results of the previous sections we conclude that the realization of (6.6) consists in the following steps:

Step 0: *Solve the auxiliary problems* (4.7).

Step 1: *Solve the problems* (5.5) *and define the piecewise linear function* ξ^j *by*

$$(\xi^j,\omega_{ki}^{(\omega)})_{\mathbf{L}_2(\gamma)} = \sum_{l=1}^{2} a_{D_l}(U_l^j+\varphi_l^{(0),h}\ ,\ \omega_{ki}^{(\gamma)}) - (f,\omega_{ki}^{(\gamma)})_{\mathbf{L}_2(D)}\ ,$$

$$k=1,\ \dots\ ,N,\quad i=1,\ \dots\ ,M. \tag{6.7}$$

Step 2: *From the equation*

$$a_{D_1}(U_1\ ,\ \psi^h) = \int_\gamma \xi^j\psi^h d\Gamma\ ,\qquad \forall\ \psi^h\in \mathbf{W}^{1,h}_{2,0(\Gamma_1)}(D_1) \tag{6.8}$$

we define the function $U_1\in\mathbf{W}^{1,h}_{2,0(\Gamma_1)}(D_1)$ *and set*

$$\xi^{j+1/2} \equiv U_1|_\gamma\ . \tag{6.9}$$

Step 3: *Compute*

$$\widetilde{U}^{j+1} = \widetilde{U}^j - \beta\xi^{j+1/2} \equiv \widetilde{U}^j - \beta U_1|_\gamma\ . \tag{6.10}$$

After Steps 1 to 3 are carried out again with $j\to j+1$. As a result we have for the domain decomposition algorithm the following rate of convergence

$$\sum_{l=1}^{2}\ (\|U_l^J - U_l\|_{\mathbf{W}_2^1(D_l)} + \|\varphi_l^J - \varphi_l^h\|_{\mathbf{W}_2^1(D_l)}) \le C|\theta|^{-J}\ ,$$

where the constant C is independent of J , and the functions U_l^J , φ_l^J are determined by (4.5) , (4.6) .

In conclusion let us estimate the values of $\widetilde{m}$, $\widetilde{M}$ from (6.2) . Let $(w^h\ ,\ \lambda^h)$ be a solution of the equation $C^h w^h = \lambda^h w^h$. Then with $v^h = (P_\omega(S_1^h)^{-1})^{1/2}w^h$ we have

$$\lambda^h = \frac{\sum_{l=1}^{2} ((S_l^h)^{-1}v^h, v^h)_{\mathbf{L}_2(\gamma)}}{((S_1^h)^{-1}v^h, v^h)_{\mathbf{L}_2(\gamma)}} = 1 + \frac{((S_2^h)^{-1}v^h, v^h)_{\mathbf{L}_2(\gamma)}}{((S_1^h)^{-1}v^h, v^h)_{\mathbf{L}_2(\gamma)}} .$$

If $\{U_l^h\}$ are the solutions of the problems

$$\left.\begin{aligned} &a_{D_l}(U_l^h , \phi_l^h) = 0 , \ U_l^h \in \mathbf{W}^{1,h}_{2,0(\Gamma_l)}(D_l) , \quad \phi_l^h \in \overset{0}{\mathbf{W}}{}_2^{1,h}(D_l) , \\ &U_l^h|_\gamma = v^h , \quad l=1,2, \end{aligned}\right\} \quad (6.11)$$

then

$$\lambda^h = 1 + \frac{a_{D_2}(U_2^h , U_2^h)}{a_{D_1}(U_1^h , U_1^h)} . \quad (6.12)$$

Introduce $\{U_l\}$ as the solutions of the problems

$$\left.\begin{aligned} &a_{D_l}(U_l , \phi_l) = 0 , \quad U_l \in \mathbf{W}_2^1(D_l) , \quad \forall \ \phi_l \in \overset{0}{\mathbf{W}}{}_2^1 (D_l) , \\ &U_l|_{\Gamma_l} = 0 , \quad U_l|_\gamma = v^h , \quad l=1,2 . \end{aligned}\right\} \quad (6.13)$$

It has been proved in [19] that there exist constants $\tilde{\tilde{m}}$, $\tilde{\tilde{M}}$, independent of v^h and of parameters of the mesh on D_l , $l=1,2$, such that

$$0 < \tilde{\tilde{m}} \leq \frac{a_{D_2}(U_2 , U_2)}{a_{D_1}(U_1 , U_1)} \leq \tilde{\tilde{M}} < \infty . \quad (6.14)$$

Taking into account the well known results concerning the convergence of U_l^h to U_l when $h \to 0$, we conclude from (6.14) , (6.12) that

$$1 + \tilde{\tilde{m}} - |\epsilon_1(h)| \leq \lambda^h \leq 1 + \tilde{\tilde{M}} + |\epsilon_2(h)| , \quad (6.15)$$

where $\{\epsilon_i(h)\}$ are some functions of h dependent of the rate of the convergence of U_l^h to U_l and for which we have $|\epsilon_i(h)|\to 0$ when $h\to 0$. The expressions for $\{\epsilon_i(h)\}$ are well known in finite element theory. In any case when the value of h is small enough we may set

$$\widetilde{m} \cong 1 + \widetilde{\widetilde{m}}, \quad \widetilde{M} \cong 1 + \widetilde{\widetilde{M}}. \tag{6.16}$$

<u>Remark 6.1:</u> The equation (6.1) has been derived using $(P_\omega(S_1^h)^{-1})^{-1/2}$. But we can also use the operators $(P_\omega(S_2^h)^{-1})^{-1/2}$, $(P_\omega(S_1^h)^{-1}+P_\omega(S_2^h)^{-1})^{-1/2}$ and analogous operators by replacing S_l^h by $S_{l,\Delta}^h$ - the Pincaré-Steklov's operator corresponding to $a_{D_{l,\Delta}}(U^h, V^h) = (\nabla U^h, \nabla V^h)_{\mathbf{L}_2(D_l)}$, $l=1,2$. As a result we will obtain domain decomposition algorithms which are similar to the iterative methods derived in [19].

<u>Remark 6.2:</u> Using equation (6.1) we can construct many algorithms based on conjugate gradient methods, methods with Chebyshev's parameters and others which will converge as geometric sequences. □

7. DOMAIN DECOMPOSITION METHODS FOR SOME ELLIPTIC PROBLEMS

In this section we apply some of the algorithms of the previous sections to two concrete boundary value problems.

7.1. DIFFUSION OF PARTICLES IN A FLUID.

Consider the problem of a diffusion of some particles in a flow of fluid with velocity $s = (u,v)$: *Given* $f(x)\in L_2(D)$, $\varphi_{(\Gamma)}(x)\in W_2^{1/2}(\partial D)$, ***find*** $\varphi(x)\in W_2^1(D)$ *such that*

$$\left.\begin{aligned} &a_D(\varphi,\psi) = (f,\psi)_{L_2(D)}, \quad \forall\psi\in \overset{0}{W}{}_2^1(D), \\ &\varphi|_{\partial D} = \varphi_{(\Gamma)}, \end{aligned}\right\} \tag{7.1}$$

where

$$a_D(\varphi,\psi) = (u \frac{\partial\varphi}{\partial x_1} + v \frac{\partial\varphi}{\partial x_2}, \psi)_{L_2(D)} + (D_{x_1} \frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial x_1})_{L_2(D)} +$$

$$+ (D_{x_2} \frac{\partial\varphi}{\partial x_2}, \frac{\partial\psi}{\partial x_2})_{L_2(D)} + (Q\varphi,\psi)_{L_2(D)},$$

$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} = 0$ in D *(equation of continuity)*,

$$\left.\begin{aligned} &0 < Q_0 \leq Q(x) \leq Q_1 < \infty, \quad 0 < D_0 \leq D_{x_1}, D_{x_2} \leq D_1 < \infty, \\ &|s| = (u^2+v^2)^{1/2} \leq |s_{max}| < \infty, \quad Q_0, Q_1, D_0, D_1, \quad |s_{max}| = \text{const}, \\ &(s,n)|_{\partial D} \equiv u\cdot n_1 + v\cdot n_2|_{\partial D} = 0 \end{aligned}\right\} \quad (7.2)$$

and $n = (n_1, n_2)$ is the outward unit vector normal at ∂D. It is easy to prove that under the conditions (7.2), then problem (7.1) has a unique solution $\varphi \in W_2^1(D)$ and the estimate (1.4) holds.

Introduce two subsets D_1, D_2, adjacent along γ. Introduce the meshes, and functional spaces described in Section 1,2 with $N=1$. Here the piecewise linear basis functions $\{\omega_{ki}(x)\}$ are denoted by $\{\omega_i\}$. Besides we introduce the following assumption: let the give data and the decompostion D_1, D_2, be such that

$$\left.\begin{aligned} I_i \equiv &\sup_{w \in W^1_{2,0(\Gamma_i)}(D_i)} \left\{ \frac{1}{2} \int_{\gamma \cap ((s,n^{(i)})<0)} |(s,n^{(i)})| w^2 d\Gamma \Big/ \left[\frac{1}{2} \int_{\gamma \cap ((s,n^{(i)})>0)} |(s,n^{(i)})| w^2 d\Gamma + \right.\right. \\ &\left.\left. + (D_{x_1} \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_1})_{L_2(D_i)} + (D_{x_2} \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_2})_{L_2(D_i)} + (Qw,w)_{L_2(D_i)} \right] \right\} \\ &\leq q_i = \text{const.} < 1, \quad i=1,2, \end{aligned}\right\} \quad (7.3)$$

where $n^{(l)} = (n_1^{(l)}, n_2^{(l)})$ is the outward unit vector normal at ∂D_l. We consider now a situation in which assumption (7.3) is valid:

Namely let γ coincide with some stream line (i.e. with some particle trajectories in the fluid). Then we will have on γ, $(s,n^{(l)}) = 0$. Therefore in (7.3) we can set $q_l \equiv 0$. In Section 2 the set γ consists of segments of straight lines. Therefore the condition $(s,n^{(l)}) = 0$ on γ may be not valid. But the following lemma holds:

<u>**Lemma 7.1:**</u> *Let* $\tilde{\gamma}$ *coincides with some stream line and* γ *be a broken line with vertices on* $\tilde{\gamma}$. *Then the condition* (7.1) *holds when* h *is small enough* (h:- maximal edge length of triangles in $\{D_l\}$).

<u>**Proof:**</u> Let x be a point on γ and $\tilde{x}$ be a point on $\tilde{\gamma}$ in which the vector $n^{(l)}(x)$ intersects $\tilde{\gamma}$ for the first time. If h is small enough then $|x-\tilde{x}| \le Ch^2$ and $|(s,n^{(l)}(x))| \le C(1+|s_{max}|)h^2$. Therefore

$$\frac{1}{2}\int_{\gamma\cap(s,n^{(l)})<0} |(s,n^{(l)})|w^2 d\Gamma \le C\,h^2\Bigg[(D_{x_1}\frac{\partial w}{\partial x_1},\frac{\partial w}{\partial x_1})_{L_2(D_l)} + (D_{x_2}\frac{\partial w}{\partial x_2},\frac{\partial w}{\partial x_2})_{L_2(D_l)} + (Qw,w)_{L_2(D_l)}\Bigg]$$

and there exists the constant $q_l < 1$ such that (7.3) holds. □

Now let γ satisfy the conditions of Lemma 7.1. Then the assumptions (1.2) and all statements of the previous sections are valid. In particular to solve problem (7.1) we can apply the algorithm from Section 4 and the results of that section still hold.

7.2 A LINEAR ELASTICITY PROBLEM.

Let $D = D_1 \cup D_2$ be the domain represented on Figure 7.1 (i.e. for simplicity let $\{D_i\}$ be rectangles) and $\lambda^{(i)}, \mu^{(i)}$ $(i=1,2)$ be positive constants. Introduce the spaces of vector-functions with $N=2$, described earlier in Sections 1,2 and set

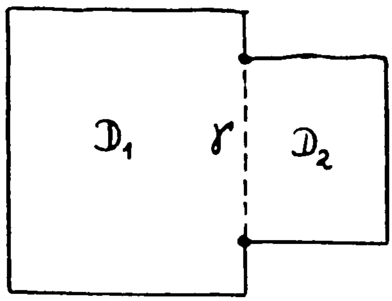

Figure 7.1

$$\varphi_{(\Gamma)} = 0 ,$$

$$f = \{f_i \text{ in } D_i ,\ i=1,2\} ,\ f_i \in \mathbf{L}_2(D_i) ,$$

$$a_D(u,v) = \sum_{i=1}^{2} a_{D_i}(u,v) ,$$

where

$$a_{D_i}(u,v) = \int_{D_i} \Big[2\mu^{(i)} \Big(\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}\Big) + \lambda^{(i)} \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial v_1}{\partial x_1} +$$

$$+ \frac{\partial v_2}{\partial x_2}\Big) + \mu^{(i)} \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)\Big(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\Big)\Big] dx , \quad x=(x_1 , x_2) , \tag{7.4}$$

$$u = (u_1 , u_2) ,\ v=(v_1 , v_2) \in \mathbf{W}_2^1(D) .$$

Consider the following plane boundary value problem of elasticity: *Given* $f(x) \in \mathbf{L}_2(D)$ *find* $\varphi(x) \in \overset{0}{\mathbf{W}}{}_2^1(D)$ *such that the relation* (1.3) *holds* $\forall\ v \in \overset{0}{\mathbf{W}}{}_2^1(D)$.

Due to the well known inequalities:

$$\|u\|^2_{\mathbf{L}_2(D_i)} \le C_{D_i} \int_{D_i} \sum_{j=1}^{2} \Big|\frac{\partial u}{\partial x_j}\Big|^2 dx , \quad u \in \mathbf{W}^1_{2,0}(\Gamma_i)\,(D_i) , \tag{7.5}$$

$$\int_{D_i} \sum_{j,k=1}^{2} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)^2 dx \geq C^{(1)}_{D_i} \|u\|^2_{\mathbf{W}^1_2(D_i)}, \quad u \in \mathbf{W}^1_{2,0(\Gamma_i)}(D_i), \tag{7.6}$$

$$\int_{D_i} \sum_{j,k=1}^{2} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)^2 dx \geq C^{(0)}_{D_i} \int_{D_i} \sum_{j=1}^{2} \left|\frac{\partial u}{\partial x_j}\right|^2 dx, \quad u \in \mathbf{W}^1_{2,0(\Gamma_i)}(D_i), \tag{7.7}$$

the bilinear forms $\{a_{D_i}(u,v)\}$ satisfy the relations (1.2) with the constants

$$C_1 = 2 \max_{i=1,2} (\lambda^{(i)} + \mu^{(i)}), \quad C_2 = \min_{i=1,2} (C^{(0)}_{D_i} \frac{\mu^{(i)}}{2}). \tag{7.8}$$

Therefore to solve the above problem we can use the domain decomposition method based on the linear optimal iterative procedure and the statement on its rate of convergence is valid. It has been proved in [19] that the constants $\tilde{\tilde{m}}$, $\tilde{\tilde{M}}$ from (6.14) may be obtained by

$$\tilde{\tilde{m}} = \frac{\mu^{(2)} C^{(0)}_{D_2} \hat{m}}{4(\lambda^{(1)} + \mu^{(1)})}, \quad \tilde{\tilde{M}} = \frac{4(\lambda^{(2)} + \mu^{(2)}) \hat{M}}{\mu^{(1)} C^{(0)}_{D_1}} \tag{7.9}$$

Here the constants $\hat{m}$, $\hat{M}$ are the bounds of the ratio $I(U^{(2)}, D_2)/I(U^{(1)}, D_1)$:

$$0 < \hat{m} \leq I(U^{(2)}, D_2)/I(U^{(1)}, D_1) \leq \hat{M} < \infty,$$

where

$$I(U^{(i)}, D_i) = \sum_{k=1}^{2} \int_{D_i} \left(\frac{\partial U^{(i)}}{\partial x_k}\right)^2 dx,$$

and where the function $U^{(i)}$ is the solution of the problem

$$\Delta U^{(i)} = 0 \quad \text{in } D_i; \quad U^{(i)}|_{\Gamma_i} = 0; \quad U^{(i)}|_{\gamma} = w \in W^{1/2}_{2,0(\Gamma_i)}(\gamma) \quad (i=1,2).$$

Obtaining practical values of $\hat{m}$, $\hat{M}$ has been described in [8]. For example, if

D_1, D_2 are symmetric with respect to γ we have $\hat{M}-\hat{m} - 1$. It is easy to determine $\hat{m}$, $\hat{M}$ for the domain represented in Figure 7.2 (see [18]).

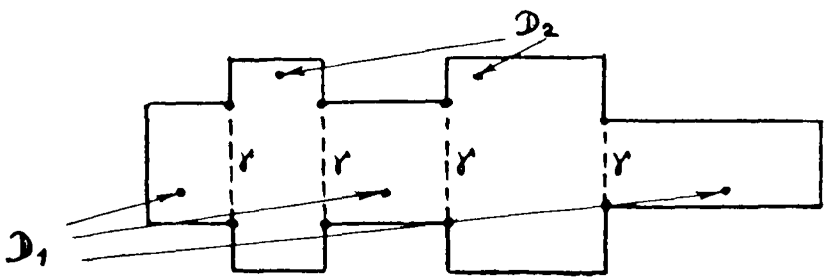

Figure 7.2

For this domain all the results of Section 6 are also valid. But here each problem in Steps 0 to 3 is divided into independent problems, which may be solved in parallel.

7.3. CONCLUSION.

To conclude let us give the estimates for the norms of $\{(S_l^h)^{-1}\}$ and investigate the convergence rate of some domain decomposition algorithms. Suppose that the basis functions $\{\omega_{ki}^{(\gamma)}\}$ are normalized and that relation (4.12) holds. Let $\{h_i\}$ be the mesh sizes on γ and $\bar{h} = \max_i h_i$. Suppose that $\bar{h} \leq C \min_i h_i$, C=const. Let h be the maximal length of the edges of the triangles in $\{D_l\}$.

Lemma 7.1. *There exists a value* $h^{(o)}$ *such that if* $h < h^{(o)}$ *then*

$$\|(S_l^h)^{-1}\|_{\mathbf{W}^{1/2,h}_{2,0(\Gamma_l)}(\gamma)\to\mathbf{W}^{-1/2,h}_{2,0(\Gamma_l)}(\gamma)} \le C<\infty \tag{7.10}$$

with the constant C *independent of* h .

<u>Proof:</u> From (5.14) , (1.2) , (5.16) we have

$$\|(S_l^h)^{-1}\widetilde{U}\|_{\mathbf{W}^{-1/2,h}_{2,0(\Gamma_l)}(\gamma)} \le C\|\widetilde{U}\|_{\mathbf{W}_2^{1/2}(\partial D_l)} ,$$

where the constant C does not depend on h if $h\le h^{(0)}$. From this inequality we obtain (7.10).

<u>Corollary:</u> Let $(S_l^h)^{-1}$ be a symmetric operator. Then the eigenvalues of this operator $(S_l^h)^{-1}$ acting on $\mathbf{L}_2(\gamma)$ belong to the closed interval $[p_0 , p_1/\bar{h}]$ with the constants $p_0 , p_1>0$ independent of h .

<u>Proof:</u> The existence of p_0 is obvious. Now, since we have, for $\widetilde{U},\widetilde{V}\in\mathbf{W}^{1/2,h}_{2,0(\Gamma_l)}(\gamma)$,

$$|((S_l^h)^{-1}\widetilde{U} , \widetilde{V})_{\mathbf{L}_2(\gamma)}|\le C\|\widetilde{U}\|_{\mathbf{W}^{1/2,h}_{2,0(\Gamma_l)}(\gamma)}\|\widetilde{V}\|_{\mathbf{W}^{1/2,h}_{2,0(\Gamma_l)}(\gamma)}$$

(C is the constant in (7.10)) and also that the following inequality holds

$$\|\sum_{k=1}^{N}\sum_{i=1}^{M} a_{ki}\omega_{ki}^{(\gamma)}\|_{\mathbf{W}^{1/2,h}_{2,0(\Gamma_l)}(\gamma)} \le \frac{\widetilde{C}}{\sqrt{h}} \left(\sum_{k=1}^{N}\sum_{i=1}^{M} a_{ki}^2\right)^{1/2}$$

(with the constant $\widetilde{C}$ independent of $\bar{h}$) we conclude that the value $p_1/\bar{h}$ with $p_1 = C\widetilde{C}^2$ can be chosen as the upper bound for the eignevalues of $(S_l^h)^{-1}$. □

Suppose that to solve the plane problem of elasticity in Section 7.2 we apply the algorithm from Section 4. Let φ_l^J $(l=1,2)$ be an approximate solution in D_l . Then taking into account the properties of $\{S_l^h\}$, the statements of Lemma 7.1 and its corallary, and well known results on the convergence of minimal residual methods we conclude that

$$\sum_{l=1}^{2} \|\varphi_l^J - \varphi_l^h\|_{W_2^1(D_l)} \leq C\left(\frac{1-\widetilde{C}h}{1+\widetilde{C}h}\right)^J , \tag{7.11}$$

where C is a constant independent of J and $\widetilde{C} = p_0/p_1$.

<u>**Final Remark:**</u> It is easy to see that the results of Lemma 7.1 and its corollary may be applied to estimate the rates of convergence of other domain decomposition algorithms in finite dimenional spaces. □

<u>Aknowledgment:</u> The editors would like to thank Lena M. Brooks for her reprocessing of the original version of this paper.

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