CHAPTER 3

A Domain Decomposition Method for a Convection Diffusion Equation with Turning Point*

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Abstract. We use asymptotic analysis to determine a domain decomposition method for a singularly perturbed convection-diffusion equation with turning points. The equation

$$-xu_x = \epsilon \Delta u \tag{1}$$

is considered on a square Ω in the plane, with Dirichlet boundary conditions prescribed on $\partial\Omega$. This equation is an idealization of the Navier-Stokes equation with velocity $-x$, parallel to the x-axis. It is known from asymptotic analysis that for small, positive ϵ (large Reynolds number), solutions of (1) very nearly satisfy the reduced equation $u_x = 0$ in subdomains which are at least a distance $C\sqrt{\epsilon}$ from $\partial\Omega$. In the vicinity of $\partial\Omega$ there may be boundary layers, depending on the boundary values. (There is no boundary layer at an inflow boundary.) Our domain decomposition method uses this asymptotic information to determine the partition into subdomains, and it also suggests the basis functions to be used in a finite-element method.

1. Introduction. Several researchers have recently used asymptotic analysis to identify good domain-decomposition strategies for singularly perturbed problems and to suggest efficient numerical algorithms on each subdomain. This work includes both time-dependent problems, such as

$$\partial_t u + a\,\partial_x u + b\,\partial_y u = \epsilon \Delta u, \tag{1.1}$$

and time-independent problems, such as

$$a\,\partial_x u + b\,\partial_y u = \epsilon \Delta u. \tag{1.2}$$

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Here ∂ denotes partial differentiation, $\partial_x = \partial/\partial x$, and Δ is the Laplace operator. In both cases the singular perturbation aspect of the problem is reflected in the fact that ϵ is a small, positive number. The basic idea of this approach is that in a large portion of the domain it is almost possible to replace, say, (1.2) by the corresponding reduced equation

$$a\,\partial_x u + b\,\partial_y u = 0.$$

There exist subdomains—boundary and internal layers—in which such a reduction is impossible. In many of the layer regions other reductions are possible. Our aim is to identify these layer regions and to use numerical methods there which are suited to the local behavior of the solution.

In this paper we present an algorithm for a problem in which there is an interaction between layer regions of two different types: a boundary layer and a layer generated by a manifold of turning points. Before describing our problem in detail, let us summarize some of the previous work on related problems.

The seminal work using asymptotic analysis to suggest numerical methods was the paper by Chin and Krasny [4] on an algorithm for solving two-point boundary-value problems for equations of the form

$$\epsilon u'' = f(x, u) \tag{1.3}$$

under the condition that $\partial f/\partial u > 0$. These problems are difficult to solve by classical numerical algorithms because the solution may have boundary layers at one or both ends of the interval. The algorithm is based on an approximation of f by a function f_h which is piecewise linear in u. For a given approximate solution u_{n-1}, the knots of a spline u_n are chosen to be the values of x for which $u_{n-1}(x)$ is a point of discontinuity for $\partial f_h/\partial u$. As basis functions for the spline u_n Chin and Krasny use either exact solutions or asymptotic approximations to solutions of

$$\epsilon u'' = f_h(x, u).$$

The method is very efficient because these splines incorporate the boundary-layer behavior of the solutions.

As an example of a numerical algorithm based on asymptotic analysis for a 2-dimensional boundary-value problem, let us summarize the work of Rodrigue and Reiter [9] on the equation

$$\partial_x u = \epsilon \Delta u \tag{1.4}$$

on a square $D = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$, with boundary conditions $u = 0$ on $y = 0$, $u = 1$ on $x = 0$ and on $y = 1$, and $\partial u/\partial x = 0$ on $x = 1$. This problem is a model for steady laminar flow of a fluid over a flat plate. We discuss this work in some detail because ours is an extension of it. The asymptotic behavior of solutions to (1.4) as $\epsilon \downarrow 0$ is well known [5]. The reduced equation for (1.4) is

$$\partial_x U = 0, \tag{1.5}$$

and the solution $U = 1$ of (1.5) fulfills the conditions on three of the four boundaries: the inflow ($x = 0$), the outflow ($x = 1$), and the top ($y = 1$). As a consequence, the solution u to (1.4) is asymptotic to U as $\epsilon \to 0$ except in a boundary-layer region in the vicinity of the bottom boundary ($y = 0$).

One feature of this boundary-layer behavior is that in a neighborhood of the origin of diameter $O(\epsilon)$ (the birth region of the boundary layer), we must keep the full equation (1.4). Elsewhere in the vicinity of the boundary $y = 0$, because the solution u of (1.4)

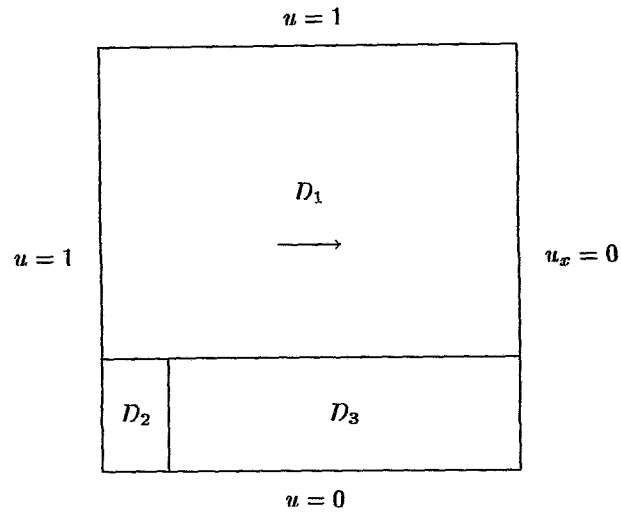

Fig. 1. Domain decomposition.

must have rapid variation in the y-direction, $\partial^2 u/\partial x^2$ is much smaller in magnitude than $\partial^2 u/\partial y^2$. Consequently, near the x-axis and away from the origin, the solution u of (1.4) is asymptotic to the solution of the boundary-layer equation

$$\partial_x v = \epsilon\, \partial_y^2 v. \tag{1.6}$$

It is known that the thickness of this boundary-layer region is $O(\sqrt{\epsilon})$.

On the basis of this asymptotic analysis the method used by Rodrigue and Reiter is as follows. The domain D is partitioned into three subdomains:

$$D_1 = \{(x,y) \mid 0 < x < 1,\ y_0 < y < 1\},$$
$$D_2 = \{(x,y) \mid 0 < x < x_0,\ 0 < y < y_0\},$$
$$D_3 = \{(x,y) \mid x_0 < x < 1,\ 0 < y < y_0\}.$$

See Fig. 1. On the basis of experience the value of y_0 was taken to be $4\sqrt{\epsilon}$. It is known from the theory that we should take $x_0 = c\epsilon$ for some positive c, but Rodrigue and Reiter determine x_0 dynamically. The following iterative method is used.

1. Take $n = 1$ and set $u_0 = 1$ on D.
2. Solve a discretized version of

$$\partial_x u_n = \epsilon \Delta u_{n-1} \tag{1.7}$$

on D_1. (This iterative scheme provides a mechanism by which the solution on $D_2 \cup D_3$ influences the solution on D_1.)

3. Use invariant embedding (a form of Gaussian elimination) to solve a discrete version of (1.4) for u_n on D_2. The boundary conditions are that $u_n = 1$ on $x = 0$, $u_n = 0$ on $y = 0$, and u_n is as determined by Step 2 on $y = y_0$. On the boundary $x = x_0$ a discrete approximation to (1.6) is used as the boundary condition, and x_0 is selected as the first vertical grid line on which $\epsilon\, \partial_x^2 u_n$ is smaller than a prescribed tolerance.
4. The equation

$$\partial_x u_n = \epsilon \left(\partial_x^2 u_{n-1} + \partial_y^2 u_n\right) \tag{1.8}$$

is solved on D_3 with boundary conditions $u_n = 0$ on $y = 0$, u_n as determined in Step 2 on $y = y_0$, and u_n as determined in Step 3 on $x = x_0$.

5. If the difference $\max_D |u_n - u_{n-1}|$ is sufficiently small, stop. Otherwise, increment n by 1, and return to Step 2.

It may be noted that in this algorithm no attempt is made to fulfill the boundary condition $\partial u/\partial x = 0$ at $x = 1$. This omission is harmless because it overlooks only a weak boundary layer in the vicinity of $x = 1$. In the work of Brown *et al.* [2] the singularly perturbed differential equation

$$a\,\partial_x u + b\,\partial_y u = \epsilon \Delta u + f(x, y) \tag{1.9}$$

is considered on a convex domain Ω with Dirichlet boundary conditions. Here the coefficients a and b may depend on x and y. (We could also permit a and b to depend on u by using an iterative scheme.) It is required, however, that a and b not vanish simultaneously—there are no turning points. The paper [2] has two main thrusts: (1) an implementation of coordinate transformations dictated by asymptotic analysis, and (2) an investigation of the stability of schemes based on (1.7).

The reduced equation for (1.9) is $a\,\partial_x u + b\,\partial_y u = f$, which may be integrated by quadrature of line integrals along the characteristic curves with boundary data specified at inflow boundaries. This process provides the solution over most of Ω, but it must be matched to boundary layers in the vicinity of the remainder of $\partial\Omega$. These boundary layers are obtained by introducing boundary-fitted coordinates and using a finite-element method of Chin-Krasny type.

In an application of this circle of ideas to time-dependent problems Chin *et al.* [3] developed a domain-decomposition method for a time-dependent equation modelling transonic laminar flow in a duct,

$$\partial_t u + u\,\partial_x u + ru = \epsilon\,\partial_x^2 u, \tag{1.10}$$

for $0 < x < L$ and $t > 0$. Here r is a smooth function of x. The boundary and initial data are such that the solution u has a shock layer internal to the domain. Outside of this shock layer one may solve the reduced equation

$$\partial_t u + u\,\partial_x u + ru = 0,$$

while the full equation (1.10) must be solved in the shock layer. An iterative method based on the equation

$$\partial_t u_n + u_{n-1}\,\partial_x u_n + ru_n = \epsilon\,\partial_x^2 u_n$$

is used to locate the shock layer. Scroggs has implemented this algorithm on an 8-processor Alliant computer.

2. Asymptotic Analysis of a problem with turning points. We consider problems in which there exists a curve of turning points, and as a model we take

$$-x\,\partial_x u = \epsilon \Delta u \tag{2.1}$$

on the unit square $\Omega = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$. For (2.1) on Ω only the portion of the boundary $x = 1$ is an inflow boundary. The direction of flow is from right to left, because the coefficient of $\partial_x u$ is negative in Ω. The characteristics of the reduced equation

$$-x\,\partial_x u = 0 \tag{2.2}$$

are parallel to the parts of the boundary $y = 0$ and $y = 1$, so we would expect to find parabolic boundary layers near the top and bottom of Ω. The boundary $x = 0$ is

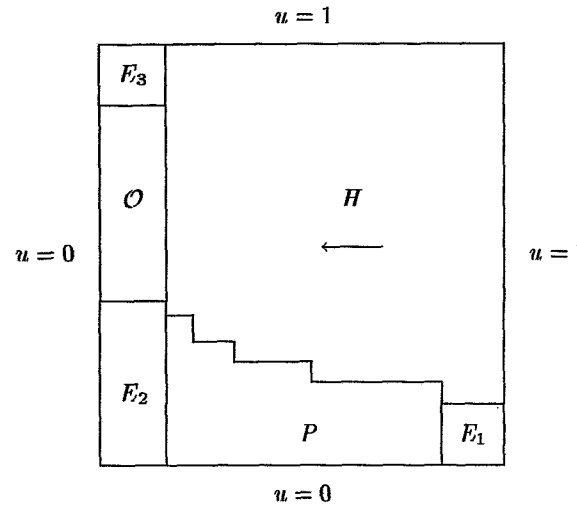

Fig. 2. Domain decomposition with turning points.

special in that it consists of turning points—the coefficient of $\partial_x u$ vanishes there. Let us remark that for this first example we have chosen a case in which the direction of flow is toward the turning points instead of away from them, because the solution may be unstable with respect to perturbation of the boundary data in the other case [1].

If for (2.1) we impose the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{if } x = 0 \text{ or if } y = 0, \\ u &= 1 \quad \text{if } x = 1 \text{ or if } y = 1, \end{aligned} \tag{2.3}$$

then the natural domain decomposition is indicated in Fig. 2. This figure is based on an asymptotic analysis of the behavior of the solution of (2.1) as $\epsilon \to 0$ as given, for example, in [6] and [7]. Let us summarize the principal features of this asymptotic behavior.

There is a parabolic boundary layer region P near the bottom boundary $y = 0$ in which (2.1) may be approximated by the boundary-layer equation

$$-x\,\partial_x u = \epsilon\,\partial_y^2 u. \tag{2.4}$$

In Fig. 2 we have drawn the region P as a sum of rectangles because that is how we implemented the domain decomposition. On the basis of asymptotic analysis it is more natural to select a curve of the family $y^2 = -C\epsilon \log x$ as the upper boundary of P because

$$\eta = \frac{y^2}{\epsilon \log x} \tag{2.5}$$

is a similarity variable for (2.4). Note that the presence of the logarithm in (2.5) indicates that the turning points induce a widening of the parabolic boundary layer. In addition, the boundary-layer approximation (2.4) breaks down near the turning points.

In the vicinity of the corner $(x, y) = (1, 0)$ there is a birth region E_1 for this boundary layer, in which we must keep the full equation (2.1). The diameter of the subdomain E_1 is of order $O(\epsilon)$ as $\epsilon \to 0$.

There is no parabolic boundary layer analogous to P near the top boundary of Ω because the solution $u = 1$ of the reduced equation (2.2) is compatible with the

boundary condition there. The subdomain on which the reduced equation (2.2) is a valid approximation to (2.1) is denoted by H in Fig. 2, and it is usually called the outer region.

The region surrounding the turning points $x = 0$ is divided into three parts, an ordinary boundary-layer region $\mathcal{O}$ on which (2.1) may be approximated by the differential equation

$$-x\,\partial_x u = \epsilon\,\partial_x^2 u \tag{2.6}$$

and two regions E_2 and E_3 in which none of the terms in (2.1) may be neglected. Because the change of variable $x = \sqrt{\epsilon}\,\xi$ in (2.6) produces a differential equation with coefficients independent of ϵ, we expect the width of the subdomain $\mathcal{O}$ to be of the order of $O(\sqrt{\epsilon})$ as $\epsilon \to 0$. The special behavior in E_2 arises from the interaction of the parabolic boundary layer in P with the turning points, and the subdomain E_3 is needed because of the incompatibility of the boundary data (2.3) in the vicinity of the corner $(x, y) = (0, 1)$.

3. Iterative Schemes in Individual Subdomains. In this section we describe the different local approximations to (2.1) which are used in the various subdomains. On many subdomains our difference scheme is an iterative method to improve a given approximate solution u_{n-1}. In Section 4 we present our global iterative method, which may be regarded as a form of matched asymptotic expansions without the algebraic manipulations normally associated with matched asymptotics.

Let us first select difference operators, given mesh sizes h_x and h_y in the x- and y-directions, respectively. On all of the subdomains we use a central finite-difference approximation to $\partial_y^2 u$,

$$D_y^2 u(x, y) = \frac{u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)}{h_y^2}. \tag{3.1}$$

On most of the subdomains we use an upstream approximation to $\partial_x u$, namely,

$$D_x^+ u(x, y) = \frac{u(x + h_x, y) - u(x, y)}{h_x} \tag{3.2}$$

and a central-difference approximation to $\partial_x^2 u$,

$$D_x^2 u(x, y) = \frac{u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)}{h_x^2}. \tag{3.3}$$

The exceptional subdomains on which we use different approximations to the partial derivatives in the x-direction are the regions E_2, $\mathcal{O}$, and E_3 adjacent to the turning points. There, in the spirit of Chin and Krasny [4], we approximate the operator $x\,\partial_x + \epsilon\,\partial_x^2$ by a finite-element method based on solutions of

$$-x\,\partial_x u = \epsilon\,\partial_x^2 u.$$

Thus, for a grid $\{x_j\}$ with $j = 0, 1, \ldots, J$ our fundamental basis spline is

$$\begin{aligned}
\phi_j(x) &= \frac{\operatorname{erf}\left\{\frac{x}{\sqrt{2\epsilon}}\right\} - \operatorname{erf}\left\{\frac{x_{j-1}}{\sqrt{2\epsilon}}\right\}}{\operatorname{erf}\left\{\frac{x_j}{\sqrt{2\epsilon}}\right\} - \operatorname{erf}\left\{\frac{x_{j-1}}{\sqrt{2\epsilon}}\right\}} \qquad \text{for } x_{j-1} < x < x_j, \\
\phi_j(x) &= \frac{\operatorname{erf}\left\{\frac{x_{j+1}}{\sqrt{2\epsilon}}\right\} - \operatorname{erf}\left\{\frac{x}{\sqrt{2\epsilon}}\right\}}{\operatorname{erf}\left\{\frac{x_{j+1}}{\sqrt{2\epsilon}}\right\} - \operatorname{erf}\left\{\frac{x_j}{\sqrt{2\epsilon}}\right\}} \qquad \text{for } x_j < x < x_{j+1}, \\
\phi_j(x) &= 0, \qquad \text{elsewhere.}
\end{aligned} \tag{3.4}$$

We define the operator L as the finite-element approximation to $-x\,\partial_x - \epsilon\,\partial_x^2$ obtained from using the basis functions ϕ_j in (3.4).

It might seem reasonable that in the outer region H, where (2.2) is the asymptotic behavior, we use an analogue of (1.7)

$$-xD_x^+ u_n = \epsilon\left(D_x^2 + D_y^2\right) u_{n-1}.$$

It turns out, however, that the stability conditions for this iterative method are quite severe, namely that

$$\epsilon < Ch_x^2 \min_H x$$

for some positive constant C. With the notation $Iu = u$ and $T_y u(x,y) = u(x, y + h_y)$, we use in H a scheme which we find to be stable under a wider range of conditions

$$\left(-xD_x^+ + \frac{2\epsilon}{h_y^2} I\right) u_n = \epsilon\left(D_x^2 + \frac{1}{h_y^2}(T_y + T_y^{-1})\right) u_{n-1}. \tag{3.5}$$

In the parabolic layer region P we use

$$\left(-xD_x^+ - \epsilon D_y^2\right) u^n = \epsilon D_x^2 u^{n-1}. \tag{3.6}$$

In the birth region of the boundary layer E_1 we simply discretize (2.1) with a fine grid,

$$-xD_x^+ u_n = \epsilon\left(D_x^2 + D_y^2\right) u_n. \tag{3.7}$$

In the ordinary layer region O we use

$$Lu_n = \epsilon D_y^2 u_{n-1}, \tag{3.8}$$

where L is the finite-element approximation to the operator $-x\,\partial_x - \epsilon\,\partial_x^2$ obtained from the splines (3.4). Finally, in regions E_2 and E_3 we use the approximation

$$Lu_n = \epsilon D_y^2 u_n. \tag{3.9}$$

4. The Global Iterative Algorithm. Our global iterative method combines the local iterations (3.5–9) as follows.

1. Make an initial domain decomposition based on asymptotic information about the location and size of the layer regions.
2. Set $n = 0$ and provide an initial approximate solution u_0. (One reasonable choice is the outer solution $u_0 = 1$.)
3. Proceed through the subdomains in the following order. Use (3.5) to find u_n on H. Use (3.7) to find u_n on E_1, using (3.6) as the boundary condition on the downstream boundary. Solve (3.6) on P to determine u_n there. Then solve (3.8) on O. Finally use (3.9) to determine u_n on E_2 and E_3.
4. Check for convergence. If we aren't finished, check u_n for consistency with (2.1) across subdomain boundaries, modify the domain decomposition if necessary, increment n, and return to Step 3.

5. Comments. The primary concerns about any numerical method are accuracy and efficiency. Our scheme derives its efficiency from the fact that difficult matrix problems are solved only on the subdomains $\bigcup_{j=1}^{3} E_j$. (We use a multigrid method

for these problems.) On the largest subdomain H we solve an uncoupled system of ordinary differential equations (3.5). This is not only fast, but it is also highly parallel. The two-point boundary-value problems (3.8) on $\mathcal{O}$ are also highly parallel. Admittedly, the algorithm (3.6) on P is quite sequential, but that is because the inherent nature of the asymptotic behavior there is a parabolic partial differential equation (2.4).

Efficiency also depends on the rate of convergence of the algorithm as $n \to \infty$. On the basis of computational experience and theory for problems with constant coefficients we have found that the rate of convergence of the scheme as a whole is limited by the behavior of (3.5) in H. In fact, for a constant-coefficient version of (3.5)

$$\left(-aD_x^+ + \frac{2\epsilon}{h_y^2} I\right) u_n = \epsilon \left(D_x^2 + \frac{1}{h_y^2}(T_y + T_y^{-1})\right) u_{n-1} \tag{5.1}$$

with $a > 0$ we have the following theorem [2] concerning the discrete l_2-norm of u_n.

Theorem 5.1. *For the iterative scheme* (5.1) *on the halfspace* $-\infty < x < 0$, $-\infty < y < \infty$ *with boundary data* $u_n(0, y) = 0$ *for every positive number* κ *there exists a continuous function* $C(\xi)$ *such that*

$$\|u_n\| \leq \kappa \|u_{n-1}\|$$

whenever

$$\frac{\epsilon}{ah_x} \leq C\left(\frac{h_x \epsilon}{ah_y^2}.\right)$$

Furthermore, there exist functions u_{n-1} *such that for any positive* δ *we have*

$$\|u_n\| \geq (1 - \delta)\kappa \|u_{n-1}\| \tag{5.2}$$

whenever

$$\frac{\epsilon}{ah_x} \geq C\left(\frac{h_x \epsilon}{ah_y^2}\right).$$

The proof is based on the Godunov-Ryabenky stability theory [8].

The significance of Theorem 5.1 for application to our problem is that the left-hand boundary of H must be sufficiently far from the y-axis for two reasons. On the one hand, we have a variable coefficient $a = x$, so that the theorem can only give heuristic information. Still, we are led by (5.2) to expect instability if x is too small in H. On the other hand, the derivative $|\partial_x^2 u|$ is large in the turning-point region, so that if we are using (3.5), then we need a small step size h_x to resolve the variations in u. This also has a destabilizing effect in (5.2). It should be noted that if the turning-point region $E_2 \cup \mathcal{O} \cup E_3$ is sufficiently wide, then neither of these effects is of concern. The solution u in H is so smooth that the mesh sizes may be taken to be large compared with $\epsilon / \min_H x$.

The criteria for determining the boundaries of P and E_j $(j = 1, 2, 3)$ are not stability, but accuracy and efficiency. If an E_j is larger than necessary, we do more computational work than we need to, but there is no degredation of accuracy. If, on the other hand, an E_j is too small, then we will have to do more global iterations in order to get convergence on the adjoining subdomains. Worse than that, we will also probably loose accuracy because fine meshes are used in the regions E_j in order to resolve the rapid variations of u there. Similar criteria govern the location of the boundary between P and H.

We recognize that classical numerical analysts are likely to regard our stability condition $\epsilon < Ch_x \min_H x$ as quite strange, but it makes sense for singular perturbation problems.

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