

A Parallel Solver for the Linear Elasticity Equations on a Composite Beam

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Abstract .

We report about parallel implementation on CRAY2 supercomputer of a parallel algorithm based on a domain decomposition method with Lagrange multiplier, for solving an ill conditioned three dimensional composite structural analysis problem, with as many as one million degrees of freedom .

We show that this method has very good features on both granularity and data dependency viewpoints. We explain the practical differences between this method and the standard domain decomposition method with Gaussian elimination of the degrees of freedom inside the subdomains .

The tests performed prove that the choice of the local solver is very important to get an efficient global method. For the studied case, it is clear that solving the local problems with a direct method is the best solution, and we give some reasons why it will be the same for many other problems .

1. Presentation of a structural analysis problem for a composite beam .

We consider the linear elasticity equations for a composite beam made of a little more than one hundred stiff fibers (carbon or iron) bound by an incompressible elastomer matrix .

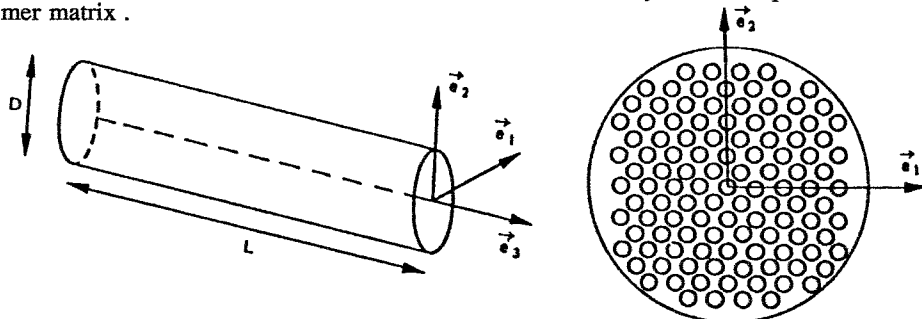


Fig. 1. Geometry of the beam .

Beam cross section .

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Homogenisation methods do not work for such a device with microscopic-scale discontinuity. But, due to the composite feature, the finite element mesh for solving the problem with discontinuous coefficients must be very refined to get a good representation of each substructure. This leads to a very large scale matrix, hence the problem can be solved only by iterative methods like the conjugate gradient method .

However, substructuring is very easy in the present case for the beam is obviously made of similar jointed composite "pencils". Hence, using a domain decomposition method for solving such a problem seems to be natural, because the decomposition is straightforward. Furthermore, all the subdomains are identical .

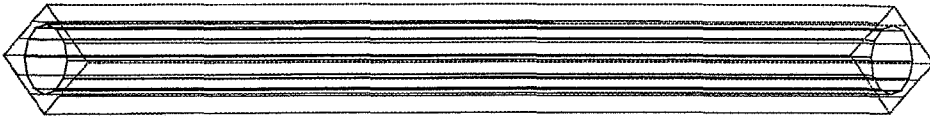


Fig. 2. A composite pencil .

It must be noticed that the problem we study is very ill conditioned for both geometrical and material reasons.

The material reasons are at first the composite feature. The Young moduli of the fiber have an order of magnitude 10^4 times the one of the elastomer. Secondly, the penalty method for the incompressibility condition in the elastomer makes the condition number increase with the penalty parameter .

The geometrical reason is linked to the fact we try to solve a beam problem, with a ratio of the length upon the width equal to 6, that makes the pure bending problem very ill conditioned .

2. The primal hybrid variational principle .

The domain decomposition method we used for solving the problem is based on the so-called primal hybrid variational principle, and consists in introducing a Lagrange multiplier to remove the continuity condition on the interface (see for instance [1] or [2]).

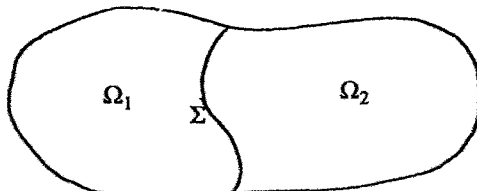
Let Ω be a bounded open subset of \mathbb{R}^3 with a smooth boundary Γ . The linear elasticity equations with homogeneous boundary conditions on Γ_0 a subset of Γ are :

$$\begin{cases} \mathbf{A}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma_0 \end{cases} \quad \text{with } (\mathbf{A}\mathbf{u})_i = - \frac{\partial(a_{ijkh}\epsilon_{kh}(\mathbf{u}))}{\partial x_j} . \quad (1)$$

The usual variational form of this problem consists in finding \mathbf{u} in $(H^1(\Omega))^3$ and satisfying the boundary condition $\mathbf{u} = 0$ on Γ_0 which minimizes the energy functional :

$$\mathbf{I}(\mathbf{v}) = \frac{1}{2}\mathbf{a}(\mathbf{v},\mathbf{v}) - (\mathbf{f},\mathbf{v}) \quad \text{with } \mathbf{a}(\mathbf{u},\mathbf{v}) = \int_{\Omega} a_{ijkh}\epsilon_{kh}(\mathbf{u})\epsilon_{ij}(\mathbf{v}) \, dx .$$

Let us consider a splitting of the domain Ω into two open subsets Ω_1 and Ω_2 with smooth boundaries Γ_1 and Γ_2 . Let us assume that the boundary of the interface $\Sigma = \Gamma_1 \cap \Gamma_2$ is included in Γ_0 so that the traces on Σ of the displacements fields \mathbf{u} satisfying the boundary condition $\mathbf{u} = 0$ on Γ_0 belong to the space $(H_0^{1/2}(\Sigma))^3$.



Solving the linear elasticity equation consists in finding two functions \mathbf{u}_1 and \mathbf{u}_2 defined on Ω_1 and Ω_2 , satisfying the boundary conditions on Γ_1 and Γ_2 which minimize the energies $\mathbf{I}_1(\mathbf{v})$ and $\mathbf{I}_2(\mathbf{v})$ with the continuity constraint : $\mathbf{v}_1 = \mathbf{v}_2$ on Σ .

The weak form of the continuity constraint is :

$$(\mathbf{v}_1 - \mathbf{v}_2, \boldsymbol{\mu})_{\Sigma} = 0 \text{ for any } \boldsymbol{\mu} \text{ in } (\mathbf{H}^{-1/2}(\Sigma))^3 .$$

One can show that the problem of minimization with constraint above is equivalent to finding the saddle-point of the Lagrangian :

$$\mathbf{L}(\mathbf{v}, \boldsymbol{\mu}) = \mathbf{I}_1(\mathbf{v}_1) + \mathbf{I}_2(\mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2, \boldsymbol{\mu})_{\Sigma} . \tag{3}$$

This is equivalent to find the fields $(\mathbf{u}_1, \mathbf{u}_2)$ in $(\mathbf{H}^1(\Omega_1))^3 \times (\mathbf{H}^1(\Omega_2))^3$ and the Lagrange multiplier $\boldsymbol{\lambda}$ in $(\mathbf{H}^{-1/2}(\Sigma))^3$ which satisfy :

$$\mathbf{L}(\mathbf{u}, \boldsymbol{\mu}) \leq \mathbf{L}(\mathbf{u}, \boldsymbol{\lambda}) \leq \mathbf{L}(\mathbf{v}, \boldsymbol{\lambda}) ,$$

for any field $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ in $(\mathbf{H}^1(\Omega_1))^3 \times (\mathbf{H}^1(\Omega_2))^3$, and any $\boldsymbol{\mu}$ in $(\mathbf{H}^{-1/2}(\Sigma))^3$.

The left inequality imposes $(\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\mu})_{\Sigma} \leq (\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda})_{\Sigma}$ and so $(\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\mu})_{\Sigma} = 0$ for any $\boldsymbol{\mu}$ in $(\mathbf{H}^{-1/2}(\Sigma))^3$, thus the continuity constraint is satisfied by the solution of the saddle-point problem .

The right inequality implies $\mathbf{I}_1(\mathbf{u}_1) + \mathbf{I}_2(\mathbf{u}_2) \leq \mathbf{I}_1(\mathbf{v}_1) + \mathbf{I}_2(\mathbf{v}_2)$ for any $(\mathbf{v}_1, \mathbf{v}_2)$ in $(\mathbf{H}^1(\Omega))^3$. It means that $(\mathbf{u}_1, \mathbf{u}_2)$ minimizes the sum of the energies on Ω_1 and Ω_2 among the fields satisfying the continuity requirement. Hence \mathbf{u}_1 and \mathbf{u}_2 are the restrictions to Ω_1 and Ω_2 of the solution of the linear elasticity equation on Ω .

The classical variational interpretation of the saddle-point problem (3) leads to the equations :

$$\begin{cases} \mathbf{A}_1 \mathbf{u}_1 + \mathbf{B}_1^* \boldsymbol{\lambda} = \mathbf{f}_1 & \text{in } \Omega_1 \\ \mathbf{u}_1 = 0 & \text{on } \Gamma_0 \cap \Gamma_1 \\ \mathbf{A}_2 \mathbf{u}_2 - \mathbf{B}_2^* \boldsymbol{\lambda} = \mathbf{f}_2 & \text{in } \Omega_2 \\ \mathbf{u}_2 = 0 & \text{on } \Gamma_0 \cap \Gamma_2 \\ \mathbf{B}_1 \mathbf{u}_1 - \mathbf{B}_2 \mathbf{u}_2 = 0 & \text{on } \Sigma \end{cases} . \tag{4}$$

where \mathbf{A}_1 and \mathbf{A}_2 are the differential operators of the linear elasticity equations on Ω_1 and Ω_2 , while \mathbf{B}_1 and \mathbf{B}_2 the trace operators onto Σ of functions belonging to $(\mathbf{H}^1(\Omega_1))^3$ and $(\mathbf{H}^1(\Omega_2))^3$.

The analysis of these equations shows that the Lagrange multiplier $\boldsymbol{\lambda}$ is in fact equal to the interaction force between the substructures along their common boundary .

On a mechanical point of view, it means that it is necessary to know the forces on the interface to obtain local independent displacements problem .

3. An iterative algorithm based on the hybrid finite element method .

A discretisation with Lagrangian finite elements for the displacements fields and for the interface constraints of the hybrid formulation (4) leads to the same equations where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1$ and \mathbf{B}_2 are the matrices associated with the discrete linear elasticity and trace operators .

$$\begin{cases} \mathbf{A}_1 \mathbf{u}_1 + \mathbf{B}_1^t \boldsymbol{\lambda} = \mathbf{f}_1 \\ \mathbf{A}_2 \mathbf{u}_2 - \mathbf{B}_2^t \boldsymbol{\lambda} = \mathbf{f}_2 \\ \mathbf{B}_1 \mathbf{u}_1 - \mathbf{B}_2 \mathbf{u}_2 = 0 \end{cases} \tag{5}$$

By substitution in the equation (5) the problem can be written with respect to $\boldsymbol{\lambda}$ only :

$$\left[\mathbf{B}_1 * \mathbf{A}_1^{-1} * \mathbf{B}_1^t + \mathbf{B}_2 * \mathbf{A}_2^{-1} * \mathbf{B}_2^t \right] * \boldsymbol{\lambda} = \mathbf{B}_1 * \mathbf{A}_1^{-1} * \mathbf{f}_1 - \mathbf{B}_2 * \mathbf{A}_2^{-1} * \mathbf{f}_2$$

Hence λ satisfies the following equation :

$$C * \lambda = b \tag{6}$$

where $C = \begin{bmatrix} B_1, -B_2 \end{bmatrix} * \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} * \begin{bmatrix} B_1' \\ -B_2' \end{bmatrix}$ and $b = \begin{bmatrix} B_1, -B_2 \end{bmatrix} * \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} * \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

The the discrete displacements spaces V_i , and the discrete Lagrange multiplier space M are chosen in order to satisfy the Babuska-Brezzi-Ladyzhenskaya condition :

$$\inf_{|\mu|_M=1} \sup_{|u_i|_{V_i}=1} (B_i * u_i, \mu) = \beta > 0 .$$

Hence the C matrix is symmetric, positive definite, and so the equation (6) has only one solution .

We refer to [1] or [2] for the now classical error estimates for the hybrid formulations of elliptic differential equations .

The resolution by the conjugate gradient method of the interface problem (6) leads to a parallel algorithm with a very good granularity .

Let μ be a vector, computing the product $\xi = C * \mu$ involves the following three steps .

Step one :

$$\text{computation of the product , } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} B_1' \\ -B_2' \end{bmatrix} * \begin{bmatrix} \mu \\ \mu \end{bmatrix} ,$$

that means computing two independent local vector-matrix products .

Step two :

$$\text{computation of the product , } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A_1^{-1} \\ A_2^{-1} \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} ,$$

that means computing the solution of two independent local sets of linear equations associated to local linear elasticity problems with Neuman boundary condition on the interface Σ .

Step three :

computation of the variation on the interface of the displacements fields w_1 and w_2 ,

$$\xi = \begin{bmatrix} B_1, -B_2 \end{bmatrix} * \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = B_1 * w_1 - B_2 * w_2$$

The simplest choice for the space M , consists in having on each interface vertex as many degrees of freedom for the multiplier as for the trace of the discrete displacements fields. In this case, the B_1 and B_2 operators are simply the operators of restrictions of the discrete displacements fields .

Then the condition, $B_1 u_1 - B_2 u_2 = 0$ on Σ , is equivalent to the condition : $u_1 = u_2$ for any degree of freedom located on Σ . So the first and third steps of the computation of the product by the dual matrix, are quite simple and involves only few operations .

In this case, it is straightforward to see that the dual matrix C is elliptic, and the solution (u_1, u_2) of the hybrid problem (5) is equal to the solution of the standard conforming problem on the whole domain .

Obviously, the main part of the computation for the resolution of the hybrid problem (5) by the conjugate gradient method consists in the step two of the matrix-vector product, i.e. the resolution of local independent elasticity problems, which can be performed in parallel. The granularity is good, because data transfers involve only the boun-

dary degrees of freedom, as the computations consist in the resolution of problems in the whole subdomains and require much more operations .

4. Comparisons with the Schur complement method .

The parallel algorithm associated with the hybrid method has the same kind of features as the Schur complement method, presented for instance in [3] or [4]. Both methods consist in solving a condensed problem on the interface, and when solving this problem by a gradient method, the main part of the computations consist in the resolution at each step of the outer iterative scheme of local independent problems in each subdomain .

The differences between the two algorithms lie in the facts that the Schur complement method is a conforming method, and that the local independent problems have Dirichlet boundary conditions for this method and Neuman boundary conditions for the hybrid method .

The first consequence is that the topology of the interface can be simpler with the hybrid method .

Two subdomains are neighbours for the hybrid method iff they have a common degree of freedom of the Lagrange multiplier. As the multiplier is associated with the weak formulation of the continuity constraint, $(v_1 - v_2, \mu)_\Sigma = 0$ for any μ in $(H^{-1/2}(\Sigma))^3$, its degrees of freedom have to be introduced only on interfaces between subdomains with a non zero integral .

Thus, for a decomposition of a two-dimensional problem into quadrangles, subdomains are neighbours for the hybrid method only if they have a common edge whereas a common vertex is enough for the conforming Schur complement method .

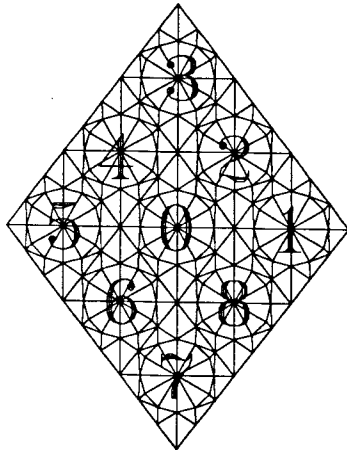


Fig. 3. Mesh of the bottom section for 9 pencils .

This feature makes the domain decomposition method with Lagrange multiplier more suitable for parallel machines with distributed memory. Furthermore, it would allow substructuring with real three-dimensional topology more easily than the Schur complement method .

The second consequence is linked with the boundary conditions of the local problems to be solved at each step of the outer gradient scheme .

With the Schur complement method, we get Dirichlet boundary conditions. So, the displacements degrees of freedom located on the interface are fixed. At the opposite, with

In the case of the decomposition of a cylinder into pencils, the topology is the same as for the two-dimensional problem for a cross section.

On the figure, we show the mesh on the bottom section of a nine pencils domain.

With the hybrid method, the pencil number 0 in the middle has only four neighbours, one for each edge, the subdomains 2,4,6 and 8, whereas all the eight surrounding pencils are neighbouring for the conforming domain decomposition method .

the hybrid method, we have prescribed stresses on the interface, that means Neuman boundary conditions .

Except for the clamped bottom section, the boundary conditions we had to assume, for the beam problem we tackled, were prescribed stresses for the top section, and free boundary conditions for the beam sides .

So, when substructuring the domain into geometrically identical subdomains, each one made of one or more pencils, we get identical local matrices for the local problems with the hybrid methods. With the Schur complement method, when substructuring in a few subdomains, for instance nine subdomains like in the figure above, the matrices of the local problems can be all different, because the fixed degrees of freedom associated with the Dirichlet boundary conditions are not the same in each subdomain .

5. Results of the tests performed with the CRAY2 .

We performed tests for domains with the same features as the composite beam presented in the first section, with less fibers but with the same kind of geometry, in order to get condition numbers of the same order of magnitude but not too many degrees of freedom .

The first lesson with these tests concerns the choice of the solver for the local independent problems .

When trying to solve them with an iterative method, the conjugate gradient method, the algorithm we got was sometimes as many as three times more expensive than the standard global diagonal scaled conjugate gradient method for the complete domain .

The explanation lies in the fact that the ratio of the length upon the width of the subdomains, here they were pencils, was greater than for the complete domain, and so was the condition number, in such a way that the conjugate gradient method was not an efficient algorithm to solve the local problems .

This problem could appear in many other cases, because when substructuring a domain to use any domain decomposition method, one try to locate the interfaces in regions in which the solution is expected to be smooth, in order to keep the number of iterations needed for the outer iterative scheme as low as possible. This can lead to substructures with such an aspect that the local matrices are ill conditioned for geometrical reasons, particularly for structural analysis problems .

Finally, for ill conditioned problems it seems safer to use a direct local solver. Moreover, as each local problem has to be solved many times, the cost of the L*U decomposition of the matrices do not justify the use of iterative solvers .

The second lesson concerns the number of subdomains .

In all tests we performed we noticed that the number of outer iterations needed to solve the global problem grew much faster than the number of subdomains .

For instance, when trying to solve the same problem for 36 pencils with 9 subdomains made of 4 pencils, the number of outer iterations can be 10 times greater than for a decomposition in 4 subdomains, each made of 9 pencils, in such a way that the global time is 5 times longer, though each iteration of the hybrid method is less expensive with more numerous and smaller subdomains .

Hence, for solving the complete problem presented in the first section we took subdomains as large as possible, according to geometric constraints and memory requirements, that meant in practice subdomains made of nine fibers, with more than one million degrees of freedom .

6. Conclusions .

The tests we performed prove the ability of the domain decomposition method with

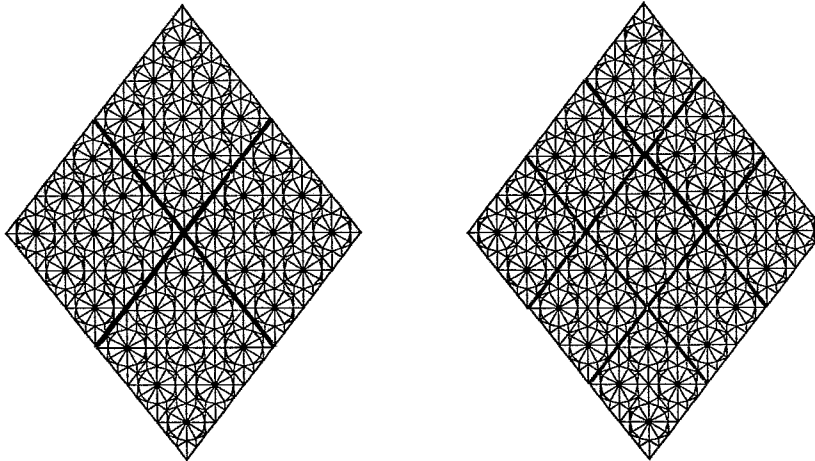


Fig. 4. Decomposition in 4 subdomains . Decomposition in 9 subdomains .

Lagrange multiplier to solve some large scale ill conditioned problems of structural analysis on parallel supercomputers .

With four subdomains, the speed-up we got on the CRAY2, with four processors and 256 megawords of memory, measured by the comparison of ellapsed times on a dedicated machine, was over 3.85 . And when using a vectorised direct local solver, each processor ran at more than one hundred megaflops, that leads to a global computation speed over 400 megaflops .

A remaining question is the problem of finding preconditioners for the dual interface problem, like the ones studied in [4] or [5] for the Schur complement method, to make this algorithm a really robust one .

References .

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