CHAPTER 14

An Additive Schwarz Algorithm for Two- and Three-Dimensional Finite Element Elliptic Problems*

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Abstract. In this paper, we present an additive variant of the Schwarz alternating method for finite element approximations of two and three dimensional elliptic problems with mixed boundary conditions. We establish that the rate of convergence of this method is optimal even when the number of subregions is large. In our analysis techniques previously developed for iterative substructuring methods are used. All these methods show great promise for parallel computers. The reported results have been recently obtained in joint work with Olof Widlund.

1. Introduction. In this paper, we will consider conforming finite element discretizations of linear, self adjoint two and three dimensional second order elliptic equations with mixed boundary conditions. Our aim is to develop and study an additive variant of the classical Schwarz alternating method, see [8], and in particular to discuss the case of many subregions. The method is described in terms of projections from the finite element space onto subspaces related to overlapping subregions which cover the region. These projections are defined by using the symmetric bilinear form associated with the original elliptic problem. In addition, a special subspace defined on a coarse triangulation is introduced. It provides a mechanism for the global transportation of information which is necessary to obtain fast convergence; cf. [9] for a discussion of this matter in the context of iterative substructuring methods. The additive form of the Schwarz algorithm is thus an iterative method for solving an equation with a symmetric and positive definite operator which is a sum of the projections. This permits the use of the conjugate gradient method. We have proved, see the theorem in section 3, that the rate of convergence is optimal

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even when the region is divided into many subregions. In the last section, we discuss the question of inexact solvers in additive Schwarz algorithms.

The methods described are interesting for parallel computations particularly for computers with many processors. The subproblems can be solved separately and at the same time. The methods also offer real benefits even when one processor is used, particularly when the subregions have simple geometry. We note that compared with iterative substructuring methods, see [1], [2], [3] and literature therein, the additive Schwarz algorithms satisfy somewhat better asymptotic bounds.

In this paper, we report on certain extensions of our report [4]. We note that our work was inspired by the paper [6] in which a variational framework for the classical multiplicative Schwarz method for continuous elliptic problems is discussed. As for earlier work on additive algorithms of this kind see [7] in which certain iterative substructuring methods on regions divided into a fixed number of subregions are discussed.

2. Statement of problems. We consider the following weak form of a linear, self-adjoint second order elliptic equation with mixed boundary conditions. Let Ω be a bounded Lipschitz region in R^n, $n = 2$ or 3. We assume that $\partial\Omega$, the boundary of Ω, is the union of two nonoverlapping sets Γ_D and Γ_N on which zero Dirichlet and arbitrary Neumann boundary conditions are given. We introduce the Sobolev space $V(\Omega) \subset H^1(\Omega)$:

$$V(\Omega) := \{v \in H^1(\Omega) : \gamma_D v = 0\}$$

where $\gamma_D v$ is the trace of v on Γ_D.

The problem takes the form: Find $u \in V(\Omega)$ such that

$$a(u,v) = \ell(v), \quad v \in V(\Omega) \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \sum_{i,j} a_{ij}(x) D_i u D_j v \, dx$$

and $\ell(v)$ is a linear continuous functional defined on $V(\Omega)$. We assume that the bilinear form $a(u,v)$ is symmetric and positive definite, i.e.

$$\begin{aligned} &a(v,u) = a(u,v), \\ &c\|u\|^2_{H^1(\Omega} \leq a(u,u), \quad u \in V \end{aligned} \tag{2.2}$$

where c is a positive constant.

The problem (2.1) is approximated by a conforming finite element method. To simplify the presentation we assume that Ω is a polyhedral (polygonal) region and that it is divided into tetrahedral (triangular) elements e_i. We assume that the curves where the boundary conditions change consist of edges of elements. We also assume that the triangulation of Ω is regular, see [2]. Let V^h denote a conforming finite element space defined on the given triangulation with zero values on Γ_D. The discrete problem is of the form:

Find $u_h \in V^h$ such that

$$a(u_h, v_h) = \ell(v_h), \quad v_h \in V^h(\Omega) \tag{2.3}$$

Our aim is to solve (2.3) by an additive version of the Schwarz alternating method.

3. The Additive Schwarz Algorithm. In this section we give a description of the additive Schwarz algorithm for solving (2.3). We assume that triangulation of Ω can be subdivided into nonoverlapping tetrahedral (triangular) subregions Ω_i, $i = 1, \ldots, N$, called substructures, in such a way that the boundary of Ω_i follows element boundaries. We assume that the substructures form a regular triangulation of Ω, with a parameter H in the sense of finite element theory. We next extend each substructures Ω_i to a larger region Ω_i'. In this way we get the partitioning of Ω into overlapping subregions Ω_i' needed in Schwarz-type domain decomposition methods. We assume that the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by a fixed fraction of H, and that $\partial\Omega_i'$ does not cut through any element. In the case when part of Ω_i' is outside of Ω we cut off that part and denote the resulting subregion by Ω_i'.

We represent the finite element space V^h as the sum of $N+1$ subspaces

$$V^h = V_0^h + V_1^h + \ldots + V_N^h \tag{3.1}$$

The first subspace V_0^h is the space of continuous, piecewise linear functions defined on the coarse triangulation with parameter H. It is also called V^H and plays a special role in the algorithm. The remaining subspaces V_i^h are associated with the subregions Ω_i' and are defined as follows. For interior subregions Ω_i'

$$V_i^h(\Omega) = \{v \in V^h \cap H_0^1(\Omega_i') : v(x) = 0, \ x \in C\Omega_i'\}$$

where $C\Omega_i'$ is the complement of Ω_i' with respect to the region Ω. In the case when $\partial\Omega_i'$ intersects $\partial\Omega$, we use the original boundary condition on $\partial\Omega_i \cap \partial\Omega$ and zero Dirichlet conditions at all other points of $\partial\Omega_i'$. We see that $V_i^h(\Omega) \subset V^h$ and that the representation (3.1) of V^h is possible.

We now introduce the projections $P_i : V^h \to V_i^h$ with respect to the bilinear form $a(u,v)$. By definition $P_i v_h \in V_i^h$ is the unique element of V^h satisfying the equation

$$a(P_i v_h, \phi_h) = a(v_h, \phi_h), \ \phi_h \in V_i^h. \tag{3.2}$$

We note that $P_i v_h$ can be computed, for any $v_h \in V^h$, at the expense of solving the finite element problem in the subspace V_i^h. For u_h, the solution of (2.3) can be computed by solving

$$a(P_i u_h, \phi_h) = \ell(\phi_h), \quad \phi_h \in V_i^h,$$

since

$$a(u_h, u_h) = \ell(\phi_h).$$

We can therefore replace the problem (2.3) by the equation

$$Pu_h \equiv (P_0 + P_1 + \cdots + P_N)u_h = g_h, \tag{3.3}$$

where the right hand side g_h can be computed as $g_h = \sum_{i=0}^{N} g_{h,i}$, $g_{h,i} = P_i u_h$. Inorder to establish that equations (2.3) and (3.3) have the same solutions, we need only show that P is invertible. This follows from our theorem given below.

The additive form of the Schwarz algorithm for solving (2.3) is simply an iterative method for solving equation (3.3). A natural choice is the conjugate gradient method; cf. [5]. Our main result, cf. [4], is that P is uniformly well conditioned.

Theorem. *For any $v_h \in V^h$ the following inequalities hold*

$$C_0 |v_h|^2_{H^1(\Omega} \leq |Pv_h|^2_{H^1(\Omega)} \leq C_1 |v_h|^2_{H^1(\Omega)} \tag{3.4}$$

where C_0 and C_1 are positive constant independent of H and h.

The derivation of a conjugate gradient method for equation (3.3) is a relatively routine matter. Details will therefore not be given here. We only note that equation (3.3) requires no further preconditioning but that it is important to work with the inner product defined by the bilinear form $a(u, v)$ since the operator P is symmetric with respect to that bilinear form but not the Euclidean inner product. In this respect this iterative method differs from the standard preconditioned conjugate gradient method, where the quadratic form associated with the preconditioner as well as the Euclidean inner product are used.

4. The Use of Preconditioners. In the algorithm described, the subproblems on Ω_i' are solved directly, see (3.2) and (3.3). These subproblems can also be solved inexactly using another bilinear form $b(u, v)$. If this problem satisfies

$$C_2 b(u, v) \leq a(u, v) \leq C_3 b(u, v), \quad u, v \in V_i^h(\Omega) \tag{3.7}$$

with positive constants C_2 and C_3 independent of H and h, then the condition number of the resulting method is bounded by $\kappa(P) C_3 / C_2$. We can, for example, use

$$b(u, v) = \int_\Omega \nabla u \cdot \nabla v \; dx$$

which satisfies (3.7). We use the bilinear form $b(u, v)$, when we solve the equations in the subspaces, and the original form $a(u, v)$ when we evaluate the residuals in the iterative method. Otherwise the algorithm is unchanged.

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