Domain Decomposition Algorithms for the Stokes Equations

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Abstract We show that the capacitance (or Schur complement) matrix C associated with a multidomain finite element approximation to the Stokes equations is symmetric and positive definite. We then propose several preconditioners which are symmetric, positive definite, and spectrally equivalent to C. Their analysis is first carried out for decomposition by two subdomains, then is extended to cover the case of strips (M adjoint subdomains, see Fig.5.1) and boxes (four subdomains sharing an internal vertex, see Fig.5.2). As stated in section 1, despite most of this paper is concerned with finite element approximation of the Stokes equations, the arguments here developed can be applied to different kind of approximations (e.g., those based on spectral methods), as well as to different kind of boundary value problems.

1. Finite dimensional approximations to boundary value problems. Let V and Q be two Hilbert spaces, with norm $\|\cdot\|$ and $|\cdot|$ respectively. We consider the problem:

$$(1.1)\qquad \begin{cases} \text{find } u \in V,\, p \in Q \text{ s.t.} & \\ a(u,v) + b(v,p) = f(v) & \forall\, v \in V \\ b(u,q) \qquad\quad = g(q) & \forall\, q \in Q \end{cases}$$

where $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are two bilinear and continuous forms, and f and g are two linear functionals defined on V and Q respectively. We assume that there exist two strictly positive constants α, β such that:

$$(1.2)\qquad \forall\, v \in V \quad a(v,v) \geq \alpha \|v\|^2 \;; \qquad \forall\, q \in Q \quad \sup_{v \in V} b(v,q)/\|v\| \geq \beta |q|$$

These properties ensure that problem (1.1) is well posed.

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We recall that (1.1) is the general setting for the variational formulation of the incompressible Stokes equations in $\Omega \subset \mathbb{R}^n$. In such case, u denotes the velocity field, p is the pressure,

$$a(u,v) = \upsilon \int_\Omega \nabla u \cdot \nabla v, \qquad b(v,q) = -\int_\Omega q \operatorname{div} v,$$

where $\upsilon > 0$ is the viscosity, $V=[H^1_0(\Omega)]^n$ and $Q = L^2(\Omega)/\mathbb{R}$ (see, e.g., [6]). The second equation of (1.1) (with $g \equiv 0$) enforces the divergence free constraint on u. Second and fourth order elliptics equations can also be cast into the same setting (with $f \equiv 0$) by letting either u or p be the primitive variable, and the others be ∇u or Δu, respectively (such formulation is suitable in view of approximation by mixed finite elements, see [4]). However, in order to keep our presentation plain, from now on we will specifically refer to (1.1) as to the variational form of the Stokes equations.

Let now V_h and Q_h be two finite dimensional subspaces of V and Q respectively. We introduce the following approximation to (1.1):

$$(1.3) \qquad \begin{cases} \text{find } u_h \in V_h,\ p_h \in Q_h & \text{s.t.} \\ a(u_h,v) + b(v,p_h) = f(v) & \forall v \in V_h \\ b(u_h,q) = g(q) & \forall q \in Q_h \end{cases}$$

We assume that the following *inf-sup condition* holds:

$$(1.4) \qquad \forall q \in Q_h \quad \sup_{v \in V_h} b(v,q)/\| v \| \geq \beta_h | q |$$

where $\beta_h > 0$ might depend on h. Under this assumption, the problem (1.3) has a unique solution, and the following error bound holds (see [3]):

$$(1.5) \qquad \| u - u_h \| + | p - p_h | \leq C(\beta_h) \{ \inf_{v_h \in V_h} \| u - v_h \| + \inf_{q_h \in Q_h} | p - q_h | \}$$

Remark 1.1 Finite element approximations to the Stokes equations that satisfy (1.4) with β_h independent of the finite element mesh size h are numerous (see, e.g., [6] and [4]). In such cases, the bound (1.5) yields optimal convergence estimates. Some Fourier-Legendre spectral approximations that satisfy (1.4) with β_h independent of the polynomial degree of the spectral solutions are also known ([7] and [11]). More generally however, spectral collocation approximation (and finite element approximation numerical integration), using can be cast in the framework (1.3) provided two discrete bilinear forms a_h and b_h are used instead of a and b. Further, for spectral Chebyshev approximation, the bilinear form intervening in the momentum equation is actually different than that used in the continuity equation. For these general cases, the inequality (1.5) becomes much more complicated (see, e.g., [6], [1]), and the forthcoming discussion and relative results should be modified accordingly.

In matrix notation, the problem (1.3) can be written as

$$(1.6) \qquad \begin{cases} \mathbf{Au} + \mathbf{B}^T\mathbf{p} = \mathbf{f} \\ \mathbf{Bu} = \mathbf{g} \end{cases}$$

where: **u** (resp. **p**) is the vector of the values of u (resp p) at the grid points,

$$a_{ij} = a(\varphi_j, \varphi_i) \ , \ b_{lm} = b(\varphi_m, q_l) \ , \ f_i = f(\varphi_i) \ , \ g_l = g(q_l)$$

and $\{\varphi_i\}$ (resp $\{q_l\}$) is the Lagrange basis of V_h (resp Q_h) relative to the grid points of the finite dimensional approximation.

2. The domain decomposition formulation and the associated capacitance matrix. We assume now that (1.1) is a boundary value problem set in an open domain Ω whose boundary is $\partial\Omega$. We make the assumption that Ω is partitioned into two disjoint subdomains Ω_1 and Ω_2 whose common boundary will be denoted by Γ. Despite most part of the forthcoming discussion applies to general approximations of the form (1.3), from now on we will explicitly refer to finite element approximations only. In this framework we require that each element of the decomposition does not cross Γ, i.e., it is contained in either Ω_1 or Ω_2. Then, for k = 1, 2 we denote by $V_{h,k}$ (resp $Q_{h,k}$) the space of the restrictions of the elements of V_h (resp Q_h) to Ω_k and by Φ_h the space of restrictions to Γ of the elements of V_h. Finally, we denote by $V^*_{h,k}$ the subspace of $V_{h,k}$ of those functions that vanish on Γ. It is proven in [9] that the single-domain finite element problem (1.3) is equivalent to the following multi-domain problem:

find $u_{h,k} \in V_{h,k}$, $p_{h,k} \in Q_{h,k}$, $k = 1, 2$ s.t.

(2.1) $a_1(u_{h,1}, v) + b_1(v, p_{h,1}) = f_1(v) \qquad \forall v \in V^*_{h,1}$

(2.2) $b_1(u_{h,1}, q) = g_1(q) \qquad \forall q \in Q_{h,1}$

(2.3) $u_{h,1} = u_{h,2} \qquad \text{on } \Gamma$

(2.4) $a_1(u_{h,1}, \rho_1\varphi) + b_1(\rho_1\varphi, p_{h,1}) + a_2(u_{h,2}, \rho_2\varphi) + b_2(\rho_2\varphi, p_{h,2}) = f_1(\rho_1\varphi) + f_2(\rho_2\varphi) \ \forall \varphi \in \Phi_h$

(2.5) $a_2(u_{h,2}, v) + b_2(v, p_{h,2}) = f_2(v) \qquad \forall v \in V^*_{h,2}$

(2.6) $b_2(u_{h,2}, q) = g_2(q) \qquad \forall q \in Q_{h,2}$

Here $\rho_k\,\varphi \in V_{h,k}$ is the interpolant extension of $\varphi \in \Phi_h$ to Ω_k, i.e., $\rho_k\,\varphi = \varphi$ on Γ, $\rho_k\,\varphi = 0$ at each finite element node internal to Ω_k, while a_k, b_k, f_k and g_k are the restrictions of a, b, f and g, respectively, to Ω_k. Actually, one has $u_{h,k} = u_{h|\Omega k}$, $p_{h,k} = p_{h|\Omega k}$, $k = 1, 2$, provided the pressures space Q_h is made of discontinuous functions across the interelement boundaries. In the case where $Q_h \subset C^\circ(\Omega)$, (2.1)-(2.6) is no more equivalent to (1.3).
The matrix representation of (2.1)-(2.6) is as follows. Denote, for each k=1,2, by $\{\varphi^k{}_i\}$, $\{q^k{}_j\}$ and $\{\psi_m\}$ the finite element Lagrange bases of $V^*_{h,k}$, $Q_{h,k}$ and Φ_h, respectively, and by $\mathbf{U}^\circ{}_k$, $\mathbf{P}_k$ and $\mathbf{U}_3$ the vectors of the corresponding finite element unknowns.
Then (2.1)-(2.6) is equivalent to the linear system of Fig.2.1, where, for k = 1, 2:

$$A_{kk}(N_k \times N_k) : (A_{kk})_{ij} = a_k(\varphi^k{}_j, \varphi^k{}_i) \ , \ B_{kk}(M_k \times N_k) : (B_{kk})_{ij} = b_k(\varphi^k{}_j, q^k{}_i)$$

(2.7) $$A_{k3}(N_k \times N_3) : (A_{k3})_{ij} = a_k(\rho_k\,\psi_j, \varphi^k{}_i) \ , \ B_{k3}(M_k \times N_k) : (B_{k3}) = b_k(\rho_k\,\psi_j, q^k{}_i)$$

$$A^k{}_{33}(N_3 \times N_3) : (A^k{}_{33})_{ij} = a_k(\rho_k\,\psi_j, \rho_k\,\psi_i), \ \ A_{33} = A^1{}_{33} + A^2{}_{33}$$

$$(\mathbf{F}_k)_i = f_k(\varphi^k{}_i), \ \ (\mathbf{F}_{k3})_i = f_k(\rho_k\,\psi_i), \ \ \mathbf{F}_3 = \mathbf{F}_{13} + \mathbf{F}_{23}$$

$$\begin{bmatrix} A_{11} & B_{11} & A_{13} & 0 & 0 \\ B^T_{11} & 0 & B_{13} & 0 & 0 \\ A^T_{13} & B^T_{13} & A_{33} & A^T_{23} & B^T_{23} \\ 0 & 0 & A_{23} & A_{22} & B_{22} \\ 0 & 0 & B_{23} & B^T_{22} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^\circ_1 \\ \mathbf{P}_1 \\ \mathbf{U}_3 \\ \mathbf{U}^\circ_2 \\ \mathbf{P}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_2 \\ \mathbf{0} \end{bmatrix}$$

Fig.2.1 The finite element system

If we define for $i = 1, 2$

$$K_i = \begin{pmatrix} A_{ii} & B_{ii} \\ B^T_{ii} & 0 \end{pmatrix} \qquad C_{i3} = \begin{pmatrix} A_{i3} \\ B_{i3} \end{pmatrix} \qquad J_i = (A^T_{i3} \quad B^T_{i3}) \ , \tag{2.8}$$

by block elimitation we deduce from the system in Fig.2.1the following Schur complement system with respect to the vector of the interface unknowns $\mathbf{U}_3$:

$$\mathbf{C}\mathbf{U}_3 = \mathbf{G} \quad , \quad \textit{with} \quad C = A^1_{33} - J_1 K_1^{-1} C_{13} + A^2_{33} - J_2 K_2^{-1} C_{23} \tag{2.9}$$

The right hand side of (2.9) is $\mathbf{G} = \sum_{i=1}^{2} \mathbf{F}_{i3} - J_i K_1^{-1} (\mathbf{F}_i, \mathbf{0})^T$. The matrix C is the *capacitance (or Schur complement) matrix.* Note that the complement is taken with respect to the interface values of the velocity only, and not to those of the pressure.

3. Functional interpretation of the capacitance matrix For any element φ of Φ_h, and for $k = 1, 2$, we look for $w_k(\varphi) \in V_{h,k}$, $\pi_k(\varphi) \in Q_{h,k}$ such that

$$\begin{cases} a_k(w_k(\varphi), v) + b_k(v, \pi_k(\varphi)) = 0 & \forall\, v \in V^*_{h,k} \\ b_k(w_k(\varphi), q) \qquad\qquad\qquad\quad = 0 & \forall\, q \in Q_{h,k} \\ w_k(\varphi) \qquad\qquad\qquad\qquad\quad\; = \varphi & \text{on } \Gamma \end{cases} \tag{3.1}$$

From now on we will refer to $(w_k(\varphi), \pi_k(\varphi))$ as to the finite element *Stokes extension* of φ to Ω_k, for $k = 1, 2$. Now set:

$$\mathcal{A}(\varphi,\psi) := \sum_{k=1}^{2} [a_k(w_k(\varphi), \rho_k\psi) + b_k(\rho_k\psi, \pi_k(\varphi))] \tag{3.2}$$

In view of (2.7)-(2.9), it is easy to show that

$$[\mathbf{C}\mathbf{x}_\varphi, \mathbf{x}_\psi] = \mathcal{A}(\varphi, \psi) \qquad \forall \varphi, \psi \in \Phi_h \tag{3.3}$$

where $\mathbf{x}_\varphi$ is a vector whose N_3 components are the values of φ at the gridpoints on Γ, $\mathbf{x}_\psi$ is defined similarly, and $[\cdot, \cdot]$ is the euclidean inner product of $\mathbb{R}^{N3}$.

Remark 3.1 For a Stokes problem, from (3.2) we deduce that $\forall\varphi\in\Phi_h$ the capacitance matrix C associates to the vector $\mathbf{x}_\varphi$ the vector $C\mathbf{x}_\varphi$ whose N_3 components are the values at the interface nodes of $\sigma_1(\varphi) + \sigma_2(\varphi)$, where

$$(3.4) \qquad \sigma_k(\varphi) := \partial w_k(\varphi)/\partial n_k - \pi_k(\varphi)\, n_k$$

(n_k = outward normal direction to $\partial\Omega_k$) is the *normal stress* on Γ associated with the Stokes extension of φ to Ω_k. Hence, equation (2.4) amounts to require the "natural" condition that the normal stress of the solution be continuous across Γ.

Proposition 3.1 *The following equality holds*

$$(3.5) \qquad \mathcal{A}(\varphi, \psi) = a_1(w_1(\varphi), w_1(\psi)) + a_2(w_2(\varphi), w_2(\psi)) \quad \forall\varphi, \psi\in\Phi_h \,.$$

Proof Using the definition (3.2) we have:

$$\mathcal{A}(\varphi,\psi) = \sum_{k=1}^{2} [a_k(w_k(\varphi),w_k(\psi)) + a_k(w_k(\varphi), \rho_k\psi - w_k(\psi)) + b_k(\rho_k\psi - w_k(\psi), \pi_k(\varphi)) + b_k(w_k(\psi), \pi_k(\varphi))]$$

By the second equation of (3.1) (with $\varphi=\psi$), the last term of the sum is zero. Moreover, the sum of the second and third term in the bracket is zero due to the first equation of (3.1) (with $v=\rho_k\psi - w_k(\psi) \in V^*_{h,k}$).

Corollary 3.1 *If the bilinear form a $(\cdot,\cdot)$ is symmetric, then the capacitance matrix C is symmetric and positive definite.*

Proof. Denoting as above by $\{\psi_j\}$ the Lagrange basis of Φ_h, from (3.2) one obtains

$$(3.6) \qquad C_{ij} = \mathcal{A}(\psi_j, \psi_i) \qquad 1 \le i, j \le N_3$$

C is therefore the matrix associated with the form $\mathcal{A}(\cdot,\cdot)$. Since the forms $a_k(\cdot,\cdot)$ are symmetric, the symmetry of C follows from (3.5). Furthermore, C is positive definite since the forms $a_k(\cdot,\cdot)$ are coercive, as stated by the first inequality of (1.4).

4. An optimal preconditioner for the capacitance matrix We introduce the reduced bilinear form on Φ_h

$$(4.1) \qquad \mathcal{B}(\varphi,\psi) := a_2(w_2(\varphi), w_2(\psi)) \qquad \forall\varphi, \psi\in\Phi_h$$

whose associated matrix is (see (2.9))

$$(4.2) \qquad B = A_{33}^2 - J_2 K_2^{-1} C_{23}$$

It is shown in [9], Lemma 6.1, that for k = 1, 2

$$\| H_k(\varphi) \|_k \le \| w_k(\varphi) \|_k \le (1 + \beta_{h,k}^{-1}) \| H_k(\varphi) \|_k \qquad \forall \varphi \in \Phi_h \tag{4.3}$$

where $\beta_{h,k}$ is the constant of the inf-sup condition (1.4) on Ω_k, $\| \cdot \|_k$ is the norm induced by the form $a_k(\cdot,\cdot)$, and $H_k(\varphi)$ is the *harmonic extension* of φ to Ω_k, i.e.,

$$H_k(\varphi) \in V_{k,h} \ : \quad a_k(H_k(\varphi), v) = 0 \qquad \forall v \in V^*_{k,h} \ , \ H_k(\varphi) = \varphi \ \text{ on } \Gamma \tag{4.4}$$

It is shown in [8] and [2] that if the finite element decomposition of Ω is quasi-uniform, then $\| H_1(\varphi)\|_1$ is uniformely equivalent to $\| H_2(\varphi) \|_2$, i.e.,

$$C_1 \| H_1(\varphi) \|_1 \le \| H_2(\varphi) \|_2 \le C_2 \| H_1(\varphi) \|_1 \qquad \forall \varphi \in \Phi_h \tag{4.5}$$

where C_1, C_2 are two constants independent of h. Using (4.3) and (4.5) we get:

$$C_1(1 + \beta_{h,1}^{-1})^{-1} \| w_1(\varphi) \|_1 \le \| w_2(\varphi) \|_2 \le C_2(1 + \beta_{h,2}^{-1}) \| w_1(\varphi) \|_1 \tag{4.6}$$

It follows (as already noticed in [9]) that if the inf-sup conditions hold in Ω_k with constants $\beta_{h,k}$ uniformely bounded from below by a constant independent of h, then $\| w_1(\varphi) \|_1$ and $\| w_2(\varphi) \|_2$ are uniformely equivalent. Hence, in view of (3.5) and (4.1) we conclude that there exists two constants K_1 and K_2 independent of h s.t.

$$K_1 \mathcal{B}(\varphi, \varphi) \le \mathcal{A}(\varphi, \varphi) \le K_2 \mathcal{B}(\varphi, \varphi) \qquad \forall \varphi \in \Phi_h \tag{4.7}$$

Thus the matrix B is *spectrally equivalent* to C, i.e.

(4.8) *the condition number of* $B^{-1} C$ *is independent of h*

Since B is symmetric, positive definite and spectrally equivalent to C, it can be used as an optimal preconditioner for conjugate gradient (or other) iterations on the capacitance system (2.9). If used with Richardson iterations, it gives rise to the generalization to Stokes equations of the Dirichlet-Neumann algorithm for elliptic equations (see [12] and [5], [8]). Clearly, the same kind of conclusion holds taking the matrix $B = A^1_{33} - J_1 K_1^{-1} C_{13}$.

5. Generalization to many subdomains We extend now the previous arguments to decompositions with several subdomains. For simplicity of exposition we will consider cartesian decompositions of "strips" and "boxes" only. For either case we will determine the associated capacitance matrix as well as several preconditioners.

5.1 Strips We consider first a domain Ω divided into M adjoining, non intersecting subdomains Ω_i. The common boundary between Ω_i and Ω_{i+1} is denoted by Γ_i (see Fig.5.1). We will assume that M is even.
The finite element multidomain problem relative to the current situation can still be defined as in (2.1)-(2.6), provided now Ω_1 denotes the set of the odd subdomains, Ω_2 that of the even ones, $\Gamma := \bigcup\{\Gamma_i, i=1,\dots,M-1\}$ and $\Phi_h := \prod\{\Phi_h(\Gamma_i), i=1,\dots,M-1\}$, where $\Phi_h(\Gamma_i)$ is

the finite element space V_h restricted to Γ_i. The capacitance matrix C relative to the current situation can be defined by means of some auxiliary matrices. For each i, denote by $\{\psi^i_m\}$ the Lagrange basis of $\Phi_h(\Gamma_i)$ at the finite element nodes $\{\xi^i_m\}$ of Γ_i. Then we define the four matrices $S^i_1,\ldots,S^i_4$ as follows:

$$(S^i_1)_{km} = a_i(w_i(\psi^i_m),\ w_i(\psi^i_k)) \tag{5.1}$$

$$(S^i_2)_{km} = a_i(w_i(\psi_m{}^{i-1}),\ w_i(\psi_k{}^{i-1})) \tag{5.2}$$

$$(S^i_3)_{km} = a_i(w_i(\psi^i_m),\ w_i(\psi_k{}^{i-1})) \tag{5.3}$$

$$(S^i_4)_{km} = a_i(w_i(\psi_m{}^{i-1}),\ w_i(\psi^i_k)) \tag{5.4}$$

where, for each $\psi \in \Phi_h(\Gamma_j)$, $(j = i\text{-}1,i)$, $w_i(\varphi)$ is the velocity field of the finite element Stokes extension of φ on Ω_i, with $w_i(\varphi) = \varphi$ on Γ_j, and $w_i(\varphi) = 0$ on $\partial\Omega_i\backslash\Gamma_j$.
If for each $\varphi \in \Phi_h(\Gamma_j)$ we denote by $\mathbf{x}_\varphi$ the vector whose components are the values of φ at the finite element nodes on Γ_j, we have

$$[S^i_4\, \mathbf{x}_\varphi, \mathbf{x}_\psi] = a_i(w_i(\psi), w_i(\varphi)) \qquad \forall\, \varphi \in \Phi_h(\Gamma_{i-1}),\ \forall\, \psi \in \Phi_h(\Gamma_i)\ .$$

Thus, S^i_4 is the algebraic representation of the finite element approximation of the Steklov-Poincaré operator $\mathcal{S} : [H^{1/2}_{00}(\Gamma_{i-1})]^2 \to [H^{-1/2}(\Gamma_i)]^2$ that associates to a vector function φ defined on Γ_{i-1} the normal stress on Γ_i of the solution to a Stokes problem in Ω_i whose right hand side is zero, and whose velocity field is equal to φ on Γ_{i-1} and is zero on $\partial\Omega_i\backslash\Gamma_{i-1}$. The other matrices defined in (5.1)-(5.3) have a similar meaning. The capacitance matrix C is block-tridiagonal and reads as

$$C = \begin{bmatrix} S^1_1 + S^2_2 & S^2_3 & & & \\ S^2_4 & S^2_1 + S^3_2 & S^3_3 & & \\ & & \cdots & & \\ & & S_4{}^{M-2} & S_1{}^{M-2} + S_2{}^{M-1} & S_3{}^{M-1} \\ & & & S_4{}^{M-1} & S_1{}^{M-1} + S_2{}^{M} \end{bmatrix} \tag{5.5}$$

Its associated bilinear form is:

$$\mathcal{A}(\varphi,\psi) = \sum_{i=1}^{M} a_i(w_i(\varphi), w_i(\psi)) \qquad \forall\, \varphi, \psi \in \Phi_h \quad . \tag{5.6}$$

Then C is symmetric and positive definite, provided a $(\cdot,\cdot)$ is symmetric.
Following what is proposed in [10] for multidomain spectral approximations to elliptic problems, we define now some preconditioners for the capacitance matrix C given in (5.5). The first preconditioner we consider has the following block diagonal structure:

$$
(5.7) \qquad B = \begin{bmatrix} S^2{}_2 & S^2{}_3 & & & & \\ S^2{}_4 & S^2{}_1 & & & & \\ & & S^4{}_2 & S^4{}_3 & & \\ & & S^4{}_4 & S^4{}_1 & & \\ & & & & S_2{}^M & S_3{}^M \\ & & & & S_4{}^M & S_1{}^M \end{bmatrix}
$$

This is precisely the generalization to the case of several subdomains of the matrix (4.2) (note that only the even subdomains are considered).The bilinear form associated with (5.7) is:

$$
(5.8) \qquad \mathcal{B}(\varphi,\psi) = \sum_{i \text{ even}} a_i(w_i(\varphi), w_i(\psi)) \qquad \forall\, \varphi, \psi \in \Phi_h
$$

(i between 2 and M in the sum) whence the matrix B is symmetric and positive definite. The equivalence between $\mathcal{A}$ and $\mathcal{B}$ can be established by proving that

$$
(5.9) \qquad \sum_{i \text{ even}} \| w_i(\varphi) \|_i^2 \textit{ is equivalent to } \sum_{i \text{ odd}} \| w_i(\varphi) \|_i^2 \ , \quad \forall\, \varphi \in \Phi_h \ .
$$

By (4.5) (case of two subdomains) and the argument used in [10], proof of theorem 4.1, we can show that the equivalence claimed in (5.9) holds with two constants independent of h but possibily depending on M^2. Therefore, we conclude that

$$
(5.10) \qquad \textit{condition number of } B^{-1} C \leq K M^2 \ \cdot \ K \textit{ independent of } h \ .
$$

To the same conclusion we can arrive taking in (5.8) the summation on all odd (rather than even) integers. The matrix (5.7) modifies accordingly.

A different block diagonal preconditioner (with M-1 blocks) is:

$$
(5.11) \qquad B = \begin{bmatrix} S^2{}_2 & & & & \\ & S^3{}_2 & & & \\ & & S^4{}_2 & & \\ & & & \ddots & \\ & & & & S_2{}^M \end{bmatrix}
$$

Its associated bilinear form is

$$
(5.12) \qquad \mathcal{B}(\varphi,\psi) = \sum_{i=1}^{M} a_i(w_i^-(\varphi), w_i^-(\psi)) \qquad \forall\, \varphi, \psi \in \Phi_h \ .
$$

where $w^-_i(\varphi)$ is the first component of a finite element Stokes extension such that $w^-_i(\varphi) = \varphi$ on Γ_{i-1} and $w_i(\varphi) = 0$ on Γ_i. By the same kind of arguments used above one can prove that the matrix B given in (5.11) is symmetric and positive definite, and that (5.10) still holds.

A symmetric situation occurs if we take the matrix B associated with a bilinear form like (5.12) with $w^+_i(\varphi)$ and $w^+_i(\psi)$ instead of $w^-_i(\varphi)$ and $w^-_i(\psi)$, respectively.

Finally, consider the *lower* bidiagonal preconditioner

$$(5.13)\qquad B = \begin{bmatrix} S^1_1 & & & & \\ S^2_4 & S^2_1 & & & \\ & S^3_4 & S^3_1 & & \\ & & & \ddots & \\ & & & S_4^{M-1} & S_1^{M-1} \end{bmatrix}$$

whose associated form is:

$$(5.14)\qquad \mathcal{B}(\varphi,\psi) = \sum_{i=1}^{M} a_i(w_i(\varphi), w_i^+(\varphi)) \qquad \forall\, \varphi, \psi \in \Phi_h \quad .$$

Its *upper*, bidiagonal counterpart is:

$$(5.15)\qquad B = \begin{bmatrix} S^2_2 & S^2_3 & & & \\ & S^3_2 & S^3_3 & & \\ & & \ddots & & \\ & & & S_2^{M-1} & S_3^{M-1} \\ & & & & S_2^M \end{bmatrix}$$

and its associated form reads as

$$(5.16)\qquad \mathcal{B}(\varphi,\psi) = \sum_{i=1}^{M} a_i\,(w_i\,(\varphi), w_i^-(\psi)) \qquad \forall\, \varphi, \psi \in \Phi_h \quad .$$

Either (5.14) and (5.16) (and consequently, the matrices (5.13) and (5.15)) fail to be symmetric. However, they are still positive, and (5.10) still holds.

Remark 5.1 Using iterative methods with block diagonal preconditioners for the capacitance system yields algorithms whose parallelism degree (i.e., the number of *independent* subproblems to be solved at each step) is equal to the number of diagonal blocks of the preconditioner.

5.2 Boxes (see Fig.5.2) We consider now the case of a box, i.e. of a domain Ω decomposed into four subdomains sharing a common internal vertex. We denote by Γ_i the common interface between Ω_i and Ω_{i+1}, i=1,...,4 (we identify Ω_5 with Ω_1).
As in the previous subsection, we introduce the capacitance matrix for the current case by means of auxiliary interface operators.We recall that ψ^r_m is the Lagrange function associated with the node ξ^r_m of Γ_r. We are not considering here neither the node corresponding to the interior vertex, nor the one belonging to $\partial\Omega$.

If N_s is the number of the interior nodes of Γ_s, we set for each i=1,...,4:

(5.17) $$(S^i_{rt})_{km} = a_i\,(w^i(\psi^r_m), w^i(\psi^t_k)) \quad k=1,\dots,N_r \;,\; m=1,\dots,N_t \;.$$

i is the index of the subdomain, r and t those of the interfaces, while k and m denote rows and columns of the matrix. The matrix S^i_{rt} is the algebraic representation of the finite element approximation of a Steklov-Poincaré operator $\mathcal{S}$. Precisely, $\mathcal{S}$ associates to a vector function φ defined on Γ_r the normal stress on Γ_t of the solution to a Stokes problem in Ω_i with zero right hand side, and with a velocity field that vanishes on $\partial\Omega_i\backslash\Gamma_r$, and coincides with φ on Γ_r. We will conventionally denote the interior vertex by Γ_0, and the corresponding node by 0. This allows us to extend the definition of the operators (5.17) to cover the case in which at least one of the indices r,t is equal to zero. In this way S^i_{0t} is a columm vector of lenght N_t, $(S^i_{r0})^T$ is a vector of lenght N_r, while S^i_{00} is a scalar. S^i_{0t} associates to the Lagrange finite element function the point 0 the normal pertaining to stress tensor (taken in the usual variational sense) of the corresponding Stokes extension (in Ω_i) at all internal nodes of the interface Γ_t. Simmetrically, S^i_{r0} associates to every Lagrange function on Γ_r the value at the point 0 of the stress tensor associated with the corresponding finite element Stokes extension in Ω_i. In terms of the above matrices and vectors the capacitance matrix C associated to the current multidomain finite element problem is (the interface unknowns are ordered as: $\mathbf{U}_1,\mathbf{U}_2,\mathbf{U}_0,\mathbf{U}_3,\mathbf{U}_4$, with $\mathbf{U}_i \in \Gamma_i$)

(5.18) $$C = \begin{bmatrix} S^1_{11}+S^2_{11} & S^2_{21} & \sigma_{10} & 0 & S^1_{41} \\ S^2_{12} & S^2_{22}+S^3_{22} & \sigma_{20} & S^3_{32} & 0 \\ \sigma_{01} & \sigma_{02} & \sigma_{00} & \sigma_{03} & \sigma_{04} \\ 0 & S^3_{23} & \sigma_{30} & S^3_{33}+S^4_{33} & S^4_{43} \\ S^1_{14} & 0 & \sigma_{40} & S^4_{34} & S^4_{44}+S^1_{44} \end{bmatrix}$$

where for convenience of notation we have set: $\sigma_{i0} = S^i_{0i} + S^{i+1}_{0i}$, $\sigma_{0i} = S^i_{i0} + S^{i+1}_{i0}$, and $\sigma_{00} = \sum_{i=1}^{4} S^i_{00}$ (as usual, a super index equal to five should be identified with 1). Its associated bilinear form is:

(5.19) $$\mathcal{A}(\varphi,\psi) = \sum_{i=1}^{4} a_i\,(w_i(\varphi), w_i(\psi)) \qquad \forall\,\varphi,\psi \in \Phi_h\;,$$

whence C is positive definite and symmetric, provided a $(\cdot,\cdot)$ is symmetric.
A block diagonal preconditioner which is the counterpart of (5.7) can be obtained from (5.18) by disregarding all matrices and vectors S^i_{rt} with a super index i odd. Such a

preconditioner is still symmetric and positive definite, however it is neither block diagonal nor spectrally equivalent to C. Actually, using the results of [12] we can show that the condition number of the corresponding preconditioned matrix grows like $K(1+\lg(H/h))^2$, where K is a positive constant, H is the maximum size of each subdomain, and h is, as usual, the finite element mesh size.

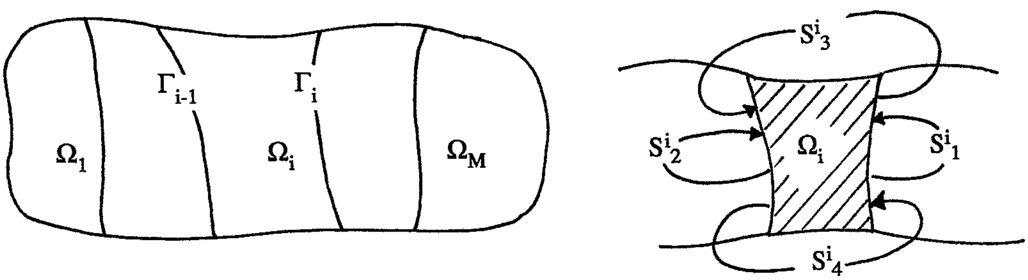

Fig.5.1 A strip and its associated interface operators

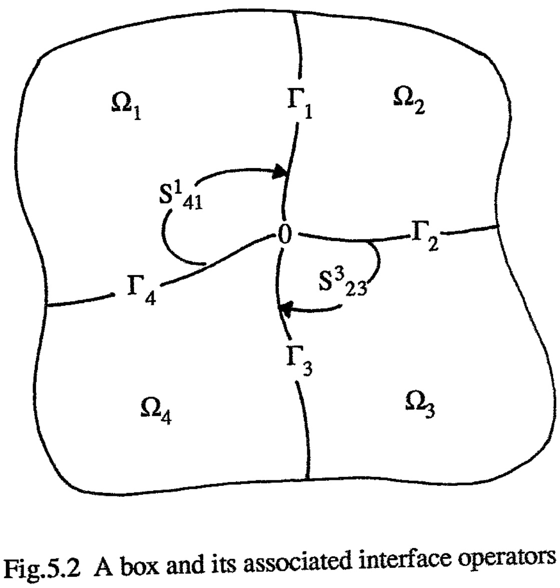

Fig.5.2 A box and its associated interface operators

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