Domain Decomposition Method and Parallel Algorithms

Kang Li-shan*

Abstract: This paper introduces the results on DDM and PA which were obtained recently by the Parallel Computation Research Group at Wuhan University.

1 Introduction

In 1980, using the Domain Decomposition Method (DDM) (see [1]) we began to design a class of asynchronous parallel algorithms, S-CR (Schwarz-chaotic relaxation), for solving mathematical physics problems while the multiprocessor system WuPP-80 was designed at Wuhan University.

In 1982, the WuPP-80, an MIMD machine with 4 processors, was put into operation at Wuhan University. By using the S-CR, a new class of asynchronous parallel DDM, many mathematical physics problems were solved on the machine and successful computing results were obtained (see [2]).

During the period from 1982 to 1985, a systematic theory on the DDM as the foundation of the asynchronous parallel algorithms for solving P.D.E.'s was developed (see [3],[4]).

In the Spring of 1986 D.J. Evans visited Wuhan University and worked with members of our Group on DDM and a series of extremely deep results of the convergence of the Schwarz alternating procedure (SAP) for the model problems were obtained (see Evans and Kang, et al. [5]—[10]). In the autumn of 1986, G. Rodrigue visited Wuhan University and suggested the use of mixed boundary conditions on the pseudo-boundaries of the subdomains. In this direction, many interesting results on the convergence of the DDM were obtained (see [13]—[17]).

In 1987, we began to study the DDM without overlapping. In this case, the symmetric DDMs are used for solving the symmetric problems in a symmetric

* Wuhan University, Wuhan, Hubei, China

domain. In this way, special kinds of parallel algorithms can be developed by which we can get the solution in two steps. These algorithms are constructed on the basis of the symmetric principle of errors. We regard this as a major breakthrough in the theory of DDM and it should influence physics, mathematics and mechanics, as well as parallel computing (see [18]—[21]).

In addition, we have also studied the algebraic DDM and DDM with other techniques (see [22]—[25]).

2 Convergence Rate of the SAP

(a) Consider the two point–boundary value problem

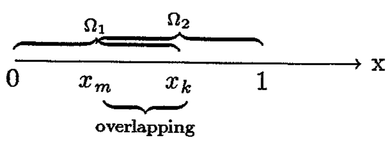
$$\left\{\begin{array}{ll} Lu \equiv -\dfrac{d^2u}{dx^2} + q^2u = f(x) & \Omega = \{x \mid 0 < x < 1\} \\ u(0) = a, \quad u(1) = b & \end{array}\right. \tag{2.1}$$

The Ω is decomposed into two subdomains

$$\Omega_1 = \{x \mid 0 < x < x_k\} \text{ and } \Omega_2 = \{x \mid x_m < x < 1\}$$

where

$$x_k > x_m, \; x_k = x_m + d.$$



SAP:

$$\left\{\begin{array}{ll} Ly^{(i+1)} = f(x) & \text{in } \Omega_1 \\ y^{(i_1)}(0) = a, \; y^{(i+1)}(x_k) = z^{(i)}(x_k) & \end{array}\right.$$

$$\left\{\begin{array}{ll} Lz^{(i+1)}(x_k) = f(x) & \text{in } \Omega_2 \\ z^{(i+1)}(x_m) = y^{(i+1)}(x_m), \; z^{(i+1)}(1) = b & \end{array}\right.$$

where $z^{(0)}(x_k)$ is the initial guess.

Theorem 2.1 *(Evans, Kang, Shao and Chen (1986)) the convergence factor of SAP is*

$$\rho_q(x_m, x_k) = \frac{\operatorname{sh} qx_m}{\operatorname{sh} qx_k} \cdot \frac{\operatorname{sh} q(1-x_k)}{\operatorname{sh} q(1-x_m)} = \rho_q(x_m, x_m + d)$$

where $\operatorname{sh} x = (e^x - e^{-x})/2$. *Moreover,*

$$\rho(x_m, x_m + d) = \frac{\operatorname{sh} qx_m}{\operatorname{sh} q(x_m + d)} \cdot \frac{\operatorname{sh} q(1 - x_m - d)}{\operatorname{sh} q(1 - x_m)} < 1,$$

and d *is the size of overlapping.*

Theorem 2.1 gives the exact relationship between the convergence factor and the geometric character of the domain decomposition. For fixed d, we have

$$\max_{x_m \in \Omega} \mathrm{sh}_q(x_m, x_m + d) = \rho_q((1-d)/2) = \left\{ \frac{\mathrm{sh}\, q[(1-d)/2]}{\mathrm{sh}\, q[(1+d)/2]} \right\}$$

This means that for fixed d, the worst case of SAP is the symmetric decomposition: $\mathrm{mes}\,\Omega_1 = \mathrm{mes}\,\Omega_2$.

For the discrete form of (2.1), we have

Theorem 2.2 *(Evans, Kang, Shao and Chen, 1986)*
The convergence factor of numerical SAP is

$$\rho_q^*(m,k) = \rho_q^*(m, m+D) = \frac{s(m)}{s(m+d)} \cdot \frac{s(N-m-D)'}{s(N_D)}$$

where $s(x) = (r_1^x - r_2^x)/2$, *and*

$$r_1 = Q + \sqrt{Q^2 - 1}, \quad r_2 = Q - \sqrt{Q^2-1}$$

and

$$Q = (2 + q^2h^2)/2.$$

It is easy to prove that $\lim_{h\to 0} \rho_q(x_m.x_k)$.

(b) For the two dimensional problem

$$(2.2) \qquad \left\{ \; -\Delta u + q^2 u = \; f \quad \text{in } \Omega = \{(x,y) \mid 0 < x < 1, \; 0 < y < 1 \; \right\}$$

we decompose Ω into two subdomains

$$\begin{aligned} \Omega_1 = &\ \{(x,y) \mid 0 < x < x_k, \; 0 < y < 1\} \\ \Omega_2 = &\ \{(x,y) \mid x_m < x < 1, \; 0 < y < 1\}. \end{aligned}$$

Theorem 2.3 *(Evans, Kang, Shao and Chen 1986) The convergence factor of the SAP is*

$$\rho_q(x_m, x_k) = \frac{\mathrm{sh}\sqrt{\pi^2+q^2}x_m \; \mathrm{sh}\sqrt{\pi^2+q^2}(1-x_k)}{\mathrm{sh}\sqrt{\pi^2+q^2}x_k \; \mathrm{sh}\sqrt{\pi^2+q^2}(1-x_m)}.$$

Theorem 2.4 *The convergence factor of the numerical SAP is*

$$\bar\rho_q(m,k) = \bar\rho_q(m, m+S) = \frac{\bar S(m)}{\bar S(m+D)} \cdot \frac{\bar S(N-m-D)}{\bar S(N-m)},$$

where

$$\begin{aligned} \bar S(x) = &\ (\bar r_1^x - \bar r_2^x)/2, \\ \bar r_1 = &\ (\bar Q + \sqrt{\bar Q^2 - 1}, \; \bar r_2 = \bar Q - \sqrt{\bar Q^2 - 1}, \\ \bar Q = &\ 1 + 2\sin^2 \tfrac{\pi h}{2} + \tfrac{h^2q^2}{2}. \end{aligned}$$

For the n-dimensional problem, we have

Theorem 2.5 *(Chen and Kang, 1987)*

$$\rho_q(x_m, x_k) = \frac{\operatorname{sh}\sqrt{(n-1)\pi^2+q^2}x_m \operatorname{sh}\sqrt{(n-1)\pi^2+q^2}(1-x_k)}{\operatorname{sh}\sqrt{(n-1)\pi^2+q^2}x_k \operatorname{sh}\sqrt{(n-1)\pi^2+q^2}(1-x_m)}.$$

(2) Neumann problems

1-dimensional problem:

$$(2.3) \qquad \begin{cases} -\dfrac{d^2u}{dx^2} + q^2u &= f(x) \quad 0 < x < 1 \\ \frac{du(0)}{dx} &= a, \quad \frac{du(1)}{dx} = b \end{cases}$$

SAP-1: If we use the Dirichlet conditions on the pseudo-boundaries, we have

Theorem 2.6 *(Kang and Evans, 1986): The convergence factor of SAP-1 is*

$$\rho_q(x_m, x_k) = \frac{\operatorname{ch} qx_m}{\operatorname{ch} qx_k} \cdot \frac{\operatorname{ch} q(1-x_k)}{\operatorname{ch} q(1-x_m)}.$$

where $\operatorname{ch} x = (e^x + e^{-x})/2.$

If $x_m \to 0$, then $\rho_q(x_m, x_m + d) \to \frac{\operatorname{ch} q(i-1)}{\operatorname{ch} qd \operatorname{ch} q} \neq 0.$
So we must change the pseudo-boundary conditions.
SAP-2: We use the Neumann conditions on the pseudo-boundaries, we have

Theorem 2.7 *(1986), The convergence factor of SAP-2 is*

$$\rho_q(x_m, x_k) = \frac{\operatorname{sh} qx_m}{\operatorname{sh} qx_k} \cdot \frac{\operatorname{sh} q(1-x_k)}{\operatorname{sh} q(1-x_m)}.$$

Similar results hold for the Dirichlet problem.

For the two-dimensional problem

$$(2.4) \qquad \begin{cases} -\Delta u + q^2 u &= f \quad \text{in } \Omega = \{(x,y) \mid 0 < x < 1,\ 0 < y < 1\} \\ \dfrac{\partial u}{\partial n} &= y \quad \text{on } \Gamma \end{cases}$$

A we have

Theorem 2.8 *The convergence factor of SAP-1 is*

$$\rho_q(x_m, x_k) = \frac{\operatorname{ch}\sqrt{\pi^2+q^2}x_m \operatorname{ch}\sqrt{\pi^2+q^2}(1-x_k)}{\operatorname{ch}\sqrt{\pi^2+q^2}x_k \operatorname{ch}\sqrt{\pi^2+q^2}(1-x_m)}.$$

Theorem 2.9 *The convergence factor of SAP-2 is*

$$\rho_p(x_m, x_k) = \frac{\operatorname{sh}\sqrt{\pi^2+q^2}x_m \operatorname{sh}\sqrt{\pi^2+q^2}(1-x_k)}{\operatorname{sh}\sqrt{\pi^2+q^2}x_k \operatorname{sh}\sqrt{\pi^2+q^2}(1-x_m)}.$$

The above results have been extended to the cases of more than two subdomains and to moving pseudo-boundaries for the purpose of balancing the load of multiprocessors.

3 Acceleration

For accelerating the convergence of SAP there are several ways.
(a) SAP with Pseudo-boundary Relaxation Factor

Consider the problem (2.1). We introduce the factor ω in the pseudo-boundary conditions as follows:

$$\begin{aligned} u^{(i+1)} &= v^{(i+1)} + \omega(v^{(i)} - v^{(i-1)}) \quad \text{on } \Gamma_1' \subset \Omega_2 \\ v^{(i+1)} &= u^{(i)} + \omega(u^{(i+1)} - u^{(i)}) \quad \text{on } \Gamma_2' \subset \Omega_1 \\ i &= 1, 2, 3, \ldots \end{aligned}$$

where $v^{(0)}$ and $v^{(1)}$ on Γ_1' are given.

Denote the convergence factor of SAP ($\omega = 1$) by ρ_q, then we have

Theorem 3.1 *(Evans, Kang, Chen and Shao, 1986)*
The optimal overrelaxation factor is given by

$$\omega_{\text{opt}} = (3/\sqrt{\rho_q})\cos((s + 4\pi)/3),$$

where

$$s = \arccos(-\sqrt{\rho_q})$$

and the corresponding convergence factor is

$$\rho_{\text{opt}} = 3(\omega_{\text{opt}} - 1)/\omega_{\text{opt}}.$$

(b) SAP with Mixed Pseudo-boundary Conditions

For one-dimensional problem (2.1) with $q = 0$, we use the mixed pseudo- boundary conditions

$$c_3 z^{(i+1)} + c_4 \frac{dz^{(i+1)}}{dx} = c_3 y^{(i+1)} + c_4 \frac{dy^{(i+1)}}{dx} \qquad \text{at } x = x_m$$

where $z^{(0)}(x_k)$ and $\frac{dz^{(0)}(x_k)}{dx}$ are the initial guesses.

Theorem 3.2 *(Lin, Wu, Rodrigue and Kang,(1987))*
The convergence factor of the SAP with parameters is

$$\rho_0 = \left| \frac{(c_1(x_k - 1) + c_2)(c_3 x_m + c_4)}{(c_1 x_k + c_2)(c_3(x_m - 1) + c_4)} \right|$$

We can easily choose the parameters c_i $(i = 1, 2, 3, 4)$ such that

$$\rho_0 = 0.$$

In these cases, the SAP in two steps gives the exact solution. If $q \neq 0$ we have

Theorem 3.3 *The convergence factor of the SAP with parameters is*

$$\rho_q = \left| \frac{(c_1 \operatorname{sh} q(x_k - 1) + c_2 q \operatorname{ch} q(x_k - 1))(c_3 \operatorname{sh} q x_m + c_4 \operatorname{ch} q x_m)}{(c_3 \operatorname{sh} q(x_m - 1) + c_4 q \operatorname{ch} q(x_m - 1))(c_1 \operatorname{sh} q x_k + c_2 q \operatorname{ch} q x_k)} \right|.$$

If we choose the parameters c_i $(i = 1, 2, 3, 4)$ such that $\rho_q = 0$, then the SAP in two steps gives the exact solution.

For two dimensional problem (2), we use the mixed pseudo–boundary conditions. We have

Theorem 3.4 *(Lin, Wu, Kang, and Rodrigue)*
The convergence factor of SAP with parameters is

$$\rho = \max_{x \in N^+} \left| \frac{(c_1 \operatorname{sh} Q_i(x_k - 1) + c_2 Q_i(x_k - 1))(c_3 \operatorname{sh} Q_i x_m + c_4 \operatorname{ch} Q_i x_m)}{(c_3 \operatorname{sh} Q_i(x_m - 1) + c_4 Q_i(x_m - 1))(c_1 \operatorname{sh} Q_i x_k + c_2 \operatorname{ch} Q_i x_k)} \right|$$

where

$$Q_i = (i^2 \pi^2 + q^2)^{\frac{1}{2}}.$$

For solving Neumann problem (4), we have

Theorem 3.5 *The convergence factor of SAP with parameters is*

$$\bar{\rho} = \max_{x \in N^+} \left| \frac{(c_1 \operatorname{sh} Q_i(x_k - 1) + c_2 Q_i(x_k - 1))(c_3 \operatorname{ch} Q_i x_m + c_4 \operatorname{sh} Q_i x_m)}{(c_3 \operatorname{ch} Q_i(x_m - 1) + c_4 Q_i(x_m - 1))(c_1 \operatorname{ch} Q_i x_k + c_2 \operatorname{sh} Q_i x_k)} \right|$$

For n–dimensional problems the conclusion is almost the same if we replace Q_i in the formulae by $\bar{Q}_i = ((n-1)i^2 + q^2)^{\frac{1}{2}}$ (see Chen and Kang(1987)).

For more than two subdomains(DDM)]

Consider the one–dimensional problem (1). The Ω is decomposed into m subdomains

$$\Omega_j = \{x \mid x_j^{(1)} < x < x_j^{(2)}\} \quad j = 1, 2, \ldots, m$$

$$x_j^{(1)} < x_{j+1}^{(1)} < x_j^{(2)} < x_{j+1}^{(2)} \quad j = 1, 2, \ldots, m; x_1^{(1)} = 0 \text{ and } x_m^{(2)} = 1.$$

Algorithm:

$$\begin{cases} -\dfrac{d^2 u_j^{(i)}}{dx^2} + q^2 u_j^{(i)} = f & \in \Omega_j \\[2ex] c_j^{(1)} u_j^{(i)} + d_j^{(1)} \dfrac{du_j^{(i)}}{dx} = c_j^{(1)} j_{j-1}^{(i)} + d_j^{(1)} \dfrac{du_{j-1}^{(i)}}{dx} & \text{at } x = x_j^{(1)} \\[2ex] c_j^{(2)} u_j^{(i)} + d_j^{(2)} \dfrac{du_j^{(i)}}{dx} = c_j^{(2)} u_{j+1}^{i-1} + d_j^{(2)} \dfrac{du_{j+1}^{(i-1)}}{dx} & \text{at } x = x_j^{(2)} \\[2ex] \qquad i = 1; \quad j = 1, 2, \ldots, m, & \\[2ex] \qquad i = 2; \quad j = m-1, m-2, \ldots, 1, \quad u_m^{(2)} = u_m^{(1)}, & \end{cases}$$

where $c^{(k)}, d_j^{(k)}$ $(k = 1, 2;\ j = 1, 1, \ldots, m)$ are parameters, $u_j^{(0)}$ and $\frac{du^{(0)}}{dx}$ on $x_j^{(2)}$ $(j = 1, 2, \ldots, m-1)$ are initial guesses.

Theorem 3.6 *(Wu, Lin, Rodrigue and Kang (1987))*
If the parameters are chosen such that

$$c_{jh}^{(1)} \operatorname{sh} qx_j^{(1)} + d_j^{(1)} q \operatorname{ch} qx_j^{(1)} = 0 \qquad j = 2, 3, \ldots, m$$

$$c_{jh}^{(1)} \operatorname{sh} qx_j^{(1)} + d_j^{(1)} q \operatorname{ch} qx_j^{(1)} = 0 \qquad j = 2, 3, \ldots, m$$

then the exact solution can be obtained in two steps.

For the discrete form (DDDM) we denote $x_j^{(1)} = J_1 h, x_j^{(2)} = J_2 h$.

Theorem 3.7 *If the parameters are chosen such that*

$$\begin{cases} hc_j^{(1)} - d_j^{(1)} + d_j^{(1)}(r_1^{J_1+1} - r_2^{J_1-1})/(r_1^{J_1} - r_2^{J_1}) & = 0 \\ hc_j^{(2)} - d_j^{(2)} + d_j^{(2)}(r_1^{J_2+1} - r_2^{J_2-1})/(r_1^{J_2} - r_2^{J_2}) & \neq 0, \end{cases}$$

where

$$r_1 = \ 1 + q^2h^2/2 + ((2qh)^2 + (qh)^4)^{\frac{1}{2}}/2$$

$$r_1 = \ 1 + q^2h^2/2 - ((2qh)^2 + (qh)^4)^{\frac{1}{2}},$$

then the exact solution can be obtained in two steps.

The similar technique can be used for accelerating the convergence of DDM used for solving the multi–dimensional problems.

(c) Extrapolation Techniques

Assume the function–sequence $\{u^{(i)}\}$ is obtained by SAP.

Denote the exact solution by u^*, and the errors by

$$e^{(i)} = u^* - u^{(i)}$$

Theorem 3.8 *(Lin, Liu and Kang (1987))*
If there exists a constant $P \neq 1$, such that

$$e^{(i+1)} = Pe^{(i)},$$

then

$$u^* = \tfrac{1}{1-P}u^{(i+1)} - \tfrac{P}{1-P}u^{(i)}.$$

For one–dimensional problems with constant coefficients, we usually can get the exact solution in two steps no matter whether the original sequence of the SAP converges.

For solving two–dimensional problem (2), if $x_k = x_m$, we use the SAP with parameters $c_1 = 1.c_2 = 0, c_3 = 0$ and $c_4 = 1$. We have

$$e^{(i+1)} = -e^{(i)}.$$

It means that the original sequence of functions does not converge. But we can get the exact solution as follows:

$$u^* = (u^{(1)} + u^{(2)})/2.$$

4 Symmetric DDM without Overlapping

(a) Symmetric Principle of Errors

Consider the linear problem

$$(*)\begin{cases} Lu = f & \text{in } \Omega \\ u = \phi & \text{on } \Gamma = \partial\Omega, \end{cases}$$

the domain Ω is symmetric w.r.t. Γ' and the operator L is symmetric w.r.t. the domain decomposition, means that $Lu(x) = L\tilde{u}(x'), \forall\ x \varepsilon \Omega_1$, where $x' \varepsilon \Omega_2$ is the symmetric point of x w.r.t. Γ'. We have the following theorem.

Theorem 4.1 *(Rao, 1987), If problem (*) has unique solution u^* then*

$$u^* = \begin{cases} (u^{(1)} + u^{(2)})/2 & \text{on } \bar{\Omega}_1 \\ (v^{(1)} + v^{(2)})/2 & \text{on } \bar{\Omega}_2, \end{cases}$$

where $v^{(i)}$ and $u^{(i)}$ satisfy the following problems:

$$\begin{cases} Lu^{(1)} = f & \text{in } \Omega_1 \\ u^{(1)} = \phi & \text{on } \Gamma \\ u^{(1)} = q & \text{on } \Gamma' \end{cases} \qquad \begin{cases} Lv^{(1)} = f & \text{in } \Omega_2 \\ v^{(1)} = \phi & \text{on } \Gamma \\ v^{(1)} = g & \text{on } \Gamma' \end{cases}$$

and

$$\begin{cases} Lu^{(2)} = f & \text{in } \Omega_1 \\ u^{(2)} = \phi & \text{on } \Gamma \\ \frac{\partial u^{(2)}}{\partial n} = \frac{\partial v^{(1)}}{\partial n} & \text{on } \Gamma' \end{cases} \qquad \begin{cases} Lv^{(2)} = f & \text{in } \Omega_2 \\ v^{(2)} = \phi & \text{on } \Gamma \\ \frac{\partial v^{(2)}}{\partial n} = \frac{\partial u^{(1)}}{\partial n} & \text{on } \Gamma' \end{cases}$$

where n is the outer normal direction of Γ'.

In 1987, the symmetric principle of errors was first discovered by Rao Chuan-xia and widely extended to many applications, especially to the parallel computing (see [17] and [18]).

The discrete form of the Symmetric Principle of Errors has been established by Shao, Wu, etc. [19].

(b) Symmetrization Principle

Denote

$f^+(x) = (f(x) + f(x'))/2, \qquad f^-(x) = (f(x) - f(x'))/2.$

Theorem 4.2 *(Lu, 1987), The solution of $(*)$ is*

$$u^*(x) = u^+(x) - u^-(x' \quad \text{on } \bar{\Omega}_1, \qquad u^*(x) = u^+(x) + u^-(x) \ \text{ in } \Omega_2,$$

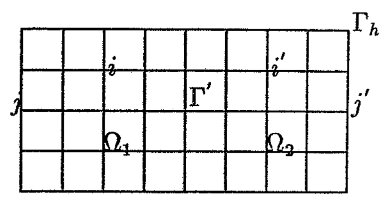
where u^+ and u^- are the solutions of problems

$$\begin{cases} Lu^{(+)} = f^+ & \text{in } \Omega_1 \\ u^{(+)} = \phi^+ & \text{on } \Gamma \\ \frac{\partial u^{(+)}}{\partial n} = 0 & \text{on } \Gamma' \end{cases}$$
$$\begin{cases} Lu^{(-)} = f^- & \text{in } \Omega_2 \\ u^{(-)} = \phi^- & \text{on } \Gamma \\ u^{(-)} = 0 & \text{on } \Gamma' \end{cases}$$

respectively.

For the discrete form of $(*)$

$$\begin{cases} L_h u_i = f_i, & \forall i \; \varepsilon \; \Omega_h \\ u_j = \phi_j, & \forall j \; \varepsilon \; \Gamma_h, \end{cases}$$



If $L_h u_{i'} = L_h u_i, \; \forall i \; \varepsilon \; \Omega_i$, i' is the symmetric point of i w.r.t. Γ', then we have

$$u_i = u_i^+ + u_i^- \quad \text{and} \quad u_{i'} = u_i^+ - u_i^-$$

where

$$\begin{cases} Lu_i^+ = f_I^+ & = (f_i + f_{i'})/2 \\ u_i^+ + \phi_j^+ & \forall j \; \varepsilon \; \Gamma_h \\ u_k^+ = u_{k'}^+ & \forall k \; \varepsilon \; \Omega_1 \end{cases} \qquad \begin{cases} Lu_i^- = f_i^- & = (f_i - f_{i'})/2 \; \forall i \; \varepsilon \; \Omega_1 \\ u_j^- = \psi_j^- & \forall j \; \varepsilon \; \Gamma_h \\ u_k^- = 0 & \forall k \; \varepsilon \; \Gamma'. \end{cases}$$

$\forall i \varepsilon \Omega_1 \bigcup \Gamma'$.

5 DDM with Other Techniques

(a) Schwarz-Projection Method for Non-linear Problems (see [25].
(b) Schwarz-Multigrid Method(see [22]).
(c) DDM-Operator Splitting Method (see [24])
(d) Numerical SAP (see [21] and [23]).

6 Numerical Experiments

The S-CR algorithms, a class of asynchronous parallel algorithms based upon the theory of DDM, were implemented and tested on the multicomputer system UwPP-80 installed at Wuhan University in 1982.

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