Iterative Solution of Elliptic Equations with Refinement: The Model Multi-Level Case*

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Abstract. A multi-level theory for AFAC [2] in a model problem case is developed that is based on *sectional* energy norm estimates for harmonic functions. This is the companion paper of [3], which treats the two-level case.

Key Words. Composite Grid, Refinement, Iterative Methods, Orthogonal Projections, Elliptic Partial Differential Operators, Finite Elements, Parallel Computation, FAC

1. Introduction. This paper assumes that the reader is familiar with the companion paper [3]. Here we consider the multi-level case for AFAC [2]. We first extend the notation of [2] (Section 2), then develop a theoretical structure for convergence (Section 3), and finally consider a model problem in a specialized geometry (Section 4).

The convergence bounds obtained here involve the relative size of successive refinement regions, that is, the maximum ratio of areas of successive refinement regions. In fact, as this maximum tends to zero, the bounds we obtain tend to the two-grid bounds developed in [3]. The multi-level theory in [1] also obtains bounds that depend on the maximum refinement ratio; this theory applies only to a modified version of AFAC, but unlike ours it does not require that this ratio be sufficiently small.

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2. Notation. For simplicity, we restrict our attention to a 2D diffusion operator with Dirichlet boundary conditions. Let Ω_1 be a polygonal domain in $\mathbf{R}^2$ and $\Omega_1 \supset \Omega_2 \supset \ldots \supset \Omega_k$ a nested sequence of polygonal, or union of disjoint polygonal, domains. Let $H_i \subset H_o^1(\Omega_i)$ be a conforming Lagrange finite element space associated with $\Omega_i, 1 \leq i \leq k$. We assume for simplicity that the element boundaries of Ω_i contain the boundary of Ω_{i+1} and that every element in Ω_{i+1} associated with H_i is the union of some elements associated with $H_{i+1}, 1 \leq i \leq k-1$. We define the i^{th} *composite grid space*

$$H_{c_i} = \sum_{j=i}^{k} H_j$$

and denote $H_c = H_{c_1}$. Consider the bilinear form

$$a(u,v) = \int_\Omega a \nabla u \cdot \nabla v \, d\Omega,$$

with the piecewise smooth diffusion coefficient $a > const > 0$ in Ω. Also consider the linear form

$$f(v) = \int_\Omega f \, v \, d\Omega.$$

The space H_c is equipped with the inner product $a(\cdot,\cdot)$ and its norm $\|\cdot\|$ defined by $\|u\| = (a(u,u))^{\frac{1}{2}}$. In what follows, references to inner products and norms (e.g., 'orthogonality' and 'unit functions') refer to these definitions.

We are interested in the iterative solution of the *discrete variational equation*

$$\mathbf{u} \epsilon H_c : \quad a(\mathbf{u}, v) = f(v) \quad \forall v \epsilon H_c. \tag{1}$$

The symbol $\mathbf{u}$ will be used to denote the solution of (1). The basic cycle of AFAC we consider here for solving (1) is as follows:

Algorithm (AFAC [2]) Let $u \epsilon H_c$ be the current approximation to $\mathbf{u}$. For each $i = 1, 2, \ldots, k$, compute u_i from

$$u_i \epsilon H_i : \quad a(u - u_i, v_i) = f(v_i) \quad \forall v_i \epsilon H_i.$$

For each $i = 1, 2, \ldots, k-1$, compute w_i from

$$w_i \epsilon H_i \cap H_{i+1} : \quad a(u - w_i, v_i) = f(v_i) \quad \forall v_i \epsilon H_i \cap H_{i+1}.$$

With $w_k = 0$, set

$$u \leftarrow u - \sum_{i=1}^{k} (u_i - w_i).$$

Our analysis of AFAC will involve the use of *discrete harmonic functions* and their associated spaces and projections. In particular, for each $i = 1, 2, \ldots, k-1$, define the *i-harmonics*

$$H_i^{i-harm} = \{u_i \epsilon H_i : \quad a(u_i, v_i) = 0 \quad \forall v_i \epsilon H_i \cap H_{i+1}\}.$$

Note that H_i^{i-harm} is just the orthogonal complement of $H_i \cap H_{i+1}$ in H_i. Let P_i^{i-harm} be the orthogonal projector mapping H_c onto H_i^{i-harm}. For each $i = 1, 2, \ldots, k-1$, define the *c_i-harmonics*

$$H_{c_i}^{harm} = \{u_i \epsilon H_{c_i} : \quad a(u_i, v_i) = 0 \quad \forall v_i \epsilon H_{c_{i+1}}\}.$$

Note that

$$H_{c_i} = H_{c_i}^{harm} \oplus H_{c_{i+1}} \tag{2}$$

is an orthogonal decomposition, $1 \leq i < k$. For convenience we define $H_{c_k}^{harm} = H_{c_k}$ and $H_{c_{k+1}} = H_{k+1} = \emptyset$ so that (2) holds for all $i = 1, 2, \ldots, k$. Note that in the decomposition $u_{c_i} = u_{c_i}^{harm} + u_{c_{i+1}}$, $u_{c_i}^{harm}$ is just the c_i–harmonic in $H_{c_i}^{harm}$ that agrees with u_{c_i} in $\Omega_i \backslash \Omega_{i+1}$. Let $P_{c_i}^{harm}$ be the orthogonal projector mapping H_c onto $H_{c_i}^{harm}$.

As in the two-level case [3], the convergence rate for AFAC depends on a measure of how much an i–harmonic deviates from being c_i–harmonic, although now we need to be more precise. In particular, first note that *local* functions with support in $\overline{\Omega}_i \backslash \Omega_{i+1}$ are both i–harmonic and c_i–harmonic. (Overbar denotes set closure.) The space of these functions we denote by H_i^{local}. Note that $H_i^{local} \subset H_i^{i-harm} \cap H_{c_i}^{harm}$. We then define *i-doubly–harmonic* functions by

$$H_i^{i-dharm} = \left\{u_i \epsilon H_i^{i-harm} : a(u_i, v_i) = 0 \quad \forall v_i \epsilon H_i^{local}\right\}.$$

We similarly define *c_i–doubly–harmonic* functions by

$$H_{c_i}^{dharm} = \left\{u_i \epsilon H_{c_i}^{harm} : a(u_i, v_i) = 0 \quad \forall v_i \epsilon H_i^{local}\right\}.$$

Note that

$$H_i^{i-harm} = H_i^{local} \oplus H_i^{i-dharm} \tag{3}$$

is an orthogonal decomposition. (We assume the trivial definitions for these spaces with $i = k$ so that (3) holds for all $i = 1, 2, \ldots, k$.) Define P_i^{local} and $P_i^{i-dharm}$ to be their corresponding orthogonal projectors, that is, P_i^{local} and $P_i^{i-dharm}$ are the respective orthogonal projectors from H_c onto H_i^{local} and $H_i^{i-dharm}$. Then one error measure we will need is given by $\epsilon = \max_{1 \leq i < k} \|(I - P_{c_i}^{harm}) P_i^{i-dharm}\|$. Note that ϵ is just the maximal two-grid estimate defined in [3]:

$$\begin{aligned}
\epsilon = \max_{1 \leq i < k} \{&\|e_{i+1}\| : e_{i+1} = (I - P_{c_i}^{harm}) u_i^{i-dharm}, \\
&u_i^{i-dharm} \epsilon H_i^{i-dharm}, \|u_i^{i-dharm}\| = 1\} \\
= \max_{1 \leq i < k} \{&\|e_{i+1}\| : e_{i+1} = (I - P_{c_i}^{harm}) u_i^{i-harm}, \\
&u_i^{i-harm} \epsilon H_i^{i-harm}, \|u_i^{i-harm}\| = 1\}.
\end{aligned}$$

We introduce the *sectional energy inner product* $a_{\Omega_j}(\cdot,\cdot) : H_c \times H_c \to \mathbf{R}$ defined for each $j = 1, 2, \ldots, k$ by

$$a_{\Omega_j}(u, v) = \int_{\Omega_j} a \nabla u \nabla v \, d\Omega.$$

The *sectional energy norm* is defined by $\|u\|_{\Omega_j} = (a_{\Omega_j}(u,u))^{\frac{1}{2}}$. We also define the *harmonic section measures* for $1 \le i \le k-2$ and $i+2 \le j \le k$ by

$$\delta_{ij} = \max\left\{ \|u_i^{i-harm}\|_{\Omega_j} : u_i^{i-harm} \epsilon H_i^{i-harm}, \|u_i^{i-harm}\| = 1 \right\}$$

and for $1 \le i \le k-1$ by

$$\delta_{i\,i+1} = \max\left\{ |a(u_i^{i-harm}, v_{i+1})| : u_i^{i-harm} \epsilon H_i^{i-harm}, v_{i+1} \epsilon H_{i+1}^{(i+1)-dharm}, \|u_i^{i-harm}\| = \|v_{i+1}\| = 1 \right\}.$$

Further, define $\delta_{i\,i} = 0$ and $\delta_{ij} = \delta_{ji}$, $i > j$, and $\delta = \max_{1 \le i \le k} \sum_{j=1}^{k} \delta_{ij}$.

3. Theory. Let $\mathbf{u}$ denote the solution of (1), u the current approximation, and $e = \mathbf{u} - u$ the *algebraic error*. Then one cycle of AFAC applied to u transforms e according to

$$e \leftarrow (I - \sum_{i=1}^{k} P_i^{i-harm})e. \tag{4}$$

Note that if the subspaces H_i^{i-harm} and $H_{c_i}^{harm}$ agree for all i, then $H_c = H_1^{1-harm} \oplus H_2^{2-harm} \oplus \ldots \oplus H_k^{k-harm}$ would be an orthogonal decomposition of H_c. This means that (4) would converge to zero in one cycle, making AFAC a direct solver. In fact, this can happen in practice, namely, for typical one-dimensional problems and higher-dimensional problems with refinement regions Ω_i wholly contained in just one element of the coarser level, H_{i-1}. However, most practical problems have $H_i^{i-harm} \ne H_{c_i}^{harm}$, for which we have the following theorem.

THEOREM 3.1. *Suppose $\gamma = \epsilon + \delta < 1$. Then a bound on the AFAC convergence factor is given by*

$$\rho(I - \sum_{i=1}^{k} P_i^{i-harm}) = \|I - \sum_{i=1}^{k} P_i^{i-harm}\| \le \gamma. \tag{5}$$

PROOF: Let u_i^{i-harm} be unit functions in H_i^{i-harm}. Let $U = (u_{ij})$ be the Gramm matrix for these functions defined by $u_{ij} = a(u_i^{i-harm}, u_j^{j-harm})$. In [2; lemma 2.3] we proved in effect that

$$\rho(I - \sum_{i=1}^{k} P_i^{i-harm}) = \sup\{\rho(I - U)\},$$

where the supremum is taken over all possible choices for u_i^{i-harm}. Thus, to bound the convergence factor for AFAC, we need only bound $\rho(I-U)$ for arbitrary choices of u_i^{i-harm}. To bound $\rho(I-U)$, decomposing by (3) and (2), we have

$$u_i^{i-harm} = \sqrt{1-\beta_i^2}\, x_i^{local} + \beta_i v_i,\ v_i = \sqrt{1-\epsilon_i^2}\, y_{c_i}^{dharm} + \epsilon_i w_{c_{i+1}}$$

where $|\beta_i| \le 1$, $|\epsilon_i| \le \epsilon$, and $x_i^{local} \epsilon H_i^{local}$, $v_i \epsilon H_i^{i-dharm}$, $y_{c_i}^{dharm} \epsilon H_{c_i}^{dharm}$, and $w_{c_{i+1}} \epsilon H_{c_{i+1}}$, which are all unit functions. Define $\theta_i = \frac{\epsilon_i}{\epsilon} a(w_{c_{i+1}}, x_{i+1}^{local})$ and note that $|\theta_i| \le 1$. Now from $a(x_i^{local}, u_{i+1}^{(i+1)-harm}) = 0$ and $a(y_{c_i}^{dharm}, x_{i+1}^{local}) = 0$, we have

$$u_{i\,i+1} = \epsilon\beta_i\sqrt{1-\beta_{i+1}^2}\,\theta_i + \beta_i\beta_{i+1} a(v_i, v_{i+1}).$$

By definition,

$$|a(v_i, v_{i+1})| \le \delta_{i\,i+1},$$

and

$$|u_{ij}| = |a(u_i^{i-harm}, u_j^{j-harm})| \le \delta_{ij}, \quad j > i+1.$$

Now since $u_{ii} = 1$ and $\beta_i\beta_{i+1} \le 1$, we can write $U = I + \epsilon T + E$, where the symmetric matrices $T = (t_{ij})$ and $E = (e_{ij})$ satisfy

$$t_{ij} = \begin{cases} \beta_i\sqrt{1-\beta_{i+1}^2}\,\theta_i & j = i+1 \\ t_{ji} & j = i-1 \\ 0 & otherwise \end{cases}$$

and

$$|e_{ij}| \le \delta_{ij}$$

for $1 \le i, j \le k$. Thus,

$$\rho(I-U) \le \epsilon\rho(T) + \delta.$$

The theorem would now be proved if we could show that $\rho(T) \le 1$. This we do by way of the following lemma.

LEMMA 3.1. *Suppose T is a $k \times k$ tridiagonal matrix of the form $T =$ tridiag $(\beta_{i-1}\sqrt{1-\beta_{i+1}^2}\,\theta_{i-1}\ \ 0\ \ \beta_i\sqrt{1-\beta_{i+1}^2}\ \theta_i)$, where β_i, θ_i satisfy $|\beta_i| \le 1$, $|\theta_i| \le 1$, $1 \le i \le k$, and $\beta_{k+1} = 0$ (for convenience). Then $\rho(T) < 1$.*

PROOF: If some $\beta_i = 0$ then T reduces to matrices of a smaller size. So, without loss of generality, we assume $\beta_i \ne 0$, $1 \le i \le k$. We shall first prove that $I+T$ is positive definite. Let a_i be the i-th pivot in the LU–decomposition of $I+T$. Then

$$a_1 = 1, \quad a_{i+1} = 1 - \frac{\beta_i^2(1-\beta_{i+1}^2)\theta_i^2}{a_i}.$$

It is easy to see by induction that $a_i \ge \beta_i^2 > 0$. Thus, $I+T$ is positive definite because all pivots are positive. Now by changing the signs of all of the β_i, we can conclude from the above that $I - T$ is also positive definite. Thus, $\rho(T) < 1$ and the lemma is proved.

4. Model Problem. The convergence bounds of the previous section depend on estimates of the "two-grid" factor ϵ and the "total section" δ. While suitable bounds on ϵ have been obtained in a fairly general setting (cf. [1-3]), there appears to be no such theory for δ. Since δ is the sum of integrals of harmonic functions over decreasingly smaller subregions, it is likely that δ can be made as small as desired by requiring that successive refinement regions cover a sufficiently small relative area. We now prove this for a model problem with special geometry.

Our model problem is the Poisson version of (1), with $a = 1$, on the unit square $\Omega = [0,1] \times [0,1]$. Our special geometry is based on uniform grids Ω_i that consist of the respective $m_i - 1$ and $n_i - 1$ interior vertical and horizontal grid lines covering the region $[0, \eta_i] \times [0,1]$, $1 \le i \le k$. We require that every other grid line of Ω_{i+1} coincide with the vertical and horizontal grid lines of Ω_i, and that

$$\frac{m_{i+1}}{\eta_{i+1}} = 2\frac{m_i}{\eta_i} \qquad \text{and} \qquad n_{i+1} = 2n_i, \ 1 \le i < k.$$

To simplify the presentation, we will assume that $m_1 = n_1 \ge 2$ and that the refinement "speed" is constant: $\eta_{i+1} = \eta\eta_i$ for $0 < \eta < 1$, $1 \le i < k$. Finally, we assume that $H_i \subset H_0^1(\Omega_i)$ is the space of continuous piecewise linear functions on a natural triangulation of $\overline{\Omega}_i$: the triangles are formed by connecting the lower left and upper right vertices of each rectangle in $\overline{\Omega}_i$.

In this section, $\|\cdot\|_2$ will be used to denote the Euclidean norm on various "nodal vectors," i.e., vectors of node values of certain functions in $H_0^1(\Omega)$. We will refer to "nodal matrices" associated with certain discrete spaces H. By this we mean the usual stiffness matrix that arises from transforming the bilinear problem (1) on H to an equation involving the nodal values of the discrete solution $\mathbf{u}$ in H. Note that the nodal matrix on the uniform space H_i is just the one that corresponds to the usual 5-point difference stencil

$$\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}.$$

The following lemmas will be used to bound nodal values of harmonic functions. The first allows us to reduce the two-dimensional problem of obtaining these bounds to a one-dimensional one.

LEMMA 4.1. *Suppose A and B are $m \times m$ symmetric positive definite block tridiagonal matrices consisting of the respective $n \times n$ blocks*

$$A_{ij} = \begin{cases} X & i = j \\ -I & |i-j| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$B_{ij} = \begin{cases} Y & i = j \\ -I & |i - j| = 1 \\ 0 & otherwise \end{cases} .$$

Suppose $X \geq Y$ (i.e., $X - Y$ is nonnegative definite), X and Y commute, and that $x = (x_1^t \cdots x_m^t)^t$ and $y = (y_1^t \cdots y_m^t)^t$ satisfy

$$Ax = s \quad \text{and} \quad By = t$$

where $s_j = t_j = 0$, $1 \leq j < q \leq m$. Suppose $x_q = y_q$. Then $\|x_j\|_2 \leq \|y_j\|_2$, $1 \leq j \leq q$.

PROOF: By performing elementary block operations on the augmented matrices $(A : s)$ and $(B : t)$, it is easy to see that

$$x_j = \left(\prod_{l=j}^{q-1} S_l^{-1} \right) x_q$$

and

$$y_j = \left(\prod_{l=j}^{q-1} T_l^{-1} \right) x_q$$

where $S_1 = X$, $S_{l+1} = X - S_l^{-1}$, $T_1 = Y$, and $T_{l+1} = Y - T_i^{-1}$, $1 \leq l < q$. By induction it is easy to see that $S_l \geq T_l$ and, hence, $S_l^{-1} \leq T_l^{-1}$. The lemma now follows from using the fact that S_l^{-1} and T_l^{-1} are rational functions of X and Y, respectively.

In the following, A will be the nodal matrix of the Laplace equation on H_i, that is, $X = diag[-1 \quad 4 \quad -1]$; B will be given by $Y = 2I$.

LEMMA 4.2. *Suppose A in Lemma 4.1 is the nodal matrix for H_i. Then $\|x_j\|_2 \leq \frac{j}{q}\|x_q\|_2$. This also holds for the nodal matrix for H_{c_i} provided the indices j and q are interpreted with respect to grid Ω_i lines.*

PROOF: Using Lemma 4.1 with $Y = 2I$, we have

$$\|x_j\|_2 \leq \gamma_j \|x_q\|_2$$

with equality for the case $n = 1$, where

$$\gamma_j = \rho \left(\prod_{l=j}^{q-1} T_l^{-1} \right)$$

is independent of n. We can thus examine the case $n = 1$ which corresponds to a simple one–dimensional case where the scalars x_l satisfy $x_j = \frac{j}{q} x_q$. This proves the first assertion. The final assertion follows from a similar argument using the fact that the presence of the interfaces $x = \eta_l$ with $l > i + 1$ cannot increase $\|x_j\|_2$ for any j (which we assume corresponds to a grid Ω_i line). This proves the lemma.

Our final lemma will be used to bound nodal values of harmonics at grid interfaces.

LEMMA 4.3. *Let $u_i^{i-dharm}$ be a unit function in $H_i^{i-dharm}$ and let w_j be the n_i–dimensional vector of its values at the nodes on the vertical grid line $x = \eta_i(1-\frac{j}{m_i})$, $1 \le j \le m_i$. Then*

$$\|w_{r_i}\|_2 \le \left(\frac{r_i l_i}{r_i + l_i}\right)^{1/2} \tag{8}$$

and

$$\|w_1\|_2 \le \left(\frac{l_i}{r_i(r_i + l_i)}\right)^{1/2} \tag{9}$$

where $r_i = m_i - l_i$ and $l_i = m_{i+1}/2$, the number of vertical grid lines of $\overline{\Omega}_i$ to the respective right and left of $x = \eta_{i+1}$. Moreover, (9) also holds if we replace the function $u_i^{i-dharm}$ by a unit function $u_{c_i}^{dharm}$ in $H_{c_i}^{dharm}$.

PROOF: By Lemma 4.2 and the fact that $u_i^{i-dharm}$ is i–doubly harmonic, we have

$$\|w_j\|_2 \le \frac{j}{r_i} \|w_{r_i}\|_2, \quad 1 \le j \le r_i, \tag{10}$$

and

$$\|w_{r_i+j}\|_2 \le \frac{l_i - j}{l_i} \|w_{r_i}\|_2. \tag{11}$$

Since $u_i^{i-dharm}$ is of unit length, so is its vector of nodal values, w, in the sense that $(Aw)^t w = 1$. Since $u_i^{i-dharm}$ is i–doubly harmonic, all but the $r_i th$ entry of Aw is nonzero. Using $j = r_i - 1$ in (10) and $j = 1$ in (11), we thus have

$$\begin{aligned} 1 &= \left(Xw_{r_i} - w_{r_i-1} - w_{r_i+1}\right)^t w_{r_i} \\ &\ge 2\|w_{r_i}\|_2^2 - \|w_{r_i-1}\|_2 \|w_{r_i}\|_2 - \|w_{r_i+1}\|_2 \|w_{r_i}\|_2 \\ &\ge \left(2 - \frac{r_i - 1}{r_i} - \frac{l_i - 1}{l_i}\right) \|w_{r_i}\|_2^2 \\ &= \left(\frac{1}{r_i} + \frac{1}{l_i}\right) \|w_{r_i}\|_2^2. \end{aligned}$$

This proves (8). Now using Lemma 4.2 with $j = 1$ and $q = r_i$, we have

$$\|w_1\|_2 \le \frac{1}{r_i} \left(\frac{r_i l_i}{r_i + l_i}\right)^{1/2}$$

which proves (9). The final assertion follows from Lemma 4.2, so the proof is complete.

THEOREM 4.1. *For the model problem and special geometry considered here, if $\eta < \frac{1}{2}$ then*

$$\delta < 2\left(\eta + \frac{\sqrt{2\eta}}{1 - \sqrt{\eta}}\right).$$

Thus, $\delta = 0(\sqrt{\eta})$.

PROOF: Let v_{i+1} and v_i be arbitrary unit functions in $H_{i+1}^{(i+1)-dharm}$ and $H_i^{i-dharm}$, respectively. Then

$$|a(v_i, v_{i+1})| \le |a_{\Omega_i \setminus \Omega_{i+2}}(v_i, v_{i+1})| + |a_{\Omega_{i+2}}(v_i, v_{i+1})|.$$

Define $\overline{v}_i \epsilon H_i$ by

$$\overline{v}(x,y) = \begin{cases} 0 & x \le \eta_{i+1} - \frac{1}{n_{i+1}} \\ v_i & x \ge \eta_{i+1} \end{cases}$$

and $\tilde{v}_i \epsilon H_i$ by

$$\tilde{v}_i(x,y) = \begin{cases} v_i(x,y) & x \le \eta_{i+2} \\ v_i(2\eta_{i+2} - x, y) & \eta_{i+2} \le x \le 2\eta_{i+2} \\ 0 & x \ge 2\eta_{i+2} \end{cases}.$$

Then $v_i - \overline{v}_i - \tilde{v}_i \epsilon H_{i+1}^{local}$ and $\|\overline{v}_i\| = \sqrt{2}\|v_i\|_{\Omega_{i+2}}$, so

$$\begin{aligned} |a(v_i,\, v_{i+1})| &= |a(\overline{v}_i + \tilde{v}_i, v_{i+1})| \\ &\le |(Jz)^t w| + \sqrt{2}\delta_{i\,i+2}, \end{aligned}$$

where z and w are the vectors of node values of v_i and v_{i+1} on $x = \eta_{i+1}$ and $x = \eta_{i+1} - \frac{1}{n_{i+1}}$, respectively, and where J is the linear interpolation operator mapping Ω_i nodal vectors on $x = \eta_{i+1}$ to Ω_{i+1} nodal vectors on $x = \eta_{i+1}$.

Note that $\|J\|_2 \le \sqrt{2}$ so that

$$|(Jz)^t w| \le \sqrt{2}\ \|z\|_2\ \|w\|_2.$$

By (8) and (9), and the relations $r_{i+1} = 2r_i$ and $\frac{l_i}{r_i + l_i} = \eta$, we have

$$\begin{aligned} \delta_{i\,i+1} &\le \sqrt{2}\left(\left(\frac{r_i l_i}{r_i + l_i}\right)\left(\frac{l_{i+1}}{r_{i+1}(r_{i+1} + l_{i+1})}\right)\right)^{1/2} + \sqrt{2}\,\delta_{i\,i+2} \\ &\le \eta + \sqrt{2}\delta_{i\,i+2}. \end{aligned} \tag{12}$$

Now let $j \ge i + 2$ and define z_q as the n_i–dimensional vector of nodal values of v_i on $x = \frac{q}{m_i}\eta_i$, $0 \le q \le m_i$. Assume for the moment that $x = \eta_j$ coincides with a grid line of Ω_i, say the p^{th} one. Let A and X be as in Lemma 4.2. It holds that

$$Xz_q - z_{q-1} - z_{q+1} = 0,\ q = 1, \ldots, l_i - 1. \tag{13}$$

Let

$$z_{l_i} = \sum_k c_k u_k,$$

where u_k are eigenvectors of X, $Xu_k = \lambda_k u_k$, $\|u_k\|_2 = 1$. Note that $\lambda_k \epsilon (2,6)$. Then by (13) we have

$$z_q = \sum_k c_k t_k(q) u_k,$$

where

$$t_k(q) = \frac{\mu_k^q - \mu_k^{-q}}{\mu_k^{l_i} - \mu_k^{-l_i}} = \frac{\sinh(q ln \mu_k)}{\sinh(l_i ln \mu_k)}$$

and where $\mu_k = \frac{\lambda_k + \sqrt{\lambda_k^2 - 4}}{2} > 1$ and $\mu_k^{-1} < 1$ are the roots of the characteristic equation $\lambda_k - \frac{1}{\mu_k} - \mu_k = 0$.

Writing $e_q = ((X - I)z_q - z_{q-1})^t z_q$, we have from the fact that z_q are values of a discrete harmonic function that

$$a_{\Omega_j}(v_i, v_i) = e_p, \; a_{\Omega_{i-1}}(v_i, v_i) = e_{l_i} \leq \|v_i\|^2 = 1.$$

Substituting for z_q, we have

$$e_q = \sum_k c_k^2 (\lambda_k t_k^2(q) + (t_k(q) - t_k(q-1))t_k(q)).$$

From the properties of hyperbolic functions, it follows that

$$\frac{t_k(p)}{t_k(l_i)} = \frac{\sinh(p ln \mu_k)}{\sinh(l_i ln \mu_k)} < \frac{p}{r}$$

and

$$\frac{(t_k(p) - t_k(p-1))t_k(p)}{(t_k(l_i) - t_k(l_i - 1))t_k(l_i)} = \frac{\cosh(\frac{2p-1}{2} ln \mu_k)}{\cosh(\frac{2l_i - 1}{2} ln \mu_k)} \frac{\sinh(p ln \mu_k)}{\sinh(l_i ln \mu_k)} < \frac{p}{l_i}.$$

Consequently,

$$\|v_i\|_{\Omega_j}^2 = a_{\Omega_j}(v_i, v_i) \leq \frac{e_p}{e_{l_i}} < \frac{p}{l_i} = \mu^{j-i-1}.$$

Hence,

$$\delta_{ij} \leq \eta^{\frac{j-i-1}{2}}, \; j > i + 1. \tag{14}$$

To see now that (14) also holds for the case that $x = \eta_j$ is not a grid line of Ω_i, it is enough to recognize that $\nabla u \cdot \nabla u$ for u in H_i is constant on triangles of Ω_i.

Now from (12) and (14) we have that

$$\delta < 2\left(\eta + \sqrt{2} \sum_{q=1}^{\infty} \eta^{q\backslash 2}\right) = 2\left(\eta + \frac{\sqrt{2\eta}}{1 - \sqrt{\eta}}\right),$$

which proves the theorem.

References

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