Iterative Solution of Elliptic Equations with Refinement: The Two-Level Case*

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Abstract. A two-level theory for FAC [16] and Asynchronous FAC [10] is developed based on the strengthened Cauchy inequality. We obtain convergence bounds that do not depend on regularity of the problem and that can be computed locally. We also establish a relationship between FAC and a closely related preconditioner introduced in [5].

Key Words. Composite Grid, Refinement, Iterative Methods, Orthogonal Projections, Elliptic Partial Differential Equations, Finite Elements, Parallel Computation, FAC

1. Introduction. The Fast Adaptive Composite grid method (FAC [16]) is an iterative method for the solution of composite grid equations using solvers on uniform grids only. It was observed to be a fast, practical method. However, its theoretical foundations have not yet been fully developed.

In brief, the simplest version of FAC can be described as a conforming finite element method using a global coarse grid and one or more (non-overlapping) local fine grids. The spaces of trial and test functions are simply the sum of standard finite element function spaces associated with all grids. To find the minimum of the energy functional, minimization is performed alternatively with respect to the spaces of coarse grid and fine grid finite element functions. Fast solvers can be applied to the resulting linear systems in many cases, especially when the grids are uniform. Several levels of refinement and parallel solution on all levels can be successfully incorporated into this scheme [10].

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In this paper, we are concerned with bounds on the convergence factor of FAC as an iterative method for the solution of the composite equations. The paper is organized as follows. In Section 2, we give some simple algebraical propositions concerning orthogonal projections and the angle between subspaces. In Section 3, we apply these abstract results to obtain a characterization of the convergence factor of FAC in terms of the cosine of the angle between certain subspaces of the space of composite grid functions. We then bound the cosine from locally computable estimates, using a technique known in multigrid literature as the '*strengthened Cauchy inequality*, which makes it possible to use bounds already available in the literature for many classes of finite elements. FAC gives rise to naturally defined preconditioners, studied in Section 4. In particular, we show that the preconditioner defined by Bramble, Ewing, Pasciak, and Schatz in [5] is equivalent to one-and-a-half steps of FAC, so an analysis of FAC applies to the preconditioner from [5] and vice versa. In Section 5, we prove a theorem relating the convergence factor of a parallel version of FAC (AFAC [17]) with the convergence factor of FAC itself.

2. Theoretical preliminaries. We begin with several simple lemmas that establish some algebraical properties of orthogonal projections. Let H be a finite dimensional linear space with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. For $u, v \in H$, $u \neq 0$, $v \neq 0$, we write $\cos(u,v) = (u,v)/||u||\,||v||$. For U, V nontrivial subspaces of H, define the cosine of the angle between U and V by $\cos(U,V) = \sup\{|\cos(u,v)| : u \in U, v \in V\}$. Further, let P_V and $P_{V^\perp}$ denote the respective orthogonal projections from H onto V and $V^\perp$, the orthogonal complement of V. Let $\rho(A)$ denote the spectral radius of a matrix A.

The following lemma is well known and is included only for completeness.

LEMMA 2.1. *Let U, V be nontrivial subspaces of H. Then*

$$\rho(P_U P_V) = ||P_U P_V P_U|| = \cos^2(U,V).$$

Proof. Let $P_U P_V u = \lambda u$, $\lambda \neq 0$. Then u is in the range of P_U, so $P_U P_V P_U u = \lambda u$. This proves that $\rho(P_U P_V) = ||P_U P_V P_U||$ because $P_U P_V P_U$ is symmetric.

To prove the second equality, because the case $\cos(U,V) = 1$ is trivial, we consider only the case $\cos(U,V) < 1$. Because the surface of the unit sphere in H is compact, there exists a $u \in U$ such that $||u|| = 1$ and $||u - P_V u||$ is minimal. Denote $v = P_V u/||P_V u||$ and $c = \cos(u,v)$. Then $|c| = \cos(U,V)$, $P_V u = cv$, and $P_U v = cu$. Consequently, $P_U P_V u = c^2 u$, so $\rho(P_U P_V) \geq \cos^2(U,V)$.

To prove the reverse inequality, note that for any $u \in U$ it holds that $||P_V u||^2 = (P_V u, u) \leq \cos(U,V)||P_V u||\,||u||$, so $||P_V u|| \leq \cos(U,V)||u||$. Similarly, $||P_U v|| \leq \cos(U,V)||v||$ for any $v \in V$. It follows that $||P_U P_V P_U|| \leq \cos^2(U,V)$. □

The next lemma allows us to estimate $\cos(U,V)$.

LEMMA 2.2. *Let U, V be nontrivial subspaces of H.*

(i) If $U \cap V = \{0\}$, then $\cos(U^\perp, V^\perp) \geq \cos(U,V)$.

(ii) If $H = U \oplus V$, then $\cos(U^\perp, V^\perp) = \cos(U,V)$.

(iii) If $X \subset U$ and $Y \subset V$ and $H = X \oplus Y$, then $\cos(U^\perp, V^\perp) \leq \cos(X,Y)$.

Proof. (i) As in the preceding proof, let $u \in U$ such that $||u|| = 1$ and $||u - P_V u||$ is minimal and denote $v = P_V u / ||P_V u||$ and $c = \cos(u, v)$. Then $|c| = \cos(U, V)$, $P_V u = cv$, and $P_U v = cu$. It follows that $u - P_V u/c \in U^\perp$, $v - P_U v/c \in V^\perp$, and $\cos(u - P_V u, v - P_U v) = c$, using elementary geometry in the two-dimensional subspace spanned by u an v. This proves (i). Propositions (ii) and (iii) follow immediately from (i). □

The next lemma about localization of the spectrum of the sum of projections will be useful in the study of AFAC.

LEMMA 2.3. *Let* $H = \bigoplus_{j=1}^{n} V_j$. *Then*

$$\lambda_{\min}\left(\sum_{j=1}^{n} P_{V_j}\right) = \min_{v_j \in V_j, ||v_j||=1, j=1,\ldots,n} \lambda_{\min}\left(G(v_1, \ldots, v_n)\right),$$

$$\lambda_{\max}\left(\sum_{j=1}^{n} P_{V_j}\right) = \max_{v_j \in V_j, ||v_j||=1, j=1,\ldots,n} \lambda_{\max}\left(G(v_1, \ldots, v_n)\right),$$

where $G(v_1, \ldots, v_n) = (g_{ij})$, $g_{ij} = (v_i, v_j)$, *is the Gram matrix of the vectors* $v_1, \cdots v_n$.

Proof. For $j = 1, \ldots, n$, let J_j be a matrix whose columns form an orthonormal basis of the space V_j. Then $P_{V_j} = J_j J_j^t$. Define $A = (J_1, \ldots, J_n)$. Then A is a square regular matrix and $\sum_{j=1}^{n} J_j J_j^t = AA^t$. Because AA^t is similar to $A^t A = A^{-1}(AA^t)A$, the spectrum of AA^t is same as the spectrum of $A^t A = (J_i^t J_j)_{i,j=1}^{n}$. Now for every $u \in H$ we may write

$$u = \begin{pmatrix} a_1 u_1 \\ \vdots \\ a_n u_n \end{pmatrix},$$

where a_j are scalars, the number of elements in each vector u_j is same as the dimension of V_j, and $u_j^t u_j = 1$. Let $a = (a_j)_{j=1}^{n}$ and $v_j = J_j u_j$. We then have $Au = \sum_{j=1}^{n} a_j v_j$ and the Rayleigh quotient

$$\frac{u^t A^t A u}{u^t u} = \frac{a^t G(v_1, \ldots, v_n) a}{a^t a}.$$

The lemma is a direct consequence of these observations. □

In the case $n = 2$, the preceding lemma gives bounds of the spectrum of the sum of two projections in terms of the angle of their ranges.

LEMMA 2.4. *Let* $H = U \oplus V$. *Then*

$$\lambda_{\max}(P_U + P_V) = 1 + \cos(U, V), \quad \lambda_{\min}(P_U + P_V) = 1 - \cos(U, V).$$

Proof. Let $u \in U$ and $v \in V$. Then the Gram matrix of u, v is

$$G(u, v) = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

where $|a| \leq \cos(U, V)$, with equality attained for some u and v. The lemma now follows from the observation that the eigenvalues of $G(u, v)$ are $1 + a$ and $1 - a$. □

3. The Fast Adaptive Composite Grid Method (FAC). For simplicity, we restrict ourselves to the case of a diffusion operator with homogeneous Dirichlet boundary conditions. Let Ω_1 be an open polygonal domain in $\mathbf{R}^2$ and $\Omega_2 \subset \Omega_1$ an open polygonal domain or the union of disjoint open polygonal domains. Let $H_{2h} \subset H_0^1(\Omega_1)$ be a conforming Lagrange finite element space associated with Ω_1 and $H_h \subset H_0^1(\Omega_2)$ be a conforming finite element space associated with Ω_2. Define the *composite grid space*

$$H_c = H_{2h} + H_h.$$

Consider the bilinear form

$$a(u,v) = \int_{\Omega_1} a \bigtriangledown u \bigtriangledown v,$$

with the diffusion coefficient a piecewise smooth and $a > \text{const} > 0$ in Ω_1. Also consider the linear form

$$f(v) = \int_{\Omega_1} fv.$$

The space H_c is equipped with the inner product $a(\cdot,\cdot)$. We are interested in the iterative solution of the *discrete variational equation*

$$\mathbf{u} \in H_c : \quad a(\mathbf{u}, v) = f(v), \quad \forall v \in H_c. \tag{1}$$

The symbol $\mathbf{u}$ will be used to denote the solution of (1).

We consider an iterative method for the solution of (1), whose basic cycle is defined as follows:

ALGORITHM 3.1. (FAC [16]) *Let $u \in H_c$ be the current approximation to* $\mathbf{u}$.

Step 1. *Compute the solution of*

$$u_h \in H_h : \quad a(u + u_h, v_h) = f(v_h), \quad \forall v_h \in H_h, \tag{2}$$

and set $u \leftarrow u + u_h$.

Step 2. *Compute the solution of*

$$u_{2h} \in H_{2h} : \quad a(u + u_{2h}, v_{2h}) = f(v_{2h}), \quad \forall v_{2h} \in H_{2h}, \tag{3}$$

and set $u \leftarrow u + u_{2h}$.

It is easy to see that the *algebraic error*

$$e = \mathbf{u} - u$$

is transformed according to

$$e \leftarrow P_{H_{2h}^\perp} P_{H_h^\perp} e. \tag{4}$$

An estimate $\rho\left(P_{H_{2h}^\perp} P_{H_h^\perp}\right) \le \text{const} < 1$ independent of h was established in [16] using *H^2-regularity* of the elliptic operator, but without restrictions on the *degree* *of refinement* in Ω_2. Here we restrict the degree of refinement — the mesh sizes of H_{2h} and H_h differ by a factor of two — and we obtain a bound without recourse to regularity using a technique well known in the multigrid literature. For the purpose of obtaining local estimates, assume that *the boundaries of the elements associated with H_{2h} contain the boundary of Ω_2 and that every element in Ω_2 associated with H_{2h} is the union of some elements associated with H_h*. We may then define

$$X = H_{2h}, \quad Y = \{u \in H_h : u = 0 \text{ on nodes of } H_{2h}\}. \tag{5}$$

It is well known, cf., [1], [4], [13], that then a bound on $\cos(X, Y)$ can be evaluated locally: Let $\{K\}$ be the elements associated with the space H_{2h}. For any such K, denote

$$a_K(u, v) = \int_K a \nabla u \nabla v.$$

Then if the inequality

$$|a_K(u, v)| \leq \gamma \sqrt{a_K(u, u)} \sqrt{a_K(v, v)}, \quad \forall u \in X, v \in Y \tag{6}$$

holds for all K with the same constant $\gamma < 1$, it follows that

$$|a(u, v)| = \left|\sum_K a_K(u, v)\right| \leq \gamma \sum_K \sqrt{a_K(u, u)} \sqrt{a_K(v, v)}$$

$$\leq \gamma \sqrt{\sum_K a_K(u, u)} \sqrt{\sum_K a_K(v, v)}.$$

This yields the so-called *strengthened Cauchy inequality*

$$|a(u, v)| \leq \gamma \sqrt{a(u, u)} \sqrt{a(v, v)}, \quad \forall u \in X, v \in Y. \tag{7}$$

Thus, γ is an upper bound on $\cos(X, Y)$.

REMARK 3.2. It is easy to see from (6) that the value of γ does not change if the diffusion coefficient a is multiplied by a different positive constant in each element. This property is usual for regularity-free estimates; for related — but different — convergence bounds with this property for multigrid methods, see [7] and [11].

REMARK 3.3. Estimates of γ for various elements and a constant diffusion coefficient a are well known, see [1], [4],[13]. Computation of γ on K reduces to the solution of a generalized eigenvalue problem for the local stiffness matrix of the element K. In the two-dimensional case, the H_h elements result from partitioning each H_{2h} element into four elements in the natural way. For this case, the optimal values of γ for triangular linear elements as computed by Maitre and Musy [13] are between $\gamma = \sqrt{3/8}$ for the unilateral triangle and $\gamma \to \sqrt{2/3}$ for the degenerate triangle with one angle close to π. The model problem with Ω_1 a rectangle divided in two rectangles Ω_2 and $\Omega_1 \setminus \Omega_2$ is studied by Fourier analysis in [14]. For triangular linear elements obtained by dividing all squares in a uniform rectangular mesh in two triangles in the same way, it holds that

$$a(u_h, u^{2h\text{-harm}}) = a_{\Omega_2}(u_h, u^{2h\text{-harm}}) \le \delta\sqrt{a(u_h, u_h)}\sqrt{a_{\Omega_2}(u^{2h\text{-harm}}, u^{2h\text{-harm}})}$$

for all $u_h \in H_h, u^{2h\text{-harm}} \in H_{2h}^{2h\text{-harm}}$, with $\delta \approx 0.669$. This bound is sharp for small h.

Denote

$$H_{2h}^{2h\text{-harm}} = \{u_{2h} \in H_{2h} : a(u_{2h}, v_{2h}) = 0, \quad \forall v_{2h} \in H_{2h} \cap H_h\}.$$

This is the orthogonal complement of $H_{2h} \cap H_h$ in H_{2h}. By analogy with the case when the diffusion coefficient a is constant, the functions from $H_{2h}^{2h\text{-harm}}$ are called $2h$–harmonic functions in Ω_2.

LEMMA 3.4. *It holds that* $P_{H_{2h}^\perp} P_{H_h^\perp} = P_{(H_{2h}^{2h\text{-harm}})^\perp} P_{H_h^\perp}$.

Proof. Let $v \in H_c$ be arbitrary and write $u = P_{H_h^\perp} v$. Define $w_{2h} = P_{H_{2h}^{2h\text{-harm}}} u$, so that

$$w_{2h} \in H_{2h}^{2h\text{-harm}} : \quad a(w_{2h}, z_{2h}) = a(u, z_{2h}), \quad \forall z_{2h} \in H_{2h}^{2h\text{-harm}}.$$

But this must hold for all $z_{2h} \in H_{2h} = H_{2h}^{2h\text{-harm}} \oplus (H_{2h} \cap H_h)$ because, for $z_{2h} \in H_{2h} \cap H_h$, $a(w_{2h}, z_{2h})$ and $a(u, z_{2h})$ are both zero by definition. Consequently, $w_{2h} = P_{H_{2h}} u$, so

$$P_{(H_{2h}^{2h\text{-harm}})^\perp} u = u - w_{2h} = P_{H_{2h}^\perp} u,$$

which proves the lemma. □

We can now exhibit a bound on the convergence factor of Algorithm 3.1.

THEOREM 3.5. *The convergence factor of Algorithm 3.1 is*

$$\rho\left(P_{H_{2h}^\perp} P_{H_h^\perp}\right) = \cos^2(H_{2h}^{2h\text{-harm}}, H_h) \le \cos^2(X, Y) \le \gamma^2,$$

where X *and* Y *are given by (5).*

Proof. The proof follows immediately from Lemma 3.4, equation (4), Lemmas 2.1 and 2.2, and from the fact that $H = H_{2h}^{2h\text{-harm}} \oplus H_h$. □

REMARK 3.6. There is, in fact, a hidden parallelism in Algorithm 3.1. In the first step, the problem in the space H_h decomposes into independent subproblems if the refinement region Ω_2 consists of several disjoint components, which is often the case in practice. Cf., a similar remark in [5].

REMARK 3.7. The work in [17] uses the regularity-based estimates of [16] to develop a theory that covers the cases of singular equations — e.g., where Neumann replaces Dirichlet boundary conditions — and inexact solvers — i.e., where the solutions of (2) and (3) are only approximated. The present theory extends to such cases in a similar way.

4. FAC as a preconditioner. We can now define a preconditioner in a natural way using one iteration of Algorithm 3.1 with initial value $u = 0$ as an approximate solver. We obtain the following algorithm.

ALGORITHM 4.1. (FAC preconditioner)

Step 1. *Compute* u_h *from*

$$u_h \in H_h, \quad a(u_h, v_h) = f(v_h), \quad \forall v_h \in H_h. \tag{8}$$

Step 2. *Compute* u_{2h} *from*

$$u_{2h} \in H_{2h}, \quad a(u_{2h}, v_{2h}) = f(v_{2h}) - a(u_h, v_{2h}), \quad \forall v_{2h} \in H_{2h}. \tag{9}$$

Step 3. *Set* $u = u_h + u_{2h}$.

The value of u obtained by this algorithm is the solution of a variational problem of the form

$$u \in H_c : \quad \tilde{b}(u, v) = f(v), \quad \forall v \in H_c. \tag{10}$$

For the next theorem, we need a simple statement about iterative methods, which we formulate as a lemma for reference.

LEMMA 4.2. *Let* $u \leftarrow \mathcal{G}(u, b) = u - B^{-1}(Au - b)$ *be an iterative method for the solution of the linear system* $Au = b$. *Then* $\mathcal{G}(0, b) = B^{-1}b$.

Because the following result is somewhat easier to formulate in terms of operators rather than bilinear forms, we let $\langle\cdot,\cdot\rangle$ be another inner product on H_c and define the operators $A, \tilde{B} : H_c \to H_c$ by

$$\left.\begin{array}{l} \langle Au, v\rangle = a(u, v) \\ \langle \tilde{B}u, v\rangle = \tilde{b}(u, v) \end{array}\right\} \forall u, v \in H_c. \tag{11}$$

The following theorem is then an immediate consequence of Lemma 4.2.

THEOREM 4.3. *It holds that*

$$I - \tilde{B}^{-1}A = P_{H_{2h}^{\perp}} P_{H_h^{\perp}} = P_{(H_{2h}^{2h\text{-harm}})^{\perp}} P_{H_h^{\perp}}.$$

Because the product $P_{(H_{2h}^{2h\text{-harm}})^{\perp}} P_{H_h^{\perp}}$ is in general not a symmetric operator (with respect to the inner product $a(\cdot,\cdot)$ on H_c), it follows that the bilinear form $\tilde{b}(\cdot,\cdot)$ is in general not symmetric. Therefore, this preconditioner cannot be used directly with the conjugate gradient algorithm. However, an effective use of such a nonsymmetric preconditioner is still possible with more general Krylov space methods, see, for example, [15].

For the application of conjugate gradients, it is therefore natural to apply an additional half-step of the FAC algorithm to obtain a symmetric preconditioner. We then get a preconditioner identical to that of Bramble, Ewing, Pasciak, and Schatz [5]. (Their work is formulated for Neumann rather than Dirichlet boundary conditions, but the present discussion can be easily adapted to that framework.)

Application of Steps 1, 2, and 1 of the FAC Algorithm 3.1 with initial value $u = 0$ yields the following algorithm.

ALGORITHM 4.4. (Symmetric FAC preconditioner)
Step 1. *Compute* u_h *from*

$$u_h \in H_h : \quad a(u_h, v_h) = f(v_h), \quad \forall v_h \in H_h. \tag{12}$$

Step 2. *Compute* u_{2h} *from*

$$u_{2h} \in H_{2h} : \quad a(u_h + u_{2h}, v_{2h}) = f(v_{2h}), \quad \forall v_{2h} \in H_{2h}. \tag{13}$$

Step 3. *Compute* w_h *from*

$$w_h \in H_h : \quad a(u_h + u_{2h} + w_h, v_h) = f(v_h), \quad \forall v_h \in H_h. \tag{14}$$

Step 4. *Set* $u = u_h + u_{2h} + w_h$.
Problem (14) in Step 3 can be rewritten as

$$w_h \in H_h : \quad a(u_{2h} + w_h, v_h) = 0, \quad \forall v_h \in H_h, \tag{15}$$

because of the definition of u_h from (12). Step 3 was introduced in [5] to make sure that the decomposition $u = u_h + (u_{2h} + w_h)$ is orthogonal, which was motivated by the following variational interpretation.

For any $u \in H_c$, we have the orthogonal decomposition

$$u = u_h + u^{h\text{-harm}}, \quad u_h \in H_h, \quad u^{h\text{-harm}} \in H^{h\text{-harm}} = H_h^{\perp}. \tag{16}$$

For $u^{h\text{-harm}}$, we can further define $u^{2h\text{-harm}} \in H_{2h}^{2h\text{-harm}}$ as the unique function in $H_{2h}^{2h\text{-harm}}$ which coincides with $u^{h\text{-harm}}$ on $\Omega_1 \setminus \Omega_2$.

LEMMA 4.5. *The result of Algorithm 4.4 is the solution of the variational problem*

$$u \in H_c : \quad b(u, v) = f(v), \quad \forall v \in H_c, \tag{17}$$

with the bilinear form $b(\cdot, \cdot)$ *defined by*

$$b(u, v) = a(u_h, v_h) + a(u^{2h\text{-harm}}, v^{2h\text{-harm}}). \tag{18}$$

Proof. We adapt the proof from [5] and provide more details. Let $u = u_h + u^{h\text{-harm}}$ be the solution of (17).

First, let $v \in H_h$. Then $v = v_h$ and (17) and (18) show that u_h satisfies (12).

Next, let $v = v_{2h} \in H_{2h} = v_h + v^{h\text{-harm}}$ and note that

$$a(u^{2h\text{-harm}}, v) = a(u^{2h\text{-harm}}, v^{2h\text{-harm}}) \tag{19}$$

because $v - v^{2h\text{-harm}} \in H_{2h} \cap H_h \perp H_{2h}^{2h\text{-harm}}$. Also,

$$a(u_h, v) = a(P_{H_h} u_h, v) = a(u_h, P_{H_h} v) = a(u_h, v_h). \tag{20}$$

Thus, we have from (17) and (18) using (19) and (20) that $u^{2h\text{-harm}}$ satisfies

$$a(u^{2h\text{-harm}}, v_{2h}) = f(v_{2h}) - a(u_h, v_{2h}), \quad \forall v_{2h} \in H_{2h}.$$

Consequently, $u^{2h\text{-harm}}$ is just u_{2h}, the solution of (13).

Finally, $u^{h\text{-harm}} = u^{2h\text{-harm}}$ on $\Omega_1 \setminus \Omega_2$, and we find the values of $u^{h\text{-harm}}$ on Ω_2 from $u^{h\text{-harm}} - u^{2h\text{-harm}} = w_h$, where w_h satisfies (15). □

Spectral equivalence of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ was established in [5] using the equality

$$a(u,v) = a(u_h, v_h) + a(u^{h\text{-harm}}, v^{h\text{-harm}})$$

(the decomposition (16) is orthogonal) and the fact that

$$(21)\quad a(u^{h\text{-harm}}, u^{h\text{-harm}}) \le a(u^{2h\text{-harm}}, u^{2h\text{-harm}}) \le Ca(u^{h\text{-harm}}, u^{h\text{-harm}}), \quad \forall u \in H_c.$$

The inequality (21) was proved in [6] using the "inverse assumption" and a rather weak form of elliptic regularity. In particular, (21) holds whenever the coefficients of the form $a(\cdot,\cdot)$ are continuous and all components of Ω_2 have Lipschitz boundary.

As above, let $\langle\cdot,\cdot\rangle$ be another inner product on H_c and define the operators $A, B : H_c \to H_c$ by

$$\left.\begin{array}{l}\langle Au, v\rangle = a(u,v)\\ \langle Bu, v\rangle = b(u,v)\end{array}\right\} \forall u, v \in H_c.$$

Now application of Lemmas 4.2 and 3.4 immediately yields the following theorem, which relates our preceding results to properties of the preconditioner $b(\cdot,\cdot)$.

THEOREM 4.6. *It holds that*

$$(22)\qquad I - B^{-1}A = P_{H_h^\perp} P_{H_{2h}^\perp} P_{H_h^\perp} = P_{H_h^\perp} P_{(H_{2h}^{2h\text{-harm}})^\perp} P_{H_h^\perp}.$$

Consequently,

$$(23)\qquad a(u,u) \le b(u,u) \le \kappa a(u,u), \quad \forall u \in H_c,$$

where

$$1 - \frac{1}{\kappa} = \rho\left(P_{H_h^\perp} P_{(H_{2h}^{2h\text{-harm}})^\perp} P_{H_h^\perp}\right) = \rho\left(P_{(H_{2h}^{2h\text{-harm}})^\perp} P_{H_h^\perp})\right) = \cos^2(H_{2h}^{2h\text{-harm}}, H_h) \le \gamma^2.$$

REMARK 4.7. It follows from (22) and from the fact that the product of projections $P_{H_h^\perp} P_{H_{2h}^\perp} P_{H_h^\perp}$ has zero eigenvalues that both inequalities in (23) are sharp, that is, each holds as an equality for some $u \in H_c$. Therefore, the present bound γ, the bound C in (21) from [5] an [6], and the bound from [16] on the convergence factor of FAC, are all in fact equivalent to different bounds on $\cos(H_{2h}^{2h\text{-harm}}, H_h)$.

5. Asynchronous FAC (AFAC). In this section, we present a variation of the FAC algorithm which decomposes into independent processes, and show the relation of its convergence properties to those of FAC. The basic cycle of this algorithm is defined as follows:

ALGORITHM 5.1. (AFAC [10]) *Let $u \in H_c$ be the current approximation to* **u**.

Step 1. *Compute u_{2h} from*

$$u_{2h} \in H_{2h}: \quad a(u - u_{2h}, v_{2h}) = f(v_{2h}), \quad \forall v_{2h} \in H_{2h}$$

Step 2. *Compute u_h and w_{2h} from*

$$\begin{aligned} u_h \in H_h: &\quad a(u - u_h, v_h) = f(v_h), \quad \forall v_h \in H_h, \\ w_{2h} \in H_{2h} \cap H_h: &\quad a(u - w_{2h}, v_{2h}) = f(v_{2h}), \quad \forall v_{2h} \in H_{2h} \cap H_h \end{aligned}$$

Step 3. *Set $u \leftarrow u - (u_{2h} - w_{2h} + u_h)$.*

REMARK 5.2. The three equations in Steps 1 and 2 can be solved independently of each other, allowing for simultaneous solvers in a multiprocessor computing system. In addition, when the equation for u_h in Step 2 are solved approximately by the full multigrid method, an approximate value of w_{2h} is available at no extra cost in the process of solving for u_h. The algorithm as presented here requires synchronization before Step 3. A completely asynchronous version of the algorithm is obtained by avoiding Step 3 and adding the replacement

$$u \leftarrow u - u_{2h}$$

into Step 1 and the replacement

$$u \leftarrow u - (u_h - w_{2h})$$

into Step 2. Then both steps can be assigned to asynchronously running processes, which requires only locking of memory locations during replacement.

LEMMA 5.3. *The error $e = \mathbf{u} - u$ is transformed by Algorithm 5.1 according to*

$$e \leftarrow e - (P_{H_{2h}^{2h\text{-harm}}} + P_{H_h})e.$$

Proof. Writing $f(v) = a(\mathbf{u}, v)$, we have

$$u_{2h} = -P_{H_{2h}}e, \quad u_h = -P_{H_h}e, \quad w_{2h} = -P_{H_{2h}\cap H_h}e.$$

The proof is concluded by noting that $P_{H_{2h}} - P_{H_{2h}\cap H_h} = P_{H_{2h}^{2h\text{-harm}}}$. □

Using Lemma 2.4, we obtain the following theorem.

THEOREM 5.4. *The convergence factor of AFAC (Algorithm 5.1) is the square root of the convergence factor of FAC (Algorithm 3.1). In particular,*

$$\rho\left(I - P_{H_{2h}^{2h\text{-harm}}} - P_{H_h}\right) = \|I - P_{H_{2h}^{2h\text{-harm}}} - P_{H_h}\| = \cos(H_{2h}^{2h\text{-harm}}, H_h) \leq \gamma.$$

Proof. By Lemma 2.4, the extreme eigenvalues of $P_{H_{2h}^{2h\text{-harm}}} + P_{H_h}$ are $1 - \cos(H_{2h}^{2h\text{-harm}}, H_h)$ and $1 + \cos(H_{2h}^{2h\text{-harm}}, H_h)$. It follows that $\rho\left(I - P_{H_{2h}^{2h\text{-harm}}} - P_{H_h}\right) = \cos(H_{2h}^{2h\text{-harm}}, H_h)$. □

REMARK 5.5. This theorem and Lemma 2.4 are related to a result of P. Bjørstad [2], which gives a more complete characterization of the spectrum of the sum of two orthogonal projections using special properties of finite element spaces. A paper containing a generalization of both Theorem 5.4 and some results from [2] is in preparation [3].

REMARK 5.6. Both FAC and AFAC algorithms generalize easily to the case of more refinement levels. Unfortunately, the theory does not carry over immediately. A theory can be developed for the multilevel algorithm in certain model situations [14]. A general multilevel theory for AFAC is the subject of current research. For a convergence bound on a modified multilevel AFAC method, see [9].

REMARK 5.7. The classical Schwarz alternating method is based on geometrical notions of partitioning and overlap, but it can be easily generalized to subspaces [12]. In this way, the refinement methods treated here can be considered as general Schwarz methods. In particular, FAC can be interpreted as the classical "multiplicative" Schwarz process applied to the subspaces H_h and H_{2h}, with "overlap" $H_{2h} \cap H_h$. AFAC corresponds to an "additive" version of the Schwarz method applied to H_{2h} and $H_h \cap (H_{2h} \cap H_h)^{\perp}$, which do not "overlap".

These refinement methods can also be viewed as block relaxation schemes with blocks corresponding to their respective spaces. Thus, FAC and AFAC can be interpreted as block Gauss-Seidel and Jacobi methods, respectively. The relationship between their convergence factors noted in Theorem 5.4 can be obtained as a direct consequence of this viewpoint, cf. [10]. Truly asynchronous FAC corresponds to the well-known method of chaotic relaxation. Finally, this interpretations shows that the symmetrization of FAC as in [5] is analogous to the usual scheme for symmetrizing relaxation that follows each sweep with another in reverse order.

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