

New Domain Decomposition Strategies for Elliptic Partial Differential Equations

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Abstract. In this paper some new strategies for numerically solving elliptic partial differential equations by domain decomposition are presented which are applicable to non-overlapped and overlapped regions of the domain.

1. Introduction. The solution of elliptic partial differential equations over a 2 dimensional region can be shown to be accomplished efficiently by a range of *Domain Decomposition* techniques in which the solution of smaller problems on subdomains can be grouped together to produce the overall solution for the whole domain. Such techniques seem ideally suited for the solution of elliptic problems on irregular domains and on the present day multiprocessor systems.

The paper considers *firstly* the case of non-overlapping regions in which the vital factor is the efficient solution of the linear systems of equations governing the variables on the interfaces between the subdomains. Here we propose a *preconditioned iterative method* called P^2CG which involves the application of 2 preconditioned stages to the well known conjugate gradient method resulting in an improved convergence rate for the method.

Secondly, the case of overlapping regions is discussed and convergence factors and acceleration strategies for the Schwarz Alternating Procedure (SAP) are presented which again leads to computationally efficient algorithms for solving problems involving partial differential equations.

2. Problem Formulation for Non-Overlapping Domains. We will first formulate our approach in the simplest case of a domain split into two

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subdomains with one interface. Consider the problem:

$$Lu = f \text{ on } \Omega , \tag{2.1}$$

with boundary condition $u = u_b$ on $\partial\Omega$,

where L is a linear elliptic operator and the domain Ω is as illustrated in Fig.2.1. We will call the interface between Ω_1 and Ω_2 , Γ .

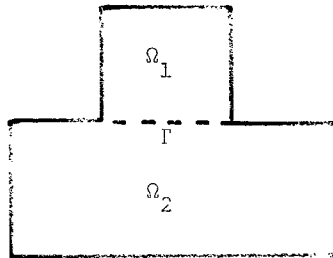


FIG. 2.1. The domain Ω and its partition

If we order the unknowns for the internal points of the subdomains first and those in the interface Γ last, then the discrete solution vector $u=(u_1, u_2, u_3)$ satisfies the linear system,

$$Au = b , \tag{2.2}$$

which can be expressed in block form as:

$$\begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} . \tag{2.3}$$

The solution of system (2.3) can now proceed as follows:

Step 1: Compute $C = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}$, (2.4)

$$w_1 = A_{11}^{-1} b_1 , \tag{2.5}$$

$$w_2 = A_{22}^{-1} b_2 , \tag{2.6}$$

and solve,

$$Cu_3 = b_3 - A_{13}^T w_1 - A_{23}^T w_2 , \tag{2.7}$$

by a suitable iterative procedure to be discussed later. Then,

Step 2: Compute $u_1 = w_1 - A_{11}^{-1} A_{13} u_3$, (2.8)

$$\text{and } u_2 = w_2^{-1} A_{22}^{-1} A_{23} u_3 \quad (2.9)$$

Note, that except for (2.7), the algorithm only requires the solution of problems with A_{11} and A_{22} , which corresponds to solving independent problems on the subdomains. The matrix C (2.4) is the Schur complement of A_{33} in A and it is sometimes called the capacitance matrix in this context. It corresponds to the reduction of the operator L on Ω to an operator on the boundary Γ .

The basic idea of the solution technique is to guess an initial solution on the boundary Γ and then to iterate to the solution of (2.7) using the preconditioned conjugate gradient method. Several alternative strategies have been proposed in which a variant of the capacitance matrix method becomes an iterative solution for the capacitance system.

In general, each iteration of the capacitance/PCG algorithm requires one solution of the smaller subproblems on domains Ω_1 and Ω_2 to form the product of the capacitance matrix C and righthand side vector. In addition, the initialisation step requires a solution of the Dirichlet problem on each of the subdomains.

3. Explicit Diagonal Block Preconditioning. We now reconsider the approach used by Evans (1984), i.e. of using a small block of fixed size, i.e., 4 points and explicitly inverting it within the iteration. Then the loss of sparseness which will inevitably occur when an explicit form of iterative method, i.e. the conjugate gradient method is used will be small and independent of the size of the given problem.

The main concern of this section is to construct new groupings of the mesh points of the network into small order groups or blocks of 4 points and to investigate their advantages when used explicitly in preconditioned iterative methods.

We consider one of the subdomains Ω_1 or Ω_2 (Fig.3.1) with Dirichlet boundary conditions.

For a large class of 2-dimensional linear elliptic differential equations in which a 5-point approximation scheme on a uniform mesh (Fig.3.2) is used, it is a simple procedure to approximate the partial derivatives of the PDE by suitable central finite difference expressions using Taylor series and quadrature techniques. Thus, the normalised finite difference equation at the point P has the form,

$$\alpha_4 u_{L,P} + \alpha_1 u_{B,P} + \alpha_2 u_{R,P} + \alpha_3 u_{T,P} = b_P \quad (3.1)$$

for all points within a coordinate square region such as in Fig.3.1.

Here we have denoted the points on the network to the Left, Top, Right, and Bottom of the representative point P in Fig.3.2 by appropriate suffixes.

For such equations, the coefficients $\alpha_1, \alpha_2, \alpha_3$ and α_4 satisfy the relationship,

$$1 \geq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad (3.2)$$

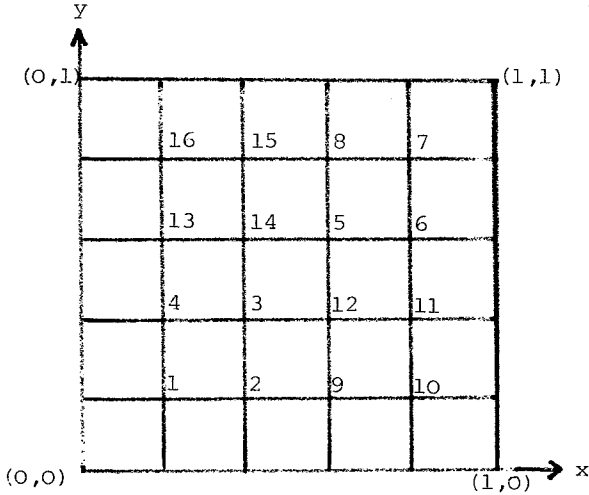


FIG. 3.1

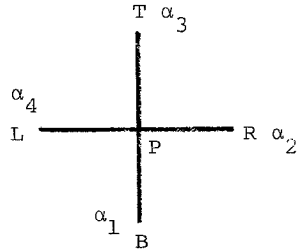


FIG. 3.2

and when grouped together by a row ordering of the points in matrix form will reduce the problem to one of solving a sparse system of linear difference equations $Au=b$ where the coefficient matrix A is of large order, $M=n^2$ non-singular, positive definite and symmetric of the well known form,

$$\begin{bmatrix} B_1 & \Lambda_3 & \circ \\ \Lambda_1 & B_2 & \circ \\ \circ & \Lambda_1 & B_r \end{bmatrix} , \text{ with } B_i = \begin{bmatrix} 1 & \alpha_2 & \circ \\ \alpha_4 & \alpha_2 & \alpha_2 \\ \circ & \alpha_4 & 1 \end{bmatrix} , i=1,2,\dots,n, \tag{3.3}$$

and $\Lambda_i = \alpha_i I$, $i=1,3$. (Varga [1]).

However, for the simple case of the unit square and $\Delta x = \Delta y = 1/5$ as illustrated in Fig.3.1, a red/black ordering of the 4-point groups will result in a coefficient matrix which has the block structure,

$$A = \begin{bmatrix} D_1 & \vdots & C_1 \\ \vdots & \vdots & \vdots \\ C_2 & \vdots & D_2 \end{bmatrix} , \tag{3.4}$$

where $D_1 = D_2 = R I_8$, with $I_8 =$ the (8×8) identity matrix and,

$$C_1 = \begin{bmatrix} R_2 & & R_3 \\ & & \\ R_1 & & R_4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} R_4 & & R_3 \\ & & \\ R_1 & & R_2 \end{bmatrix}, \quad (3.5)$$

with,

$$R_0 = \begin{bmatrix} 1 & \alpha_2 & 0 & \alpha_3 \\ \alpha_4 & 1 & \alpha_3 & 0 \\ 0 & \alpha_1 & 1 & \alpha_4 \\ \alpha_1 & 0 & \alpha_2 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (3.6)$$

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \end{bmatrix}$$

and

$$R_4 = \begin{bmatrix} 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_4 & 0 \end{bmatrix}$$

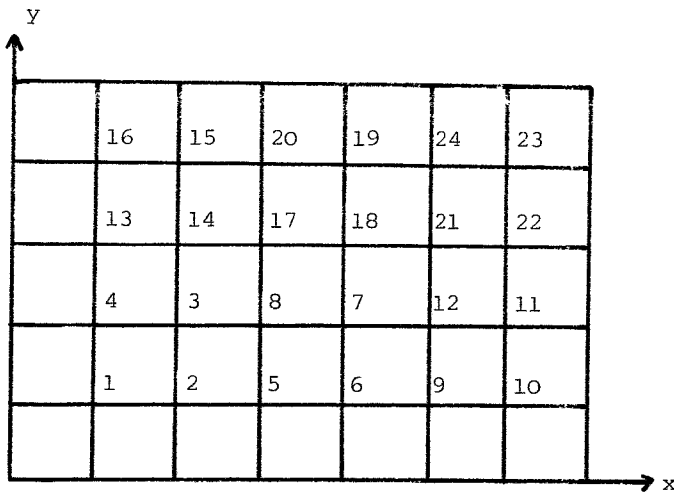


FIG. 3.3

Further for larger meshes, it is a simple matter to deduce that when the 4-point blocks are taken in natural row ordering (Fig.3.3) and because the block equations only refer to adjacent blocks then the coefficient matrix has the familiar structure, i.e.

$$A \equiv \begin{bmatrix} B_1 & C_1 & & & \\ A_1 & B_2 & C_2 & & \\ & & & \ddots & \\ & & & & C_{k-1} \\ & & & & A_k & B_k \end{bmatrix}, \quad (3.7)$$

where $C_i \equiv R_3 I$, $A_i \equiv R_1 I$ and,

$$B_i \equiv \begin{bmatrix} R_0 & & & & \\ & R_2 & & & \\ & & & \ddots & \\ & & & & R_2 \\ & & & & R_4 & R_0 \end{bmatrix}, \quad i=1,2,\dots,k.$$

Now any explicit block iterative method can be considered as a point preconditioned iterative method applied to a transformed matrix $A^E = [\text{diag}\{R_0\}]^{-1} A$ and vector $b^E = [\text{diag}\{R_0\}]^{-1} b$. Since R_0 is a small order diagonal submatrix which can be easily inverted. The matrix A^E and vector b^E (where the superfix E denotes the 'explicit' form) can be evaluated explicitly from a new computational molecule (Fig.3.4) and is a preconditioned form (diagonally block scaled) of the original matrix A. To determine the preconditioned linear system $A^E u = b^E$, with $A^E = I - L^E - U^E$, we proceed as follows:

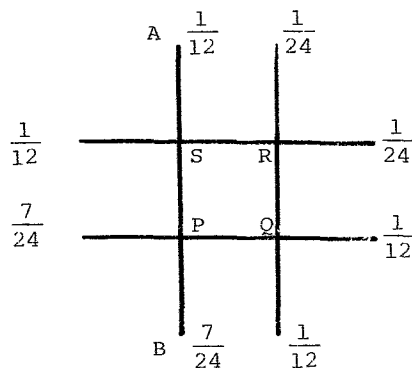


FIG. 3.4

The matrix $[\text{diag}\{R_0\}^{-1}]$ is simply $\text{diag}\{R_0^{-1}\}$ and the inverse of R_0 given by,

$$R_0^{-1} = (d^{-1}) \begin{bmatrix} \alpha_5 & -\alpha_2 \alpha_6 & 2\alpha_2 \alpha_3 & -\alpha_3 \alpha_7 \\ -\alpha_4 \alpha_6 & \alpha_5 & -\alpha_3 \alpha_7 & 2\alpha_3 \alpha_4 \\ 2\alpha_1 \alpha_4 & -\alpha_1 \alpha_7 & \alpha_5 & -\alpha_4 \alpha_6 \\ -\alpha_1 \alpha_7 & 2\alpha_1 \alpha_2 & -\alpha_2 \alpha_6 & \alpha_5 \end{bmatrix}, \quad (3.8)$$

where $d = (\alpha_1 \alpha_3 - \alpha_2 \alpha_4)^2 - 2(\alpha_1 \alpha_3 + \alpha_2 \alpha_4) + 1$ and $\alpha_5 = 1 - \alpha_1 \alpha_3 - \alpha_2 \alpha_4$, $\alpha_6 = 1 + \alpha_1 \alpha_3 - \alpha_2 \alpha_4$, $\alpha_7 = 1 + \alpha_2 \alpha_4 - \alpha_1 \alpha_3$.

The block structure of A^E is the same as that of A (i.e., eqn. (3.7)) with the submatrices R_0 replaced by identity matrices, and the submatrices R_i , replaced by $R_0^{-1} R_i$, $i=1,2,3,4$. In addition, where R_i , $i=1,2,3,4$ has a column or row of zeros so does $R_0^{-1} R_i$, and where an element α_i occurs as the (p,q) th element of R_i , the q th column of $R_0^{-1} R_i$ is the p th column of R_0^{-1} , multiplied by α_i . So, for example,

$$R_0^{-1} R_1 = (d^{-1}) \begin{bmatrix} 0 & 0 & -\alpha_1 \alpha_2 \alpha_6 & \alpha_1 \alpha_5 \\ 0 & 0 & \alpha_1 \alpha_5 & -\alpha_1 \alpha_4 \alpha_6 \\ 0 & 0 & -\alpha_1^2 \alpha_7 & 2\alpha_1^2 \alpha_4 \\ 0 & 0 & 2\alpha_1^2 \alpha_2 & -\alpha_1^2 \alpha_7 \end{bmatrix}. \quad (3.9)$$

Similarly,

$$R_0^{-1} R_2 = (d^{-1}) \begin{bmatrix} -\alpha_2^2 \alpha_6 & 0 & 0 & 2\alpha_2^2 \alpha_3 \\ \alpha_2 \alpha_5 & 0 & 0 & -\alpha_2 \alpha_3 \alpha_7 \\ -\alpha_1 \alpha_2 \alpha_7 & 0 & 0 & \alpha_2 \alpha_5 \\ 2\alpha_1 \alpha_2^2 & 0 & 0 & -\alpha_2^2 \alpha_6 \end{bmatrix},$$

$$R_0^{-1} R_3 = (d^{-1}) \begin{bmatrix} -\alpha_3^2 \alpha_7 & 2\alpha_2 \alpha_3^2 & 0 & 0 \\ 2\alpha_3^2 \alpha_4 & -\alpha_3^2 \alpha_7 & 0 & 0 \\ -\alpha_3 \alpha_4 \alpha_6 & \alpha_3 \alpha_5 & 0 & 0 \\ \alpha_3 \alpha_5 & -\alpha_2 \alpha_3 \alpha_6 & 0 & 0 \end{bmatrix},$$

and

$$R_O^{-1}R_4 = (d^{-1}) \begin{bmatrix} 0 & \alpha_4\alpha_5 & -\alpha_4\alpha_3\alpha_7 & 0 \\ 0 & -\alpha_4^2\alpha_6 & 2\alpha_3\alpha_4^2 & 0 \\ 0 & 2\alpha_1\alpha_4^2 & -\alpha_4^2\alpha_6 & 0 \\ 0 & -\alpha_1\alpha_4\alpha_7 & \alpha_4\alpha_5 & 0 \end{bmatrix}$$

For the model Laplacian problem and a square grid $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1/4$ so that,

$$R_O^{-1} = \frac{1}{6} \begin{bmatrix} 7 & 2 & 1 & 2 \\ 2 & 7 & 2 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 1 & 2 & 7 \end{bmatrix} \text{ and } R_O^{-1}R_1 = -\frac{1}{24} \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} ,$$

from which we can establish the computational stencil at the point P to be as in Fig.3.4.

Further analyses will yield analogous relationships for the remaining points Q,R and S of the 4-point block.

The use of this computational stencil to derive the solution of self-adjoint P.D.E.'s is given in Evans [2].

Finally, the transformed matrix A^E has the block form, given by eqn. (3.7), where,

$$B_i = \begin{bmatrix} I & R_O^{-1}R_2 & & & \\ R_O^{-1}R_4 & I & R_O^{-1}R_2 & & \\ & & & \ddots & \\ & & & & R_O^{-1}R_2 & \\ & & & & & I \end{bmatrix}, \quad (3.10)$$

$$C_i = R_O^{-1}R_3, \quad A_i = R_O^{-1}R_i, \quad i=1,2,\dots,k.$$

Thus, in this section we have shown that when the system of difference equations,

$$A^E \underline{u} = \underline{b}^E, \quad (3.11)$$

is derived from the new computational molecule given by Fig.3.4 then the matrix A^E has already been block diagonally scaled by the 4×4 block matrix R_O . Thus, this can be regarded as a form of diagonal scaling or preconditioning.

4. Conjugate Gradient Acceleration. Acceleration strategies to obtain the solutions of (3.11) have been shown to yield superior convergence rates in block JOR and SOR schemes, (Evans, [2]).

However, in this section we are concerned with the application of

Conjugate Gradient (CG) method to the Domain Decomposition technique.

Since the matrix A given in (3.1) is symmetric and positive definite then we can apply the C.G. algorithm in the form,

$$\begin{aligned}
 & \left. \begin{aligned} r^{(n)} &= b - Au^{(n)} \\ d^{(n)} &= -r^{(n)} \end{aligned} \right\} n=1 \\
 & \sigma_n = r^{(n)} \cdot d^{(n)} / d^{(n)} \cdot Ad^{(n)} \\
 & u^{(n+1)} = u^{(n)} + \sigma_n d^{(n)} \\
 & r^{(n+1)} = r^{(n)} - \sigma_n Ad^{(n)} \\
 & \tau_{n+1} = -r^{(n+1)} \cdot Ad^{(n)} / d^{(n)} \cdot Ad^{(n)} \\
 & d^{(n+1)} = r^{(n+1)} + \tau_{n+1} d^{(n)} \\
 & \text{until } |u^{(n+1)} - u^{(n)}| < 5 \times 10^{-6}, \text{ otherwise}
 \end{aligned} \tag{4.1}$$

Further we can immediately see that since the system $A^E u = b^E$ has been *explicitly* obtained from Fig.3.4, then in this case, the Conjugate Gradient algorithm can also be applied directly to A^E to form a preconditioned C.G. method to yield an immediate improvement. Thus we have for the *first* preconditioning stage,

$$\begin{aligned}
 & \left. \begin{aligned} r^{(n)} &= b^E - A^E u^{(n)} \\ d^{(n)} &= -r^{(n)} \end{aligned} \right\} \\
 & \sigma_n = r^{(n)} \cdot d^{(n)} / d^{(n)} \cdot A^E d^{(n)} \\
 & u^{(n+1)} = u^{(n)} + \sigma_n d^{(n)} \\
 & r^{(n+1)} = r^{(n)} - \sigma_n A^E d^{(n)} \\
 & \tau_{n+1} = -r^{(n+1)} \cdot A^E d^{(n)} / d^{(n)} \cdot A^E d^{(n)} \\
 & d^{(n+1)} = r^{(n+1)} + \tau_{n+1} d^{(n)} \\
 & \text{until } |u^{(n+1)} - u^{(n)}| < 5 \times 10^{-6} \text{ otherwise}
 \end{aligned} \tag{4.2}$$

Next a *second* preconditioning stage can be applied *implicitly* when we premultiply eqn.(3.11) by a non-singular matrix M^{-1} where M^{-1} is an approximate inverse of A. Thus, (3.11) is transformed into the pre-conditioned form,

$$M^{-1} A^E u = M^{-1} b, \tag{4.3}$$

where M is a conditioning matrix such that its inverse is easily computed.

Let us consider the possible forms for the matrix M:

a) Splitting of A^E

If we assume that A^E can be written (without loss of generality) as

$A^E = I - L^E - U^E$ where L^E and U^E are strictly lower and upper triangular matrices which can be derived from Fig.3.4 then a suitable sparse preconditioning form for M given by Evans [3],[4] is,

$$M = (I - \omega L^E)(I - \omega U^E) \quad (4.4)$$

b) Approximate factorisation of A^E

Alternatively, we can consider the factorisation of A^E into easily invertible lower and upper triangular matrix factors for which some standard methods are well known, i.e. LU , LL^T or LDU . However in the factorisation procedure of a sparse matrix, large numbers of 'fill-ins' occur (zero entries which are replaced often by small insignificant real numbers inherently incorporating a round-off error). Such numbers also greatly increase the computational work and storage requirements of the matrix. (Evans [5]).

Thus, in order to reduce the fill-in terms of the triangular factors L, U to a minimum, many researchers have considered approximate factorisation techniques of the form,

$$M = L_r U_r \approx A^E \quad (4.5)$$

where L_r, U_r denote the corresponding sparse triangular factors in which r non-zero off-diagonal 'fill-in' vectors have been retained.

Finally, the (Preconditioned)² Conjugate Gradient method (P²CG) is specifically formulated to efficiently solve the algebraic system $A^E u = b^E$ resulting from the above discretization. The P²CG method is an iterative algorithm in which the following steps are computed at each iteration. For $k > 1$,

$$\left. \begin{aligned} r^{(n)} &= M^{-1} b - M^{-1} A^E u^{(n)} \\ d^{(n)} &= -r^{(n)} \end{aligned} \right\} n=1$$

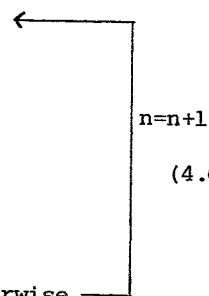
$$\sigma_n = r^{(n)} \cdot d^{(n)} / d^{(n)} \cdot M^{-1} A^E d^{(n)}$$

$$u^{(n+1)} = u^{(n)} + \sigma_n d^{(n)}$$

$$r^{(n+1)} = r^{(n)} - \sigma_n M^{-1} A^E d^{(n)}$$

$$\tau^{(n+1)} = -r^{(n+1)} \cdot M^{-1} A^E d^{(n)} / d^{(n)} \cdot M^{-1} A^E d^{(n)}$$

$$d^{(n+1)} = r^{(n+1)} + \tau_{n+1} d^{(n)}$$



to be continued until $|u^{(n+1)} - u^{(n)}| < 5 \times 10^{-6}$, otherwise

The convergence of the P²CG method is determined both by the clustering of the eigenvalues and the condition number of $M^{-1} A^E$, and thus critically depends on the selection of M in (4.4) and (4.5). Good preconditioners M are symmetric and positive definite and significantly reduce the condition number of the system, are less expensive in solving $M r^{(n)} = (b - A^E u^{(n)})$, rather than $Au = b$, and do not significantly increase the amount of storage relative to the storage needed to solve $Au = b$. By appropriately preconditioning the system we greatly reduce the amount of work expended in the computation of the solution u .

5. Convergence of the Preconditioned Conjugate Gradient Method. For the Conjugate Gradient method we have that,

$$\|u-u^{(n)}\|_A \leq \|p_n(u-u^{(0)})\|_A \leq \max_j |p_n(\lambda_j)| \|u-u^{(0)}\|_A, \quad \forall p_n \in \hat{P}_n, \quad (5.1)$$

where \hat{P}_n is the set of polynomials $p_n(z) = \sum_{j=0}^n \beta_j z^j$, $\beta_j \in \mathbb{R}$ of degree at most n with $\beta_0=1$.

To estimate the reduction of the initial error $\|u-u^{(0)}\|$ after n steps it is sufficient to construct a polynomial p_n of degree at most n such that $p_n(0)=1$ and p_n is as small as possible on the interval $[\lambda_1, \lambda_M]$ containing the eigenvalues of A , i.e. so that the quantity, i.e. the convergence rate

$$\gamma_n = \max |p_n(z)|, \quad z \in [\lambda_1, \lambda_M], \quad (5.2)$$

is as small as possible.

The best polynomial is well known in approximation theory and is the Chebyshev polynomial with the corresponding value of γ_n

$$\gamma_n = 2 \left| \frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1} \right|^n, \quad n=0,1,2. \quad (5.3)$$

Thus, for a given $\epsilon > 0$ to satisfy,

$$\|u-u^{(n)}\|_A \leq \epsilon \|u-u^{(0)}\|_A,$$

it is sufficient to choose n such that $\gamma_n \leq \epsilon$ or

$$n > \frac{1}{2} \sqrt{\kappa(A)} \log 2/\epsilon, \quad (5.4)$$

where $\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}}$ with $\lambda_{\max} = \max_j \lambda_j$ and $\lambda_{\min} = \min_j \lambda_j$ is the condition number.

We thus conclude that the required number of iterations for the conjugate gradient method is proportional to $\sqrt{\kappa(A)}$.

Since in a typical FD/FE application, we have $\kappa(A) = O(h^{-2})$ then the required number of iterations for the C.G. method would be $O(h^{-1})$.

Now, if $\rho(J)$ and $\rho(L)$ and $\rho(2B)$ denote the dominant eigenvalues of the point 1 line and 2×2 block Jacobi schemes respectively then,

$$\rho(J) = 1 - \pi^2 h^2 / 2, \quad \rho(L) = 1 - \pi^2 h^2, \quad \rho(2B) = 1 - \pi^2 h^2,$$

confirming that the matrix A^E substituted for A in the C.G. schemes will bring about an improvement of $\sqrt{2}$ (Evans [2]). Numerical experiments in progress confirm these expectations.

6. Schwarz Alternating Procedure for Overlapping Domains. We now consider the Schwarz Alternating Procedure (SAP) to a domain Ω which is decomposed into 2 overlapping sub-domains Ω_1 and Ω_2 in which the values along the internal boundaries $\Gamma_i, i=1,2$ are extrapolated (over-relaxed) by an acceleration factor ω .

Consider then the 2-dimensional Dirichlet problem,

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega = \{(x,y) \mid 0 < x < 1, 0 < y < 1\}, \\ u|_{\Gamma} &= \phi \end{aligned} \tag{6.1}$$

where f and ϕ are known. The domain Ω (Fig.6.1) is decomposed into two overlapping subdomains $\Omega_1 = \{(x,y) \mid 0 < x < x_k, 0 < y < 1\}$ and $\Omega_2 = \{(x,y) \mid x_l < x < 1, 0 < y < 1\}$, where $x_l < x_k$ and $x_k = x_l + d$.

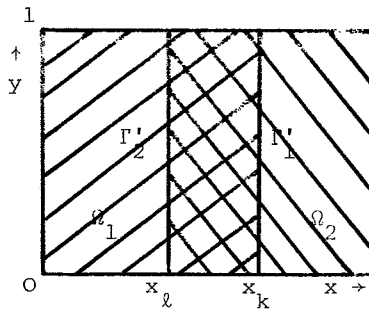


FIG.6.1

If we denote the boundaries of Ω_i by $\Gamma_i (\Gamma_i \subset \Gamma)$ and $\Gamma'_i (\Gamma'_i \subset \Omega_j), (i \neq j), i=1,2, \Gamma'_1$ and Γ'_2 are called the pseudo-boundaries along which we extrapolate.

Applying the Schwarz Alternating Procedure (SAP) with pseudo-boundary relaxation factor ω for solving problem (6.1), we obtain the following subproblems,

$$\left. \begin{aligned} -\Delta u^{(i+1)} &= f, \text{ in } \Omega_1 \\ u^{(i+1)} &= \phi, \text{ on } \Gamma_1 \\ u^{(i+1)} &= v^{(i-1)} + \omega(v^{(i)} - v^{(i-1)}) \text{ on } \Gamma'_1 \end{aligned} \right\} \tag{6.2}$$

$$\left. \begin{aligned} -\Delta v^{(i+1)} &= f, \text{ in } \Omega_2, \\ v^{(i+1)} &= \phi, \text{ on } \Gamma_2, \\ v^{(i+1)} &= u^{(i)} + \omega(u^{(i+1)} - u^{(i)}), \text{ on } \Gamma_2', \text{ } i=0,1,2,\dots, \end{aligned} \right\} \quad (6.3)$$

where, $v^{(-1)}(x_k, y)$, $v^{(0)}(x_k, y)$ and $u^{(0)}(x_l, y)$ are the initial guesses.

The errors $e^{(i)}(x, y) = u^*(x, y) - u^{(i)}(x, y)$ and $E^{(i)}(x, y) = u^*(x, y) - v^{(i)}(x, y)$ satisfy the following subproblems,

$$\left. \begin{aligned} -\Delta e^{(i+1)} &= 0, \text{ in } \Omega_1, \\ e^{(i+1)} &= 0, \text{ on } \Gamma_1, \\ e^{(i+1)} &= E^{(i-1)} + \omega(E^{(i)} - E^{(i-1)}), \text{ on } \Gamma_1' \end{aligned} \right\} \quad (6.4)$$

and,

$$\left. \begin{aligned} -\Delta E^{(i+1)} &= 0, \text{ in } \Omega_2, \\ E^{(i+1)} &= 0, \text{ on } \Gamma_2, \\ E^{(i+1)} &= e^{(i)} + \omega(e^{(i+1)} - e^{(i)}), \text{ on } \Gamma_2', \text{ } i=0,1,2,\dots, \end{aligned} \right\} \quad (6.5)$$

where $E^{(-1)}(x_k, y)$, $E^{(0)}(x_k, y)$ and $e^{(0)}(x_l, y)$ are the initial errors.

If we assume,

$$\text{and, } E^{(i)}(x_k, y) = \sum_{m=1}^{\infty} b_m^{(i)} \sin m\pi y, \text{ } i=-1,0. \quad (6.6)$$

$$e^{(0)}(x_l, y) = \sum_{m=1}^{\infty} a_m^{(0)} \sin m\pi y, \quad (6.7)$$

then the solutions of problems (6.4) and (6.5) are given by,

$$e^{(i+1)}(x, y) = \sum_{m=1}^{\infty} a_m^{(i+1)} \gamma_m(x, x_k) \sin m\pi y, \quad (6.8)$$

and,

$$E^{(i+1)}(x, y) = \sum_{m=1}^{\infty} b_m^{(i+1)} \gamma_m(1-x, 1-x_l) \sin m\pi y, \quad (6.9)$$

where,

$$\gamma_m(x, x_k) = \frac{\text{sh } m\pi x}{\text{sh } m\pi x_k}.$$

According to (6.8) and (6.9) and the pseudo-boundary conditions of problems (6.4) and (6.5) we have,

$$\begin{aligned} e^{(i+1)}(x_k, y) &= \sum_{m=1}^{\infty} a_m^{(i+1)} \sin m\pi y = (1-\omega)E^{(i-1)}(x_k, y) + \omega E^{(i)}(x_k, y) \\ &= \sum_{m=1}^{\infty} \gamma_m(1-x_k, 1-x_l) [(1-\omega)b_m^{(i-1)} + \omega b_m^{(i)}] \sin m\pi y, \end{aligned}$$

i.e.,

$$a_m^{(i+1)} = \gamma_m(1-x_k, 1-x_\ell) [(1-\omega)b_m^{(i-1)} + \omega b_m^{(i)}], \quad i=0,1,2,\dots, \quad (6.10)$$

and in the same way we get,

$$b_m^{(i+1)} = \gamma_m(x_\ell, x_k) [(1-\omega)a_m^{(i)} + \omega a_m^{(i+1)}], \quad i=0,1,2,\dots. \quad (6.11)$$

From (6.10) and (6.11) we have,

$$a_m^{(i+1)} = \rho_m(x_\ell, x_k) [\omega^2 a_m^{(i)} - 2\omega(\omega-1)a_m^{(i-1)} + (\omega-1)^2 a_m^{(i-2)}], \quad i=2,3,4,\dots, \quad (6.12)$$

and,

$$b_m^{(i+1)} = \rho_m(x_\ell, x_k) [\omega^2 b_m^{(i)} - 2\omega(\omega-1)b_m^{(i-1)} + (\omega-1)^2 b_m^{(i-2)}], \quad i=1,2,3,\dots, \quad (6.13)$$

where, for $m=1,2,3,\dots$

$$\rho_m(x_\ell, x_k) = \gamma_m(x_\ell, x_k) \cdot \gamma_m(1-x_k, 1-x_\ell), \quad (6.14)$$

and, $a_m^{(0)}, b_m^{(-1)}$ and $b_m^{(0)}$ are known,

$$\begin{aligned} a_m^{(1)} &= \gamma_m(x_\ell, x_k) [(1-\omega)b_m^{(-1)} + \omega b_m^{(0)}], \\ b_m^{(1)} &= \gamma_m(1-x_k, 1-x_\ell) [(1-\omega)a_m^{(0)} + \omega a_m^{(1)}], \\ a_m^{(2)} &= \gamma_m(x_\ell, x_k) [(1-\omega)b_m^{(0)} + \omega b_m^{(1)}]. \end{aligned} \quad (6.15)$$

Thus, from (6.8), (6.9), (6.12) and (6.13) we have,

$$\begin{aligned} \| |e^{(i+1)}(x,y) | \|_{L_2}^2 &\leq \| |e^{(i+1)}(x_k,y) | \|_{L_2}^2 \\ &= \int_0^1 [e^{(i+1)}(x_k,y)]^2 dy \\ &= \frac{1}{2} \sum_{m=1}^{\infty} [a_m^{(i+1)}]^2, \quad (6.16) \\ &= \frac{1}{2} \sum_{m=1}^{\infty} \{ \rho_m(x_\ell, x_k) [\omega^2 a_m^{(i)} - 2\omega(\omega-1)a_m^{(i-1)} \\ &\quad + (\omega-1)^2 a_m^{(i-2)}] \}^2 \\ &\leq \frac{1}{2} \sum_{m=1}^{\infty} \{ \rho_1 \omega^2 |a_m^{(i)} - 2a_m^{(i-1)} + a_m^{(i-2)}| \}^2, \quad (6.17) \end{aligned}$$

and,

$$\| |E^{(i+1)}(x,y) | \|_{L_2}^2 \leq \frac{1}{2} \sum_{m=1}^{\infty} \{ \rho_1 \omega^2 |b_m^{(i)} - 2ab_m^{(i-1)} + a^2 b_m^{(i-2)}| \}^2, \quad (6.18)$$

where $a=(\omega-1)/\omega$, and,

$$\rho_1 = \rho_1(x_\ell, x_k) = \left(\frac{\text{sh } \pi x_\ell}{\text{sh } \pi x_k} \right) \left(\frac{\text{sh } \pi(1-x_k)}{\text{sh } \pi(1-x_\ell)} \right). \quad (6.19)$$

If we let $a_m^{(i)} = r^i$ (or $b_m^{(i)} = r^i$), then the characteristic equation of (6.12) (or (6.13)) is obtained in the form,

$$r^3 - \rho_m (x_\ell, x_k) \omega^2 (r-a)^2 = 0. \quad (6.20)$$

From (6.16) and (6.17) we can see that the convergence-rate is determined by the low-frequency component (see Evans, [6],[7]). So we define the convergence factor of procedures (6.2) and (6.3) as,

$$r(\rho, \omega) = \max_{1 \leq j \leq 3} |r_j|, \quad (6.21)$$

where r_1, r_2, r_3 are the roots of the cubic equation,

$$r^3 - \rho_1 \omega^2 (r-a)^2 = 0, \quad 0 \leq \rho_1 < 1. \quad (6.22)$$

By making the substitution,

$$r = (R + \frac{\rho \omega^2}{3}), \quad (6.23)$$

the reduced cubic equation is obtained in the form,

$$R^3 + R(2a - \frac{1}{3}\rho\omega^2)\rho\omega^2 + (\frac{2}{3}\rho\omega^2 a - \frac{2}{27}\rho^2\omega^4 - a^2)\rho\omega^2 = 0, \quad (6.24)$$

$$\text{or } R^3 + bR + c = 0, \quad (6.25)$$

$$\text{with, } b = \rho\omega^2(2a - \frac{1}{3}\rho\omega^2), \quad (6.26)$$

$$c = \rho\omega^2(\frac{2}{3}\rho\omega^2 a - \frac{2}{27}\rho^2\omega^4 - a^2),$$

where $a = (\omega-1)/\omega$, and $\rho_1 =$ convergence factor at $\omega=1$.

The maximum rate of convergence of (6.22) is given by the value of ω which allows the discriminant of (6.25) to be zero, i.e.,

$$c^2/4 + b^3/27 = 0. \quad (6.27)$$

By substituting the values of b and c of (6.26) into (6.27), we obtain the expression which when simplified yields the relation,

$$a^3(27a - 4\rho\omega^2) = 0. \quad (6.28)$$

Thus, the optimum value of ω , i.e. ω_0 , which yields the maximum rate of convergence is given by the nonlinear relation $\omega^2 = 27a/4$ or the cubic equation,

$$\omega^3 - \frac{27}{4}\omega + \frac{27}{4\rho} = 0, \quad (6.29)$$

which is already in its reduced form.

Now for $\omega > 1$ the discriminant of (6.29) is ≤ 0 so the solution of (6.29) is given by the trigonometric relations,

$$\begin{aligned} \omega_1 &= 2\sqrt{-\beta/3} \cos(s/3), \\ \omega_2 &= 2\sqrt{-\beta/3} \cos(s/3 + 2\pi/3), \end{aligned} \quad (6.30)$$

$$\text{and } \omega_3 = 2\sqrt{-\beta/3} \cos(s/3 + 4\pi/3),$$

with,
$$s = \arccos(\beta/2\sqrt{-\beta^3/27}) , \tag{6.31}$$

where $\beta = -27/4$.

The optimum value of ω which maximises the convergence rate ($1 < \omega < 2$) is then given by,

$$\omega_0 = 2\sqrt{-\beta/3} \cos(s/3 + 4\pi/3) , \tag{6.32}$$

or
$$\omega_0 = (3/\sqrt{\rho}) \cos[(s+4\pi)/3] , \tag{6.33}$$

with $s = \arccos(-\sqrt{\rho})$.

The optimum convergence factor ρ_0 at the optimum value ω_0 is determined from the solution of (6.25) at which the discriminant $c^2/4 + b^3/27 = 0$.

The roots of (6.25) are given by,

$$\begin{aligned} R_1 &= A+B \\ R_2 &= -(A+B)/2 + 0.5(A-B)\sqrt{3i} , \\ R_3 &= -(A+B)/2 - 0.5(A-B)\sqrt{3i} , \end{aligned} \tag{6.34}$$

where

$$\begin{aligned} A &= 3\sqrt{-c/2 + \sqrt{c^2/4 + b^3/27}} , \\ B &= 3\sqrt{-c/2 - \sqrt{c^2/4 + b^3/27}} . \end{aligned}$$

Also, at the optimum value ω_0 , relation (6.28) holds from which we can establish that,

$$A_0 - B_0 = -3a_0/4 , \tag{6.35}$$

where $a_0 = (\omega_0 - 1)/\omega_0$.

Thus the optimum factor ρ_0 is given by the solution of (6.22) which is related to the solution of (6.25) by the transformation (6.23) and is,

$$\rho_0 = 3a_0/4 + \rho\omega_0^2/3 , \tag{6.36}$$

and by virtue of (6.28) we have,

$$\rho_0 = 3a_0 = 3(\omega_0 - 1)/\omega_0 .$$

Finally, the range of convergence of the Schwarz alternating method with overrelaxation factor ω on the pseudo boundaries can be determined as follows.

From an examination of the sign of the discriminant ($c^2/4 + b^3/27$) of (6.25) it can be established that at the ends of the convergence range, there exists one real root and two complex conjugate roots of (6.23). Thus, if we denote the roots of (6.23) as $\alpha_1, \alpha_2, \pm i\gamma_2$ and using the theory of equations, the following relations can be established,

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= \rho\omega^2 \\ 2\alpha_1\alpha_2 + (\alpha_2^2 + \gamma_2^2) &= 2a\rho\omega^2 \\ \alpha_1(\alpha_2^2 + \gamma_2^2) &= a^2\rho\omega^2 \end{aligned} \tag{6.37}$$

Now the range of convergence is determined by the roots $|\alpha_2 \pm i\gamma_2| = 1$ and consequently the relations (6.37) become,

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= \rho\omega^2 \\ 2\alpha_1\alpha_2 + 1 &= 2a\rho\omega^2 \\ \alpha_1 &= a^2\rho\omega^2, \end{aligned} \tag{6.38}$$

to yield, $(2a\rho\omega^2 - 1)/a^2\rho\omega^2 = \rho\omega^2(1-a)$, (6.39)

or $\rho\omega^2 = 1/[a(1+a)]$.

By the substitution of $a=(\omega-1)/\omega$, this relation simplifies to the quadratic equation,

$$(\omega-1)(2\omega-1) = 1/\rho \text{ ,} \tag{6.40}$$

for the determination of the values $\omega = \omega_f$ for which the Schwarz alternating method is convergent and is,

$$\omega_f = 2(1-1/\rho)/(3 \pm \sqrt{9-8(1-1/\rho)}) \text{ .} \tag{6.41}$$

Convergence factor ρ_1 at $\omega=1$	Optimal overrelaxation parameter ω	Optimal convergence factor $\rho_0 = \gamma(\rho, \omega_0)$	Convergence range	
			Lower ω_f	Upper ω_f
0.6	1.127	0.339	-0.196	1.696
0.65	1.144	0.378	-0.162	1.662
0.7	1.163	0.421	-0.131	1.631
0.75	1.185	0.468	-0.104	1.604
0.8	1.210	0.520	-0.079	1.579
0.85	1.240	0.581	-0.057	1.557
0.9	1.279	0.654	-0.036	1.536
0.95	1.334	0.752	-0.017	1.517
0.98	1.389	0.841	-0.007	1.507
0.99	1.419	0.887	-0.003	1.503

TABLE 6.1

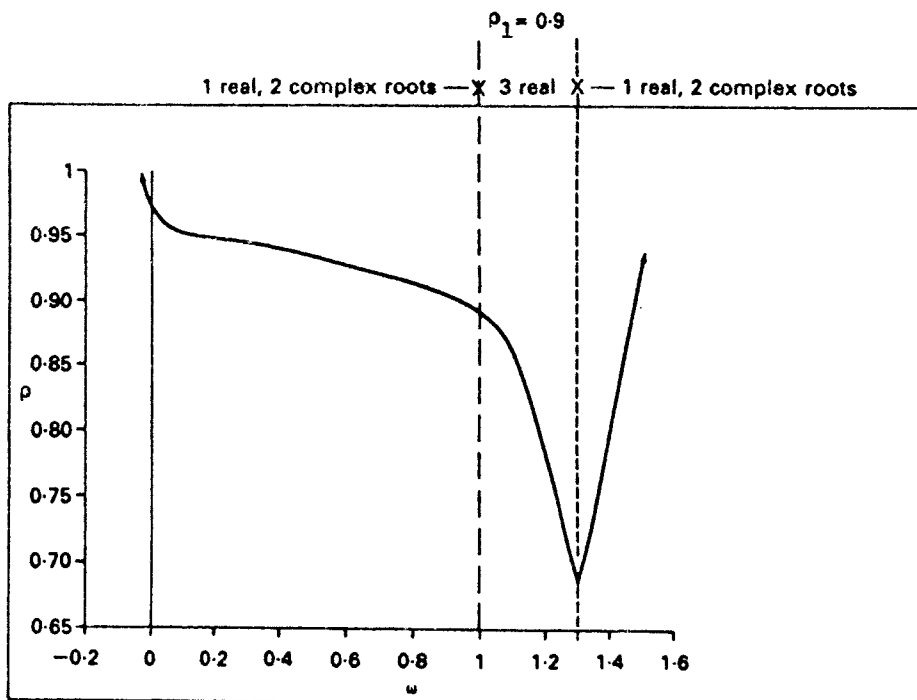


FIG. 6.2

Some typical values of the optimal overrelaxation factor ω_0 , the maximum convergence factor ρ_0 and the convergence range of the Schwarz alternating method for various values of ρ are given in Table 6.1. Also, a plot of ρ against ω is given in Figure 6.2 for $\rho_1=0.9$ at $\omega=1$.

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