On the Schwarz Alternating Method II: Stochastic Interpretation and Order Properties

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Abstract. We continue here a systematic investigation of convergence properties of the Schwarz alternating method and related decomposition methods. Our study here is based upon the maximum principle and the stochastic interpretation of the Schwarz alternating procedure.

Introduction.

This paper is a sequel of [36] and part II of a series of papers devoted to the mathematical study of various decomposition methods (*domain decomposition methods*) for the solution of various linear or nonlinear partial differential equations. In the recent years, the applications of iterative methods solving subproblems or problems in subdomains to the numerical analysis of boundary value problems have been developed by various authors and a partial list of contributions to this general theme can be found in the bibliography.

Parts I and II of this series of papers are devoted to the study of the classical Schwarz alternating method (that we recall in section I below). In some sense, even if many interesting and important variants have been introduced recently, the Schwarz algorithm remains the prototype of such methods and also presents some properties (like "robustness", or indifference to the type of equations considered...) which do not seem to be enjoyed by other methods. In part I [36], we studied the Schwarz alternating method from a variational view-point (iterated projections in an Hilbert space) and obtained various convergence results. In some sense, with such a variational viewpoint, one is naturally led to variants based upon "control-calculus of variations" considerations (as in [10], [11], [12] ...) which, at least for Laplace equations, are a bit faster for computing applications.

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On the other hand, as it was originally proved by Schwarz [1], the Schwarz alternating method also converges, say for Laplace equations, because of the strong maximum principle for harmonic functions (see for instance the paragraph on Schwarz method in [37]). Let us observe, by the way, that the Schwarz alternating method seems to be the only domain decomposition method converging for two entirely different reasons : variational characterzation of the Schwarz sequence and maximum principle.

This paper is a systematic study of such properties of Schwarz alternating method. First, we recall in section 1 from [38] the stochastic interpretation of Schwarz method in terms of successive exit times from the subdomains of the underlying diffusion processes (Brownian motion in the case of Laplace equation). This interpretation shows that Schwarz method is intimately (even if simply) connected with the deep structure of Laplace's equation.

Next, in section 2, we present a convergence proof for Schwarz method for uniformly elliptic equations in the case of overlapping domains : as we will see the convergence is geometrical and we will indicate an estimate on the rate of convergence. In section 3, we study the same question when we relax the condition of overlapping, allowing the "boundaries of the two subdomains" to touch at the boundary of the original domain". As we will see, if the situation of section 2 is not basically modified for Dirichlat boundary conditions (in this case, our analysis is a minor extension of Schwarz original convergence proof), we will show that drastic changes occur for Neumann boundary conditions. Next, in section 4, we observe that if we start with a subsolution (respectively a supersolution) of the full problem, Schwarz alternating method creates an increasing (respectively decreasing) sequence of subsolutions (respectively supersolutions) and this will allow to prove convergence in some geometrical situations where the condition that the two subdomains overlap is somehow relaxed (including the case of Neumann boundary conditions considered in section 3). Section 5 is devoted to equations which are no more uniformly elliptic like degenerate elliptic equations or parabolic equations (heat equation for instance) possibly with a time discretization : in the cases when the two subdomains strictly overlap, we will show that geometrical convergence is still true, and this will be a consequence of the fact that Schwarz lemma is still true for degenerate equations (and therefore is not always related to the strong maximum principle). Next, we will present in section 6 another convergence proof based upon the stochastic interpretation : this proof will be purely probabilistic and will give some hint on the way to "optimize the domain splitting" in order to obtain the fastest convergence. Section 7 will a brief presentation of the applications of Schwarz alternating methods when the original domain is split into more than two subdomains (section 8).

Let us also mention that we will study in the remaining parts of this series of papers some variants of Schwarz alternating method that we will introduce to take care of the geometrical situation when the domain is split (decomposed) in two (or more) subdomains separated only by an interface (n-1 -dimensional manifold) - in particular, these subdomains do not overlap at all.

Summary

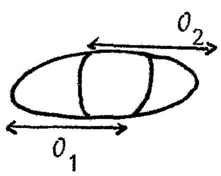
1. Presentation of Schwarz method and stochastic interpretation.

We consider a bounded, open domain $\mathcal{O}$ in $\mathbb{R}^N$ and we assume (to simplify) that $\mathcal{O}$ is smooth and connected. We then decompose $\mathcal{O}$ in two subdomains $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

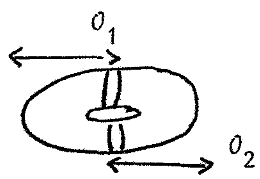
(1) $$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$$

and we denote by $\Gamma = \partial\mathcal{O}$, $\Gamma_1 = \partial\mathcal{O}_1$, $\Gamma_2 = \partial\mathcal{O}_2$, $\gamma_1 = \partial\mathcal{O}_1 \cap \mathcal{O}_2$, $\gamma_2 = \partial\mathcal{O}_2 \cap \mathcal{O}_1$, $\mathcal{O}_{12} = \mathcal{O}_1 \cap \mathcal{O}_2$, $\mathcal{O}_{11} = \mathcal{O}_1 \cap \overline{\mathcal{O}}_2^c$, $\mathcal{O}_{22} = \mathcal{O}_2 \cap \overline{\mathcal{O}}_1^c$.

Various decompositions are possible as it can be seen from the following figures

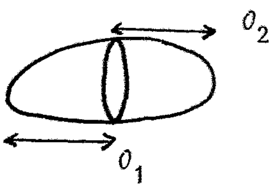


1.a

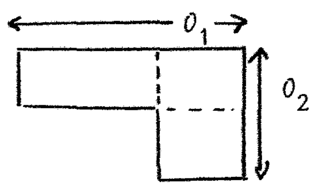


1.b

1.c



2.a



2.b

We will always assume to simplify that γ_1, γ_2 are smooth...

We will also say that $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap if $\overline{\mathcal{O}}_{11}$ and $\overline{\mathcal{O}}_{22}$ do not intersect. Observe that this is the case in figures 1.a, 1.b, 1.c but not in figures 2.a or 2.b.

Next, suppose that we want to solve the following model problem

(2) $$-\Delta u = f \quad \text{in } \mathcal{O} \quad , \quad u = 0 \quad \text{on } \partial\mathcal{O}$$

where f is a given function say in $C(\overline{\mathcal{O}})$ (or in $H^{-1}(\mathcal{O})$ ...) . The Schwarz alternating procedure consists in solving successively the following problems : let u^o be any initialization say in $C_o(\overline{\mathcal{O}})$ (i.e. continuous functions on $\overline{\mathcal{O}}$ vanishing on $\partial\mathcal{O}$), we obtain u^{2n+1} $(n \geq 0)$ and u^{2n} $(n \geq 1)$ by solving respectively

(3) $$-\Delta u^{2n+1} = f \quad \text{in } \mathcal{O}_1 \quad , \quad u^{2n+1} = u^{2n} \quad \text{on } \partial\mathcal{O}_1$$

(4) $$-\Delta u^{2n} = f \quad \text{in } \mathcal{O}_2 \quad , \quad u^{2n} = u^{2n-1} \quad \text{on } \partial\mathcal{O}_2 \quad ,$$

and $u^{2n+1} \in C(\overline{\mathcal{O}}_1)$, $u^{2n} \in C(\overline{\mathcal{O}}_2)$, $u^{2n+1} = 0$ on $\Gamma_1 \cap \Gamma$, $u^{2n} = 0$ on $\Gamma_2 \cap \Gamma$. In fact, (3),(4) require that u^{2n}, u^{2n+1} are defined on $\overline{\mathcal{O}}$ and we extend obviously u^{2n+1} and u^{2n} respectively to $\overline{\mathcal{O}}$ by u^{2n} and u^{2n-1} so that $u^{2n+1}, u^{2n} \in C_o(\overline{\mathcal{O}})$ and $u^{2n+1} \equiv u^{2n}$ on $\overline{\mathcal{O}}_{22}$, $u^{2n} \equiv u^{2n+1}$ on $\overline{\mathcal{O}}_{11}$.

In [36] , we explained how u^{2n+1}, u^{2n} correspond to successive projections in subspaces of $H^1_o(\mathcal{O})$. We want to present now a different interpretation of the sequence $(u^n)_n$ in terms of successive exit times from $\overline{\mathcal{O}}_1$ resp. $\overline{\mathcal{O}}_2$. To this end, we consider any standard probability space (Ω, F, F^t, P) equipped with a Brownian motion B_t continuous and F^t adapted and we introduce the following stopping times

(5) $$\tau_x = \inf \{t \geq 0 \ , \ x+B_t \notin \mathcal{O}\}$$

(6) $$\tau_x^{2n+1} = \inf \{t \geq \tau_x^{2n} \ , \ x+B_t \notin \mathcal{O}_1\} \quad \text{for } n \geq 0$$

(7) $$\tau_x^o \equiv 0 \ , \ \tau_x^{2n} = \inf \{t \geq \tau_x^{2n-1}, x+B_t \notin \mathcal{O}_2\} \quad \text{for } n \geq 1$$

for all $x \in \overline{\mathcal{O}}$. As it is well-known, the solution u of (2) is given by the formula

(8) $$u(x) = E \int_o^{\tau_x} f(x+B_t)\, dt \quad , \quad \forall\, x \in \overline{\mathcal{O}} \quad .$$

At this point, let us mention once for all that B_t is not really a Brownian motion but $B_t = \sqrt{2}\, B'_t$ and B'_t is a Brownian motion. Then, we claim that we have the

<u>Lemma 1.</u> For all $x \in \overline{\mathcal{O}}$ and for all $n \geq 0$, we have

(9) $$u^n(x) = E \int_o^{\tau_x^n} f(x+B_t)\, dt + u^o\left(x+B_{\tau_x^n}\right)$$

Remarks. i) From the definitions (5),(6),(7), one has immediately

(10) $$\tau_x^n \uparrow_n \ , \quad \tau_x^n \leq \tau_x \quad \text{for all } n \geq 0 \text{ and for all } x \in \overline{\mathcal{O}} .$$

ii) Recall also (see for instance [39]) that if λ_1 is the first eigenvalue of $-\Delta$ in $H_o^1(\mathcal{O})$ then we have

$$\sup_{x \in \overline{\mathcal{O}}} E\left[e^{\lambda\tau_x}\right] < \infty \qquad \text{for all } \lambda \in (0,\lambda_1)$$

hence in particular $\tau_x < \infty$ a.e., for all $x \in \overline{\mathcal{O}}$.

Proof of Lemma 1. By induction, suppose that (9) holds for n and let us prove the same formula for $n+1$. Without loss of generality we may assume that n is even. Then, if $x \notin \overline{\mathcal{O}}_1$, $\tau_x^{n+1} = \tau_x^n$ and $u^{n+1}(x) = u^n(x)$ therefore (9) is proved in this case. On the other hand, if $x \in \overline{\mathcal{O}}_1$, we recall from standard facts that u^{n+1} is given by

(11) $$u^{n+1}(x) = E\int_o^{\tau'_x} f(x+B_t)dt + u^n\left(x+B_{\tau'_x}\right) \ , \quad \forall\, x \in \overline{\mathcal{O}}_1$$

where $\tau'_x = \inf\,(t \geq 0\ ,\ x+B_t \notin \mathcal{O}_1)$.

Then, using formula (9) for u^n and the Markov property, (11) immediately yields (3) for u^{n+1} . Δ

2. Convergence proof via the maximum principle : overlapping domains.

All throughout this section, we assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap and that $\partial\mathcal{O}_1 \cap \partial\mathcal{O}$, $\partial\mathcal{O}_2 \cap \partial\mathcal{O}$ are both nonempty. The main convergence result is given by the

Proposition 2. There exist $k_1,k_2 \in (0,1)$ which depend only respectively of $(\mathcal{O}_1,\gamma_2)$ and $(\mathcal{O}_2,\gamma_1)$ such that for all $n \geq 0$

(12) $$\sup_{\overline{\mathcal{O}}_1} |u-u^{2n+1}| \leq k_1^n\, k_2^n \sup_{\gamma_1} |u-u^o|$$

(13) $$\sup_{\overline{\mathcal{O}}_2} |u-u^{2n}| \leq k_1^n\, k_2^{n-1} \sup_{\gamma_2} |u-u^o| \quad .$$

Remarks. 1) We will give below some estimates on k_1,k_2 .

2) As we will see from the proof below, it is not necessary to take $u^o \in C_o(\overline{\mathcal{O}})$: for instance, it is enough to consider $u^o \in C(\overline{\mathcal{O}})$ and we then define u^1 by (3) replacing the boundary conditions on $\partial\mathcal{O}_1$ as follows

$$u^1 = 0 \quad \text{on } \partial\mathcal{O}_1 \cap \partial\mathcal{O} \ , \quad u^1 = u^o \quad \text{on } \partial\mathcal{O}_1 \cap \mathcal{O}_2 \ .$$

Then, $u^2 \in C_o(\overline{\mathcal{O}})$ and the estimates (12),(13) still hold.

3) A similar result with the same proof holds for different boundary conditions on $\partial\mathcal{O}$ or for more general uniformly elliptic second-order operators. More precisely, the same result holds if we replace Dirichlet boundary condition by Neumann or oblique derivative or Robin type or mixed type boundary condition. Furthermore, the operator $-\Delta$ may be replaced by any divergence free operator

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_i b_i \frac{\partial}{\partial x_i} + c$$

where $a_{ij}, b_i, c \in L^\infty$, $c \geq 0$ and

$$(14) \qquad \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{, a.e. } x \in \mathcal{O}, \text{ for some } \nu > 0$$

or by a nondivergence operator $-\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_i b_i\frac{\partial}{\partial x_i} + c$ where $a_{ij} \in C(\overline{\mathcal{O}})$ (for instance) satisfy (14), $c, b_i \in L^\infty$, $c \geq 0$.

4) A priori, the rate of convergence given by $(k_1k_2)^{1/2}$ is different from the one obtained by the variational (projections in Hilbert spaces) argument of [36] since the "errors" $u-u^{2n+1}$, $u-u^{2n}$ are estimated in different norms (L^∞ here, instead of H^1 in [36]).

5) It is possible to deduce from Proposition 2 the geometric convergence of u^n to u in H^1 : in fact, more generally, even if u_o is not smooth (say $u_o \in H^1_o(\mathcal{O}) + C_o(\overline{\mathcal{O}})$) one can show easily by interior elliptic regularity - using thus the fact that $\mathcal{O}_1$ and $\mathcal{O}_2$ do overlap - that there exists $n_o \geq 1$ (depending only on the dimension N) such that $u-u^n \in H^1_o(\mathcal{O}) \cap C_o(\overline{\mathcal{O}})$ for $n \geq n_o$. Then, denoting by $\tilde{u}^n = u^{n_o+n}$ for $n \geq 0$, one sees that $\tilde{u}^n$ is a new "Schwarz sequence" corresponding to the new initialization $\tilde{u}^o = u^{n_o}$ hance Proposition 2 applies to $u-\tilde{u}^n$ and one still gets geometric convergence of u^n to u in $L^\infty(\mathcal{O})$. Furthermore, we claim that u^n converges geometrically to u in $H^1_o(\mathcal{O})$: indeed, introducing (as in [36]) $\zeta_1, \zeta_2 \in W^{1,\infty}(\mathcal{O})$ such that $0 \leq \zeta_1, \zeta_2 \leq 1$, $\zeta_1+\zeta_2 \equiv 1$ on $\overline{\mathcal{O}}$, $\zeta_i \equiv 1$ on $\overline{\mathcal{O}}_{ii}$, $\zeta_i \equiv 0$ on $\overline{\mathcal{O}}_{jj}$ for all $i \neq j \in \{1,2\}$, we first obtain multiplying the equation satisfied by $u-u^n$ (for instance) by $\zeta_2^2(u-u^{2n})$

$$\int_{\mathcal{O}} \zeta_2^2|\nabla(u-u^{2n})|^2 = -2\int_{\mathcal{O}} \zeta_2\,\nabla\zeta_2\cdot\nabla(u-u^{2n})(u-u^{2n})$$
$$\leq C\left(\int_{\mathcal{O}} \zeta_2^2|\nabla(u-u^{2n})|^2\right)^{1/2} k_1^n k_2^{n-1} .$$

Hence,

$$|\zeta_2(u-u^{2n})|_{H^1_o(\mathcal{O})} \leq C\,k_1^n\,k_2^{n-1} .$$

Then, observing that $u-u^{2n+1} - \zeta_2(u-u^{2n}) \in H_o^1(\mathcal{O})$, we deduce easily

$$\int_{\mathcal{O}_1} |\nabla(u-u^{2n+1})|^2 = \int_{\mathcal{O}_1} \nabla(u-u^{2n+1}) \cdot \nabla\{\zeta_2(u-u^{2n})\}$$

or

$$\int_{\mathcal{O}_1} |\nabla(u-u^{2n+1})|^2 \leq \int_{\mathcal{O}_1} |\nabla\{\zeta_2(u-u^{2n})\}|^2 .$$

And since $u-u^{2n+1} = u-u^{2n} = \zeta_2(u-u^{2n})$ on $\mathcal{O}-\mathcal{O}_1$, we deduce finally

$$\int_{\mathcal{O}} |\nabla(u-u^{2n+1})|^2 \leq \int_{\mathcal{O}} |\nabla\{\zeta_2(u-u^{2n})\}|^2$$

i.e.

$$|u-u^{2n+1}|_{H_o^1(\mathcal{O})} \leq C\, k_1^n k_2^{n-1} ,$$

proving thus our claim. Notice also that this proof remains valid for general uniformly elliptic second-order operators with straightforward bounds for the first-order terms.

6) In the case when, for example, $\overline{\mathcal{O}}_2 \subset \mathcal{O}$ and thus $\partial\mathcal{O}_2 \cap \partial\mathcal{O} = \emptyset$ then the same result with the same proof holds provided we take $k_2 = 1$.

Δ

We now turn to the proof of Proposition 2 which is an immediate consequence of the following standard lemma.

Lemma 3. Let $w \in L^\infty(\mathcal{O}_1)$ be continuous on $\overline{\mathcal{O}}_1 - \{\overline{\partial\mathcal{O}_1 \cap \mathcal{O}} \cap \partial\mathcal{O}\}$, satisfy

$$(15) \qquad -\Delta w = 0 \text{ in } \mathcal{O}_1 , \quad w = 0 \text{ on } \partial\mathcal{O}_1 - \overline{\partial\mathcal{O}_1 \cap \mathcal{O}} , \quad w = 1 \text{ on } \partial\mathcal{O}_1 \cap \mathcal{O} .$$

Then,

$$k_1 = \sup\{w(x) \,/\, x \in \partial\mathcal{O}_2 \cap \overline{\mathcal{O}}\} \in (0,1) .$$

Remarks. 1) Of course, a similar lemma holds for $\mathcal{O}_2$.

2) An estimate on k_1 is given below.

3) In one dimension, if $\mathcal{O}_1 = (a,b)$, $\mathcal{O}_2 = (c,d)$ with $a < c < b < d$, then $w(x) = \frac{x-a}{b-a}$ and $k_1 = \frac{c-a}{b-a}$.

Once Lemma 3 is proven, Proposition 2 follows easily from the maximum principle since $|u-u^{2n+1}| \leq \sup_{\gamma_1} |u-u^{2n}| w$, hence $\sup_{\gamma_2} |u-u^{2n+1}| \leq k_1 \sup_{\gamma_1} |u-u^{2n}|$ and thus $\sup_{\gamma_2} |u-u^{2n+1}| \leq$ $\leq k_1^{n+1} k_2^n \sup_{\gamma_1} |u-u^o|$, $\sup_{\gamma_1} |u-u^{2n}| \leq k_1^n k_2^n \sup_{\gamma_1} |u-u^o|$. Then, (12) and (13) follows from the maximum principle which yields

$$\sup_{\overline{\mathcal{O}}_1} |u-u^{2n+1}| = \sup_{\gamma_1} |u-u^{2n+1}| = \sup_{\gamma_1} |u-u^{2n}| \text{ for } n \geq 0,$$

$$\sup_{\bar{\mathcal{O}}_2} |u-u^{2n}| = \sup_{\gamma_2} |u-u^{2n}| = \sup_{\gamma_2} |u-u^{2n-1}| \text{ for } n \geq 1 .$$

Finally, let us conclude by mentioning that Lemma 3 is an immediate consequence of the strong maximum principle. Δ

It is difficult (apparently) to obtain a sharp estimate on k_1 since k_1 depends very much on the geometries of $\mathcal{O}_1$ and $\mathcal{O}_2$. However, it is possible to estimate k_1 asymptotically when γ_1 "converges" to γ_2. In order to avoid rather unpleasant technicalities, we will consider only the case of figure 1.c (even if a similar analysis can be performed in general situations) i.e. $\gamma_1, \gamma_2 \subset \mathcal{O}$. Then, we just observe that if ν is the unit outward normal to γ_1 by Hopf maximum principle we have $\frac{\partial w}{\partial \nu} \geq \kappa > 0$ on γ_1. Hence, if $\varepsilon = \max_{y \in \gamma_2} d(y,\gamma_1)$, we have up to the second-order terms (ε^2 terms) $k_1 \cong 1-\kappa\varepsilon$. Observe also that if γ_1 "goes to" $\partial\mathcal{O}$, κ behaves like dist $(\gamma_2,\partial\mathcal{O})^{-1}$. We will not push further here this kind of estimates.

3. Convergence proof via the maximum principle : weakly overlapping domains.

We now turn to the case when $\mathcal{O}_1$ and $\mathcal{O}_2$ do not overlap as it is the case in figures 2.a or 2.b. We still assume, of course, that $\gamma_1 \cap \gamma_2 = \emptyset$ and we now assume that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are not tangent at points of $\partial\mathcal{O}$ (belonging to $\bar{\gamma}_1 \cap \bar{\gamma}_2$). Then, we claim that Proposition 2 still holds in this case. Of course, we just have to explain why lemma 3 is still valid. Since $w < 1$ in $\mathcal{O}$, it is enough to show that

$$\lim\sup \{w(y) \;/\; y \in \gamma_2 \;,\; d(y,\partial\mathcal{O}) \to 0\} < 1 \quad .$$

And this follows easily from potential theory. Observe that in two dimensions it is possible to identify this limit : indeed, if θ_1 is the "angle of $\partial\mathcal{O}_1$" at a point y_o belonging to $\bar{\gamma}_1 \cap \bar{\gamma}_2$ ($\theta_1 = \pi$ if $\mathcal{O}_1$ is smooth) and θ_2 is the "angle between $\partial\mathcal{O}$ and γ_2 at this point when $w(y) \to \frac{\theta_2}{\theta_1}$ (< 1) as y goes to $y_o, y \in \gamma_2$ (see figure 2 below)

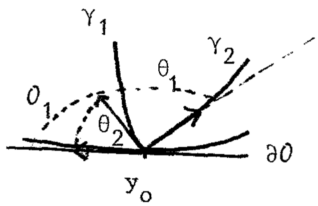


Figure 2.

We now make some comments on the analogues in this case of the remarks following Proposition 2 : first of all, the extension (Remark 3) to more general uniformly elliptic operators is still valid here. Next, if we replace Dirichlet boundary conditions by Neumann boundary conditions on $\partial\mathcal{O}$ (replacing $-\Delta$ by $-\Delta+c$ with $c > 0$), then lemma 3 is no more true and in fact it is not difficult to show that since $u^n = u^o$ at points $y \in \bar{\gamma}_1 \cap \bar{\gamma}_2$, the Schwarz sequence does not converge anymore uniformly on $\bar{\mathcal{O}}$ to the solution u (choose u^o such that $u^o(y_o) \neq u(y_o)$ at some point $y_o \in \bar{\gamma}_1 \cap \bar{\gamma}_2$). This difficulty may also be seen from a variational view-point since (with the notations of [36]) one does not have anymore V $(= H^1(\mathcal{O})) = V_1$ $(= \{u \in H^1(\mathcal{O})\ ,\ u \equiv 0 \text{ on } \mathcal{O}_{22}\}) + V_2$ $(= \{u \in H^1(\mathcal{O})\ ,\ u \equiv 0 \text{ on } \mathcal{O}_{11}\})$. However, one still has $V = \overline{V_1+V_2}$ (see [36]) and this ensures (cf. [36]) that u^n converges in $H^1(\mathcal{O})$ to u . We will also prove convergence in this case by a different method in section 4.

The analogues of Remarks 4 and 5 are still true here : observe only for Remark 5 that one has to introduce cut-off functions ζ_1, ζ_2 as in [36] involving a singularity at points $y \in \bar{\gamma}_1 \cap \bar{\gamma}_2$.

4. Sub and supersolutions.

In this section, we want to explain a striking property of Schwarz alternating method namely that if u^o is a subsolution of (2) (resp. supersolution) then u^n is also for all $n \geq 0$ a subsolution of (2) (resp. supersolution) and furthermore u^n is an increasing (resp. decreasing) sequence. Next, we will give some applications of this observation to convergence properties even in cases when the usual analysis (either the variational one as in [36] , or the one made in the preceding sections 2-3) fails. We begin with the

Theorem 4. Let $u^o \in C(\bar{\mathcal{O}})$ satisfy

(16) $\qquad - \Delta u^o \leq f$ in $\mathcal{D}'(\mathcal{O})$, $u^o \leq 0$ on $\partial\mathcal{O}$
(resp.
(17) $\qquad - \Delta u^o \geq f$ in $\mathcal{D}'(\mathcal{O})$, $u^o \geq 0$ on $\partial\mathcal{O}$).

Then, for all $n \geq 1$, $u^n \in C_o(\bar{\mathcal{O}} - \partial\mathcal{O} \cap \bar{\gamma}_1 \cap \bar{\gamma}_2)$ is bounded and also satisfies (16) (resp. (17)). Furthermore, we have

(18) $\qquad u^n \leq u^{n+1}$ on $\bar{\mathcal{O}}$ for $n \geq 0$, $u^n \underset{n}{\uparrow} u$ uniformly on $\mathcal{O}$
(resp.
(19) $\qquad u^n \geq u^{n+1}$ on $\bar{\mathcal{O}}$ for $n \geq 0$, $u^n \underset{n}{\downarrow} u$ uniformly on $\mathcal{O}$).

Remarks. 1) Notice that no assumptions on $\mathcal{O}_1, \mathcal{O}_2$ are made except $\gamma_1 \cap \gamma_2 = \emptyset$, $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$. In particular, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ can "meet tangentially" at $\partial\mathcal{O}$, case which was excluded in section 3.

2) It is possible to relax the regularity requirements on u^o as follows : $u^o \in C(\overline{\mathcal{O}}) + H^1(\mathcal{O})$, $(u^o)^+ \in C_o(\overline{\mathcal{O}}) + H_o^1(\mathcal{O})$.

3) The same result holds for Neumann type boundary conditions (and in fact more general ones as well).

Proof. By the maximum principle, we have $u^1 \geq u^o$ in $\overline{\mathcal{O}}_1$ since $u^1 \geq u^o$ on $\partial\mathcal{O}_1$ and u^o satisfies (16). Since $u^1 = u^o$ on $\overline{\mathcal{O}}-\overline{\mathcal{O}}_1$, we also have $u^1 \geq u^o$ in $\overline{\mathcal{O}}$. Next, we have to show

$$- \Delta u^1 \leqq f \qquad \text{in } \mathcal{D}'(\mathcal{O}) \quad .$$

This is clearly the case in $\mathcal{O}_1$ by definition of u^1 and it is also the case in $\mathcal{O}-\overline{\mathcal{O}}_1$ since u^1 agrees with u^o there and u^o satisfies (16). Thus, we just have to check that this claim is still true "across γ_1 ". But this follows from a general observation due to H. Berestycki and P.L. Lions [40] since $u^1 \geq u^o$ in $\mathcal{O}$ and $u^1 = u^o$ on γ_1 . We prove the corresponding properties of u^n for $n \geqq 2$ similarly.

Hence, $u^n(x)$ is an increasing sequence for each n . Since, by the maximum principle $u^n \leqq u$ in $\overline{\mathcal{O}}$ for all $n \geqq 0$, $u^n(x)$ converges to some $\tilde{u}$ and $\tilde{u}$ is a bounded function on $\overline{\mathcal{O}}$. Furthermore, since $u^n \leqq \tilde{u} \leqq u$ in $\overline{\mathcal{O}}$ and $u \in C_o(\overline{\mathcal{O}})$, $\tilde{u}$ vanishes on $\partial\mathcal{O} - \bar{\gamma}_1 \cap \bar{\gamma}_2$ and is continuous at points of $\partial\mathcal{O} - \bar{\gamma}_1 \cap \bar{\gamma}_2$. In addition, by elliptic regularity, u^{2n+1} converges uniformly on compact subsets of $\mathcal{O}_1$ and u^{2n} converges uniformly on compact subsets of $\mathcal{O}_2$. And since u^n is increasing and $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$, this implies that u^n converges uniformly to $\tilde{u}$ on compact subsets of $\mathcal{O}$, therefore $\tilde{u} \in C(\overline{\mathcal{O}})$.

Finally, $\tilde{u}$ satisfies

$$- \Delta\tilde{u} = f \quad \text{in } \mathcal{D}'(\mathcal{O}_1) \quad , \quad - \Delta\tilde{u} = f \quad \text{in } \mathcal{D}'(\mathcal{O}_2)$$

since u^{2n+1}, u^{2n} satisfy respectively these equations. But $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$, therefore we deduce

$$- \Delta\tilde{u} = f \qquad \text{in } \mathcal{D}'(\mathcal{O})$$

hence $\tilde{u} \equiv u$ in $\mathcal{O}$ and we conclude. Δ

Theorem 4 is clearly a convergence result, however it is restricted to some special initial choices of u^o . Nevertheless, we can deduce the general case.

Corollary 5. Let $u^o \in C(\overline{\mathcal{O}})$. Then, u^n converges uniformly to u on $\overline{\mathcal{O}}$.

Proof. It is enough to introduce $\bar{u}^o, \underline{u}^o \in C(\overline{\mathcal{O}})$ satisfying

$$- \Delta \bar{u}^{o} \geqq f \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \bar{u}^{o} \geqq u^{o} \quad \text{in } \bar{\mathcal{O}}$$

$$- \Delta \underline{u}^{o} \leqq f \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \underline{u}^{o} \leqq u^{o} \quad \text{in } \bar{\mathcal{O}}$$

(the existence of $\bar{u}^{o}, \underline{u}^{o}$ is an easy exercise). Then, the Schwarz sequences generated by $\bar{u}^{o}, \underline{u}^{o}$ denoted respectively by $\bar{u}^{n}, \underline{u}_{n}$ satisfy by the maximum principle

$$\bar{u}^{n} \geqq u^{n} \geqq \underline{u}_{n} \qquad \text{in } \bar{\mathcal{O}}$$

and we conclude applying Theorem 4 to $\bar{u}^{n}$ and $\underline{u}_{n}$. Δ

Let us emphasize that Corollary 5 applies to arbitrary decompositions of $\mathcal{O}$ into $\mathcal{O}_1, \mathcal{O}_2$ (such that $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$, $\gamma_1 \cap \gamma_2 = \emptyset$) and that the same result holds for Neumann boundary conditions even if $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$, cases which were not always covered by the arguments given in sections 2-3.

In fact, using these arguments, one can even allow some "interior non overlapping", more precisely, in two dimensions for example, one can allow γ_1 and γ_2 to intersect at, say, a finite number of points like in figure 3 below.

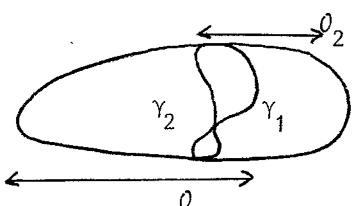


Figure 3.

Indeed, if we repeat the proof of Theorem 4, we find $\tilde{u} \in C(\mathcal{O} - S)$ where S is the finite set $S = \gamma_1 \cap \gamma_2$, $\tilde{u}$ is bounded on $\mathcal{O}$, $\tilde{u}$ vanishes on $\partial\mathcal{O} - \bar{\gamma}_1 \cap \bar{\gamma}_2$ and $- \Delta \tilde{u} = f$ in $\mathcal{D}'(\mathcal{O} - S)$. Then, since $\tilde{u}$ is bounded, the possible singularities at S are easily shown to be removable, hence

$$- \Delta \tilde{u} = f \qquad \text{in } \mathcal{D}'(\mathcal{O})$$

and $\tilde{u} \equiv u$ on $\mathcal{O} - S$. Therefore, the same results as Theorem 4 and Corollary 5 hold in this more general situation provided we replace $\mathcal{O}$ by $\mathcal{O} - S$ in the convergence statements.

5. Degenerate equations and time-dependent problems.

We begin with degenerate second-order elliptic equations. We want to explain in this case how the preceding arguments show that the Schwarz alternating method does converge even for degenerate operators like

$$(20) \qquad A = -\sum_{i,j=1}^{N} a_{ij}\partial_{ij} - \sum_{i=1}^{N} b_i\partial_i + c$$

where $a_{ij} = \sum_{k=1}^{m} \sigma_{ik}\sigma_{jk}$, σ_{ik}, b_i are Lipschitz on $\bar{\mathcal{O}}$, c is continuous on $\mathcal{O}$ $(1 \leqq i,j \leqq N$, $1 \leqq k \leqq m)$ and

$$(21) \qquad c \geqq c_o > 0 \quad \text{in } \bar{\mathcal{O}} .$$

We will not state precise results since to do so we would need to detail the way boundary conditions are imposed : this technical point can be handled using classical theory (see J.J. Kohn and L. Nirenberg [41] , D.W. Stroock and S.R.S. Varadhan [42]) or the more recent theory of viscosity solutions (see H. Ishii and P.L. Lions [46] for boundary conditions and uniqueness).

We just want to observe here that the method of sub and supersolutions given in section 4 gives the convergence in the general case (Dirichlet or Neumann boundary conditions and $\gamma_1 \cap \gamma_2 = \emptyset$) using the theory of viscosity solutions and in particular the fact that if u^o is a viscosity supersolution then $u^1 \leqq u^o$ in $\bar{\mathcal{O}}_1$ and thus u^1 is a viscosity supersolution in $\mathcal{O}$ (see M.G. Crandall and P.L. Lions [43] ; M.G. Crandall, L.C. Evans and P.L. Lions [44] ; P.L. Lions [45] for more details on viscosity solutions). On the other hand, the method of section 3 to prove geometrical convergence fails for general degenerate elliptic equations - it is easy to build first-order operators yielding counter-examples - since it requires some form of strong maximum principle.

Finally, it is possible to adapt the method of section 2 i.e. in the case when $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap. This is due to the following lemma.

<u>Lemma 6.</u> There exists some $\mu > 0$ depending only on c_o , diam $\mathcal{O}_1$ and bounds on b,a such that if $w \in C(\bar{\mathcal{O}}_1)$ solves (in viscosity sense or in classical sense if $u \in C^2(\mathcal{O}_1)$)

$$(22) \qquad Aw = 0 \text{ in } \mathcal{O}_1 \quad , \quad \sup_{\gamma_1} \bar{w} \leqq 1 \quad , \quad w|_{\partial\mathcal{O}_1 \cap \partial\mathcal{O}} = 0$$

then we have

$$(23) \qquad \sup_{\gamma_2} w \leqq \exp(-\mu\delta^2)$$

where $\delta = \text{dist}(\gamma_1,\gamma_2) = \text{Inf}\{|x-y| \ / \ x \in \gamma_1 , y \in \gamma_2\} > 0$.

<u>Remarks.</u> 1) The same result holds for Neumann boundary conditions.

2) The proof below gives some precise bound on μ .

3) In some sense, going from uniformly elliptic operator A to a degenerate one amounts to replace δ by δ^2 in the convergence rate.

<u>Proof of Lemma 6.</u> Let $x_o \in \gamma_2$. Without loss of generality, we may

assume that $x_o = 0$ and we denote by $\rho = \text{dist}(x_o, \gamma_1) = \text{Inf}\{|x_o - y| \ / \ y \in \gamma_1\}$. Then, we introduce

$$\bar{w}(x) = \exp \mu\{|x|^2 - \rho^2\}$$

where $\mu > 0$ will be determined later on. We then compute $A\bar{w}$ to find

$$A\bar{w} = \left(-2\mu \operatorname{Tr} a - 4\mu^2 \sum_{i,j=1}^{N} a_{ij}x_i x_j + 2\mu \sum_{i=1}^{N} b_i x_i + c\right) \bar{w}$$

while clearly $\bar{w} \geqq 0$ on $\partial \mathcal{O}_1$ and $\bar{w} \geqq 1$ on γ_1. Hence, choosing small enough so that

$$c_o \geqq 2\mu \|b\|_\infty (\text{diam } \mathcal{O}_1) + 2\mu \operatorname{Tr} a + 4\mu^2 \|\sigma\|_\infty^2 (\text{diam } \mathcal{O}_1)^2$$

we deduce $A\bar{w} \geqq 0$ in $\mathcal{O}_1$. Therefore, by the maximum principle, we have

$$w \leqq \bar{w} \quad \text{in } \mathcal{O}_1 \quad .$$

Hence, in particular, $\sup_{\gamma_2} w \leqq \sup_{x_o \in \gamma_2} \exp\{-\mu \, \text{dist}(x_o, \gamma_2)^2\}$ and (23) is proven. Δ

A very particular case of the remarks made above is the case of parabolic equations with a spatial domain decomposition in $\mathcal{O}_1, \mathcal{O}_2$ i.e. for example

(24) $$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \mathcal{O} \quad , \quad u|_{\partial \mathcal{O} \times [0,T]} = 0 \ , \quad u|_{t=0} = u_o \text{ in } \bar{\mathcal{O}}$$

where for instance $u_o \in C_o(\bar{\mathcal{O}})$, $f \in C(\bar{\mathcal{O}} \times [0,T])$. Then, if $u^o \in (\bar{\mathcal{O}} \times [0,T])$ satisfies (and this is not really necessary) $u^o|_{t=0} = u_o$ in $\bar{\mathcal{O}}$, $u^o|_{\partial \mathcal{O} \times [0,T]} = 0$, we define u^n by (for $n \geqq 0$)

(25) $$\frac{\partial u^{2n+1}}{\partial t} - \Delta u^{2n+1} = f \quad \text{in } \mathcal{O}_1 \quad , \quad u^{2n+1}|_{t=0} = u_o \quad \text{in } \bar{\mathcal{O}}_1 \ ,$$
$$u^{2n+1}|_{\partial \mathcal{O}_1 \times [0,T]} = u^{2n}|_{\partial \mathcal{O}_1 \times [0,T]}$$

(26) $$\frac{\partial u^{2n+2}}{\partial t} - \Delta u^{2n+2} = f \quad \text{in } \mathcal{O}_2 \ , \quad u^{2n+2}|_{t=0} = u_o \quad \text{in } \bar{\mathcal{O}}_2 \ ,$$
$$u^{2n+2}|_{\partial \mathcal{O}_2 \times [0,T]} = u^{2n+1}|_{\partial \mathcal{O}_2 \times [0,T]}$$

and we extend u^{2n+1}, u^{2n+2} to $\bar{\mathcal{O}} \times [0,T]$ by u^{2n}, u^{2n+1} respectively. Then, all the arguments introduced in sections 2-3-4 apply in this case.

A more interesting situation occurs when we combine Schwarz alternating method with a time discretization. Four possible combinations were proposed in Part I [36]. Let us just mention here that using maximum principle arguments as in the preceding sections, one can prove convergence and error bounds for these methods. We will not pursue this direction here to restrict the length of this paper.

6. Convergence proof via the stochastic interpretation.

Using the notations of section 1, we already observe that since $\tau_x^n \uparrow_n$ and $\tau_x^n \leqq \tau_x < \infty$ a.s. , $\tau_x^n \uparrow_n \sigma \leqq \tau_x$ where σ is a stopping time. Furthermore, since the Brownian paths i.e. the trajectories of $x+B_t$ are continuous (in t) a.s. we have

$$x+B_\sigma \in \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2 \quad \text{a.s.}$$

Therefore, if $\gamma_1 \cap \gamma_2 = \emptyset$, this implies that $x \in B_\sigma \in \partial\mathcal{O}$ a.s. i.e. $\sigma \geqq \tau$ and we conclude

$$\sigma = \tau_x \quad \text{a.s.}$$

And thus we have proven immediately that :

1) for all $x \in \overline{\mathcal{O}}$, $\tau_x^n \uparrow_n \tau_x$ a.s.,

2) hence, by Lebesgue's lemma, $u^n(x) \underset{n}{\to} u(x)$ for all $x \in \overline{\mathcal{O}}$,

3) this convergence is uniform in $\overline{\mathcal{O}}$. For the last claim, we have to show

$$\sup_{x \in \overline{\mathcal{O}}} E\left[\tau_x - \tau_x^n\right] \underset{n}{\to} 0 \quad . \tag{27}$$

But this follows from 2) and Dini's lemma, choosing $u^o \equiv 0$, $f \equiv 1$ so that $u^n(x) = E\left[\tau_x^n\right]$.

This (striking) proof of convergence clearly adapts to more general situations (general elliptic operators, possibly degenerate, other boundary conditions...). Next, we wish to push these stochastic arguments in order to obtain some estimates on the rate of convergence of u^n to u , that is the rate of convergence of τ_x^n to τ_x . In order to simplify the presentation, we will always assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap i.e. $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$. We first prove the

<u>Theorem 7.</u> There exist $k_1, k_2 \in (0,1)$ depending only on $\mathcal{O}_1, \mathcal{O}_2$ such that

$$\sup_{x \in \bar{\gamma}_j} P\left[\sigma_x^i < \tau_x\right] \leqq k_i \quad \text{for } i,j = 1,2 \ , \ i \neq j \ , \tag{28}$$

where σ_x^i is the first exit time from $\mathcal{O}_i$. In particular, we deduce

$$\sup_{x \in \overline{\mathcal{O}}} P\left[\tau_x > \tau_x^{2n+1}\right] \leqq (k_1 k_2)^n \tag{29}$$

$$\sup_{x \in \overline{\mathcal{O}}} P\left[\tau_x > \tau_x^{2n+2}\right] \leqq k_1^n k_2^{n+1} \quad . \tag{30}$$

<u>Remarks.</u> 1) As usual, the proof given below is valid for general boundary conditions and general uniformly elliptic equations i.e. general nondegenerate diffusion processes.

2) Recalling that $E\left[e^{\lambda\tau_x}\right] \in C(\overline{\mathcal{O}})$ for all $\lambda < \lambda_1$ (where λ_1

is the first eigenvalue of $-\Delta$ in $H_o^1(\mathcal{O})$ - see for instance [39] - we deduce easily some estimate on the convergence of u^n to u since

$$|u^n(x)-u(x)| \leq C\, E\,[\tau_x-\tau_x^n] + C\, P\,[\tau_x > \tau_x^n]\ ,\ \forall\, x \in \overline{\mathcal{O}}$$

and

$$E\,[\tau_x-\tau_x^n] \leq P\,[\tau_x > \tau_x^n]^{1/\alpha}\, E\,[\tau_x^{\alpha'}]^{1/\alpha'}$$

$$\text{for all}\quad \alpha > 1\ ,\quad \alpha' = \frac{\alpha}{\alpha-1}\ .$$

3) As we will see in the proof below, the crucial estimate (28) follows from some knowledge of the support of diffusion processes. Such knowledge - and the available additionnal informations such as "invariant measures"... - should play an important role in an attempt to determine "the optimal decomposition of $\mathcal{O}$ into $\mathcal{O}_1$ and $\mathcal{O}_2$ ". Δ

Proof of Theorem 7. The estimates (29) and (30) follow easily from (28) and the strong Markov property. Next, using the continuity of σ_x^i, τ_x in x , we see that in order to prove (28) we just have to show for all $x \in \bar{\gamma}_j$ $(i = 1,2)$

$$P\left(x + B_{\sigma_x^i} \notin \bar{\gamma}_i\right) > 0 \qquad \text{with}\quad j \neq i\ .$$

To this end, we choose a continuous trajectory (function) ω from $[0,\infty)$ into $\mathbb{R}^N$ such that $\omega(0) = x$, $\omega(1) \in \partial\mathcal{O}_i - \bar{\gamma}_i$, $\omega(t) \notin \overline{\mathcal{O}}_i$ for all $t > 1$, $\omega(t) \in \mathcal{O}_i$ for all $t \in (0,1)$. Then, by a famous result due to D.W. Stroock and S.R.S. Varadhan [47] , we have for all $T < \infty$, $\varepsilon > 0$

$$P\left[\sup_{t\,\in\,[0,T]} |x+B_t-\omega(t)| \leq \varepsilon\right] > 0\ .$$

We next choose $T = 2$, ε small enough so that $\varepsilon >$ dist $(\omega(2),\overline{\mathcal{O}}_i)$, $\varepsilon < \min_{t\,\in\,[0,T]}$ dist $(\omega(t),\bar{\gamma}_i)$. Then, denoting by

$$\Omega_\varepsilon = \left\{\sup_{t\,\in\,[0,T]} |x + B_t - \omega(t)| \leq \varepsilon\right\}\ ,$$

we see that on Ω_ε , $x+B_T \notin \overline{\mathcal{O}}_i$ hence $\sigma_x^i \leq T$ and thus $x+B_{\sigma_x^i} \notin \bar{\gamma}_j$. And our claim is proven. Δ

It is also possible to give a pure probabilistic proof of the convergence result implied by Lemma 6 for degenerate elliptic equations. Recall that $\mathcal{O}_1$ and $\mathcal{O}_2$ overlap. It is easy to check that Lemma 6 follows from the following inequality

$$E\,[e^{-c_o\sigma_x^1}\, 1_{\sigma_x^1 < \tau_x}] \leq e^{-\mu\delta^2}\ ,\qquad \forall\, x \in \gamma_2\ ,$$

(where σ_x^1, τ_x are defined as before but correspond now to the relevant diffusion process), inequality which can be proved directly by probabilistic arguments. The mere fact that the right-hand side is strictly less than 1 is obvious intuitively : indeed if $\sigma_x^1 < \tau_x$, this means that the diffusion process starting from $x \in \gamma_2$ exists from $\mathcal{O}_1$ by γ_1 and since $\delta = \text{dist } (\gamma_1,\gamma_2) > 0$, it "takes some time" to cross that distance. The phenomenon involved here is much less subtle than in Theorem 7.

7. Nonlinear problems.

First of all, we want to observe that all the convergence arguments given in sections 2-5 are still valid for monotone nonlinear equations of the form

$$(31) \qquad - \Delta u + \beta(u) \ni f \quad \text{in } \mathcal{O}, \qquad u|_{\partial\mathcal{O}} = 0$$

where β is a maximal monotone graph such that $0 \in \text{Dom } (\beta)$ - of course, (31) means in particular that $u(x) \in \text{Dom } (\beta)$ a.e. in $\mathcal{O}$. A particular an important example is given by $\beta(t) = \emptyset$ if $t < 0$, $\beta(0) = [0,\infty)$, $\beta(t) = 0$ if $t > 0$: it corresponds to variational inequalities or obstacle problems (in this case the obstacle is 0 but more general functions could be considered as well). We then define the Schwarz method exactly as in section 1 replacing only the equations for u^n by the above equation (31)... And all the convergence results are easily adapted to this case. Notice also that obstacle problems have a stochastic interpretation in terms of optimal stopping, more precisely in the above example we have

$$u(x) = \inf_{\theta} \left\{ E \int_0^{\theta} f(x+B_t)dt \;/\; \theta \text{ stopping time} \right\} .$$

Then, it is not difficult to show that the Schwarz sequence u^n is then given by

$$u^n(x) = \inf_{\theta} \left\{ E \int_0^{\theta\wedge\tau_x^n} f(x+B_t)dt + u^o\left(x + B_{\theta\wedge\tau_x^n}\right) \;/\; \theta \text{ stopping time} \right\}$$

Hence, we can also use the probabilistic approach developed in section 6.

The other class of nonlinear second-order equations that can be analysed by our method is the class of fully nonlinear elliptic possibly degenerate second-order equations

$$(32) \qquad F(D^2u,Du,u,x) = 0 \qquad \text{in } \mathcal{O} \quad .$$

This class contains as particular cases the first-order Hamilton-Jacobi equations - F is then independent of D^2u - which are the fundamental partial differential equations for optimal deterministic control and deterministic games (Bellman and Isaacs' equations), the Hamilton-Jacobi-Bellman equations of optimal stochastic control - F convex or concave in D^2u - and the Isaacs' equations of stochastic differential games. For all those problems, we can use the theory of viscosity solutions (as in [46]) to prove (similarly as we did in the preceding sections) convergence results for the Schwarz alternating method.

Notice also that the arguments of section 6 can be used together with the control or games interpretation of solutions to provide other convergence proofs.

We will give only one (extreme) example : we consider the Eikonal equation

(33) $$|\nabla u| = f \quad \text{in } \mathcal{O} , \qquad u = 0 \quad \text{on } \partial\mathcal{O}$$

where $f > 0$ in $\overline{\mathcal{O}}$, $f \in C(\overline{\mathcal{O}})$ and u is the unique viscosity solution of (33) – see M.G. Crandall and P.L. Lions [43] . For simplicity, we choose $u^o \equiv 0$. And we build the Schwarz sequence as follows

(34) $$|\nabla u^{2n+1}| = f \quad \text{in } \mathcal{O}_1 \quad , \quad u^{2n+1} = u^{2n} \quad \text{on } \partial\mathcal{O}_1$$

(35) $$|\nabla u^{2n+2}| = f \quad \text{in } \mathcal{O}_2 \quad , \quad u^{2n+2} = u^{2n+1} \quad \text{on } \partial\mathcal{O}_2$$

and we extend as usual u^{2n+1}, u^{2n+2} to $\overline{\mathcal{O}}$ by u^{2n}, u^{2n+1} respectively. Actually, some care is required to build such a sequence. Recall that we assume that $\gamma_1 \cap \gamma_2 = \emptyset$. To actually build (34),(35) we argue inductively and we assume by induction that $0 = u^o \leq u^1 \leq \ldots \leq u^n$ in $\overline{\mathcal{O}}$ and u^n is a viscosity subsolution of (33), then by comparison results on viscosity solutions, we deduce that $u^{n+1} \in C_o(\overline{\mathcal{O}})$, $u^{n+1} \geq u^n$ in $\overline{\mathcal{O}}_1$ if n is even, $\overline{\mathcal{O}}_2$ if n is odd. The fact that u^{n+1} viscosity solution of (34) or (35) exists follows from the results of P.L. Lions [48] . Then, one can check that u^{n+1} is also a viscosity subsolution of (33). Since u^n is bounded in Lipschitz norm and is increasing with respect to n , we deduce that u^n converges uniformly to some $\tilde{u} \in C_o(\overline{\mathcal{O}})$. Furthermore, by the stability results of viscosity solutions, $\tilde{u}$ is a viscosity solution of (33) in $\mathcal{O}_1$ and in $\mathcal{O}_2$ therefore in $\mathcal{O}$. And $\tilde{u} \equiv u$, proving thus the convergence.

8. Multidomains decomposition.

We want now to consider extensions of the classical Schwarz alternating method to the case of a decomposition of $\mathcal{O}$ into more than two subdomains namely

$$\mathcal{O} = \bigcup_{i=1}^{m} \mathcal{O}_i$$

where $m \geq 1$, $\mathcal{O}_1,\ldots,\mathcal{O}_m$ are open sets. We begin with the simple geometrical situation where

(36) $$\overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j \cap \overline{\mathcal{O}}_k = \emptyset \quad \text{whenever} \quad 1 \leq i,j,k \leq m , \quad i \neq j \neq k \neq i \ .$$

Some examples where (36) holds are given by figures 4.a,4.b below while some examples where (36) does not hold is given by figure 5 below.

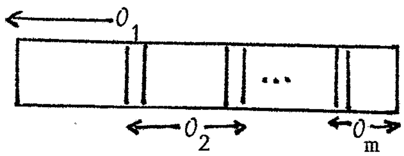


Figure 4.a

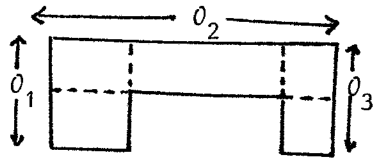


Figure 4.b

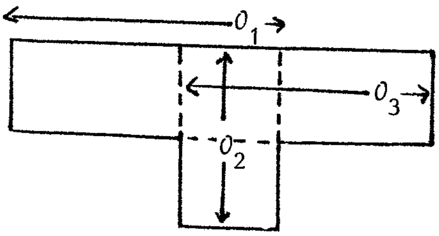


Figure 5.

Then, for all $1 \leqq i \neq j \leqq m$, we denote by $\gamma_{ij} = \partial O_j \cap O_i$. We next introduce a (parallel) extension of Schwarz alternating method. For each $i \in \{1,\dots,m\}$, we choose $u_i^o \in C(\bar{O}_i)$ such that $u_i^o = 0$ on $\partial O_i \cap \partial O$ and we build inductively sequences $u_i^n \in C(\bar{O}_i)$ for $n \geqq 1$, $1 \leqq i \leqq m$, as follows

$$(37)\qquad \begin{cases} -\Delta u_i^{n+1} = f \quad \text{in } O_i \quad , \quad u_i^{n+1} = 0 \quad \text{on} \quad \partial O \cap \partial O_i \ , \\ u_i^{n+1} = u_j^n \quad \text{on } \gamma_{ji} \quad \text{for all } j \neq i \ . \end{cases}$$

We begin with the analysis of the convergence of u_i^n to $u|_{\bar{O}_i}$ in the case when O_i and O_j overlap for all $i \neq j$, $(\bar{\gamma}_{ij} \cap \bar{\gamma}_{ji} = \emptyset)$ or when O_i and O_j weakly overlap for all $i \neq j$ (if $\bar{\gamma}_{ij} \cap \bar{\gamma}_{ji} \neq \emptyset$, $\bar{\gamma}_{ij}$ and $\bar{\gamma}_{ji}$ "make a positive angle" at intersection points). In both cases by the arguments of section 2-3, we find some $k \in (0,1)$ such that for all $1 \leqq i \neq j \leqq m$ we have

$$\sup_{\gamma_{ij}} |u-u_i^{n+1}| \leqq k \max_{\ell \neq i} \sup_{\gamma_{\ell i}} |u-u_\ell^n| \quad .$$

In particular, this yields

$$\max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^{n+1}| \leqq k \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^n|$$

hence

$$(38)\qquad \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^n| \leqq k^n \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^o| \quad .$$

And since, by the maximum principle, we have for all i

$$|u-u_i^{n+1}| \leqq \max_{j \neq i} \max_{\gamma_{ji}} |u-u_j^n| \qquad \text{in } \overline{\mathcal{O}}_i$$

we deduce from (38) the following inequality

$$\max_i \sup_{\mathcal{O}_i} |u-u_i^{n+1}| \leqq k^n \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^o| \quad . \tag{39}$$

It is also worth explaining how it is possible to adapt the arguments of section 4 to this case, proving thus in particular the convergence in the general case when $(\mathcal{O}_i)_{1 \leqq i \leqq m}$ only satisfy $\gamma_{ij} \cap \gamma_{ji} = \emptyset$ for all $1 \leqq i \neq j \leqq m$. Indeed, let $u^o \in C(\overline{\mathcal{O}})$ be a supersolution (resp. subsolution) of (2) and choose $u_i^o = u^o|_{\overline{\mathcal{O}}_i}$ for all $1 \leqq i \leqq m$. Then, we claim that for all $1 \leqq i \neq j \leqq m$, $n \geqq 0$ we have

$$u_i^{n+1} \leqq u_j^n \quad \text{in } \overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j \quad , \quad u_i^{n+1} \leqq u_i^n \quad \text{in } \overline{\mathcal{O}}_i \quad . \tag{40}$$

This can be show easily by induction since once the second claim is proven, the first one follows remarking that u_i^{n+1} and u_j^n solve the same equation in $\mathcal{O}_i \cap \mathcal{O}_j$, $u_i^{n+1} = u_j^n$ on γ_{ji} and $u_j^n = u_i^{n-1} \geqq u_i^n$ on γ_{ij} . From (40), we deduce easily that u_i^n converges as n goes to $+\infty$ to some u_i and $u_i = u_j$ on $\mathcal{O}_i \cap \mathcal{O}_j$ for all $1 \leqq i \neq j \leqq m$. This allows to define some $\tilde{u}$ which is bounded, continuous on $\overline{\mathcal{O}} - \bigcup_{i \neq j} (\partial\mathcal{O}_i \cap \partial\mathcal{O}_j)$ and satisfies

$$-\Delta\tilde{u} = f \qquad \text{in } \mathcal{O} \quad .$$

From this we deduce that $\tilde{u} \equiv u$ and the convergence is proven in this case.

A final remark concerning the situation when (36) holds is the possibility of allowing more flexibility in the iterative method (37) and more precisely of replacing the boundary condition on each γ_{ji} by

$$u_i^{n+1} = u_j^p \quad \text{with } p = p(n,i,j) \text{ , on } \gamma_{ji} \text{ for all } j \neq i \ . \tag{41}$$

We then assume that $p(n,i,j)$ is nondecreasing with respect to n , $p(n,i,j) \leqq n$ for all $n \geqq 0$, $p(n,i,j) \to +\infty$ as $n \to +\infty$, for all $1 \leqq i \neq j \leqq m$. For instance, the first proof we made above is modified as follows

$$\max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^{n+1}| \leqq k \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^{p(n,i,j)}|$$

hence

$$\overline{\lim_n} \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^{n+1}| \leqq k \overline{\lim_n} \sup_{\gamma_{i_o j_o}} |u-u_{i_o}^{r(n)}|$$

for some $1 \leq i_o \neq j_o \leq m$, and for some sequence $r(n) \underset{n}{\to} +\infty$ as $n \to +\infty$. And this yields : $\overline{\lim_n} \max_{i \neq j} \sup_{\gamma_{ij}} |u-u_i^{n+1}| = 0$, and the convergence is proven.

We now turn to more general geometrical situations than (36). An example is given by figure 5 and another meaningful example is given by the following.

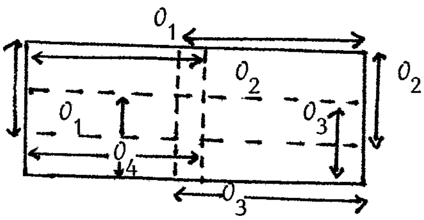


Figure 6.

We will need some restrictions on the decomposition : in two dimensions we will need to assume that $\partial O_i \cap \partial O_j \cap \overline{O}_k$ is a finite set for all i,j,k distinct in $\{1,\dots,m\}$. In higher dimensions, similar conditions involving the dimensionality of such intersection sets are needed but we will skip them for the sake of simplicity.

We next have to explain how we modify (37) in this general situation : we will use the same equation as in (37) but we modify the boundary condition as follows

$$(42)\quad \begin{cases} u_i^{n+1} = u_{j_1}^n & \text{on } \partial O_i \cap O_{j_1} \cap \left[\bigcup_{j \neq i,j_1} O_j^c\right] \text{ for all } j_i \neq i \\ u_i^{n+1} = u_{j_1 j_2}^n & \text{on } \partial O_i \cap O_{j_1} \cap O_{j_2} \cap \left[\bigcup_{j \neq i,j_1,j_2} O_j^c\right] \\ & \qquad \text{for all } j_1 \neq j_2 \neq i \neq j_1 \\ u_i^{n+1} = u_{j_1 \dots j_{m-1}}^n & \text{on } \partial O_i \cap O_{j_1} \cap O_{j_{m-1}} \\ & \qquad \text{where } \{i,j_1,\dots,j_{m-1}\} = \{1,\dots,m\} \end{cases}$$

and $u_{j_1 \dots j_k}^n$ (for all $k \geq 2$) is chosen to satisfy (arbitrarily)

$$(43)\qquad \text{Min}\left(u_{j_1}^n,\dots,u_{j_k}^n\right) \leq u_{j_1 \dots j_k}^n \leq \text{Max}\left(u_{j_1}^n,\dots,u_{j_k}^n\right) .$$

To prove the convergence of this method, we first observe that if we choose $\bar{u}_o, \underline{u}_o$ supersolution of (2) (resp. subsolution of (2)) such that $\bar{u}_o \geq u_i^o$ on $\overline{O}_i$, $\underline{u}_o \leq u_i^o$ on $\overline{O}_i$ for all $1 \leq i \leq m$, then it is easy to show

$$(44)\qquad \underline{u}_i^n \leq u_i^n \leq \bar{u}_i^n \quad \text{for all } n \geq 0 ,\ i \in \{1,\dots,m\}$$

where $\underline{u}_i^n, \bar{u}_i^n$ are the sequences generated by the above iterative method with the special choices

$$(45)\qquad \begin{cases} \bar{u}^n_{j_1\cdots j_k} = \operatorname{Max}\left(\bar{u}^n_{j_1},\ldots,\bar{u}^n_{j_k}\right) \\ \underline{u}^n_{j_1\cdots j_k} = \operatorname{Min}\left(\underline{u}^n_{j_1},\ldots,\underline{u}^n_{j_k}\right) \end{cases}$$

for all $n \geqq 0$, $k \geqq 2$, $j_1\cdots j_k$ distinct in $\{1,\ldots,m\}$. Therefore, it is enough to prove the convergence of $\underline{u}^n_i, \bar{u}^n_i$ and we will do so for $\bar{u}^n_i$ (for instance). Exactly as in section 4, because $\bar{u}^o$ is a supersolution, one checks easily that $(\bar{u}^n_i)_n$ is bounded in $L^\infty(\mathcal{O}_i)$ and

$$(46)\qquad u \leqq \bar{u}^{n+1}_i \leqq \bar{u}^n_i \quad \text{in } \mathcal{O}_i \text{ for all } n \geqq 0 , \; i \in \{1,\ldots,m\}.$$

Therefore, $\bar{u}^n_i$ converges uniformly on compact subsets of $\mathcal{O}_i$ to some $\bar{u}_i \in L^\infty(\mathcal{O}_i) \cap C(\mathcal{O}_i)$ which is continuous and vanishes on $\partial\mathcal{O}_i \cap \partial\mathcal{O}$ except maybe at a finite number of points and which satisfies

$$-\Delta u_i = f \quad \text{in } \mathcal{O}_i \quad , \quad u \leqq u_i \quad \text{in } \mathcal{O}_i \quad .$$

Furthermore, there exists a finite set S contained in $\mathcal{O}$ such that for all $i,j_1,\ldots,j_k$ distinct in $\{1,\ldots,m\}$, for all $k \in \{1,\ldots,m-1\}$, $u_i, u_{j_i},\ldots,u_{j_k}$ are continuous on $\partial\mathcal{O}_i \cap \mathcal{O}_{j_1} \cap \ldots \cap \mathcal{O}_{j_k} \cap \left(\bigcup_{j \neq i,j_1,\ldots,j_k} \mathcal{O}^c_j\right) \cap S^c$ and we have there :
$u_i = \max\left(u_{j_1},\ldots,u_{j_k}\right)$.

We then introduce $w = \max\left(u_{i_1},\ldots,u_{i_k}\right)$ on $\bar{\mathcal{O}}_{i_1} \cap \ldots \cap \bar{\mathcal{O}}_{i_k} \cap \left(\bigcup_{i \neq i_1,\ldots,i_k} \mathcal{O}^c_i\right)$ for all $i_1,\ldots,i_k$ distinct in $\{1,\ldots,m\}$, $k \in \{1,\ldots,m\}$ and we observe that w is bounded on $\mathcal{O}$, continuous on $\partial\mathcal{O}$ except maybe at a finite number of points and satisfies

$$-\Delta w \leqq f \quad \text{in } \mathcal{D}'(\mathcal{O}-S) \quad , \quad u \leqq u_i \leqq w \quad \text{in } \mathcal{O}_i \quad \text{for all } i \in \{1,\ldots,m\} .$$

Because w is bounded and S is finite this yields

$$-\Delta w \leqq f \qquad \text{in } \mathcal{D}'(\mathcal{O})$$

and together with the boundary condition, we deduce $w \leqq u$ in $\mathcal{O}$. Hence $w \equiv u$ in $\mathcal{O}-S$ and in particular $u_i \equiv u$ in $\mathcal{O}_i$, proving thus the convergence.

In some very particular geometrical stituations like the one given by figure 5, we can improve the above convergence proof : indeed, in this case, the same proof as in the case when (36) holds applies and we obtain in this way geometrical convergence.

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