CHAPTER 9

Optimal Iterative Refinement Methods*

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Abstract. We consider the solution of the linear systems of algebraic equations which arise from elliptic finite element problems defined on composite meshes. Such problems can systematically be built up by introducing a basic finite element approximation on the entire region and then repeatedly selecting subregions, and subregions of subregions, where the finite element model is further refined in order to gain higher accuracy. We consider conjugate gradient algorithms, and other acceleration procedures, where, in each iteration, problems representing finite element models on the original region and the subregions prior to further refinement are solved. We can therefore use solvers for problems with uniform or relatively uniform mesh sizes, while the composite mesh can be strongly graded.

In this contribution to the theory, we report on new results recently obtained in joint work with Maksymilian Dryja. We use a basic mathematical frame work recently introduced in a study of a variant of Schwarz' alternating algorithm. We establish that several fast methods can be devised which are optimal in the sense that the number of iterations required to reach a certain tolerance is independent of the mesh size as well as the number of refinement levels. This work is also technically quite closely related to previous work on iterative substructuring methods, which are domain decomposition algorithms using non-overlapping subregions.

1. Introduction. In this paper, we consider the solution of the large linear systems of algebraic equations which arise when working with elliptic finite element approximations on composite meshes. In this contribution to the theory, we report on new results recently obtained in joint work with Maksymilian Dryja.

Finite element models on composite meshes can systematically be built up inside a frame work of conforming finite elements; cf. Ciarlet [3]. We do so to be able to use

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a number of technical tools which are available primarily in the conforming case. We begin by introducing a basic finite element approximation on the entire region and we then repeatedly select subregions, and subregions of subregions, where the finite element model is further refined. We solve the resulting linear system by using an iterative method such as the conjugate gradient, Richardson or Chebyshev method. We accelerate the convergence by solving so called standard problems. These correspond to the finite element models on the original region, prior to any refinement, and those on the subregions prior to further refinement. We can thus use solvers for problems with uniform or relatively uniform mesh sizes, while the composite mesh can be strongly graded.

This approach offers a number of advantages. An existing code can be upgraded, in order to increase the accuracy locally, without a radical redesign of the data structures etc., since we can use the old code to solve one or several of the standard problems. Issues of data structures and geometry are generally simpler if we design programs for composite mesh problems in terms of simpler standard problems. The use of simple standard problems also tends to improve the performance of the programs on vector machines. Finally we note that if each of the standard problems has appreciatively fewer degrees of freedom than the composite model, then we might benefit from solving a number of smaller problems rather than one large one. In other words, we might view our approach as a divide-and-conquer strategy.

We study two families of algorithms which we call multiplicative and additive respectively. The so called additive algorithms are particularly well suited for parallel computing in that in each iteration step we can simultaneously solve all the standard problems on the the different levels of refinement. Synchronization between the processors is required once in each iteration, namely when the residual is computed and assembled. One or a group of processors can therefore be assigned in a straightforward way to each of the standard problems. As we will see, we can view the multiplicative algorithms as preconditioned conjugate gradient methods, while the additive variant involves the solution of a transformed equation with the same solution as the original one using no further preconditioning.

The systematic study of the methods under consideration goes back at least to 1983, when the Fast Adaptive Composite (FAC) method was introduced by McCormick [12]. Issues related to the implementation of this method on parallel computers led to the introduction of Asynchronous FAC (AFAC) methods [8] a few years later. Numerical experiments have now been reported for model cases and recently Richard Ewing, Steve McCormick and others have begun to test the algorithms for more difficult problems arising in industry. The convergence of the FAC method is discussed in McCormick et al. [12], [13] under a certain additional regularity assumption. An important contribution to the theory is given by Bramble, Ewing, Pasciak and Schatz [2], who outlined a proof of optimality for a two level FAC algorithm. They require no additional regularity in their proof. In recent papers Bjørstad [1] and Mandel and McCormick [11], develop a theory for both multiplicative and additive algorithms, using two levels. In particular, they obtain interesting results concerning the relationship between the spectra of the two methods. In a recent paper, Mandel and McCormick [10] establish optimality for a multi-level AFAC algorithm for a special model problem.

Our study began with the discovery that FAC and AFAC methods have a structure quite similar to that of the classical Schwarz procedure, see Schwarz [14], and an additive variant thereof recently considered in Dryja [4] and Dryja and Widlund [5]. Our earlier work was in turn inspired by a recent paper by P.-L. Lions [9] in which a variational frame work for the classical, multiplicative Schwarz' method is developed for continuous elliptic problems. We were able to establish a rate of convergence which is independent of the number of degrees of freedom as well as the number of subregions for a special kind of additive Schwarz algorithm, see Dryja [4] and Dryja and Widlund [5]. In our view, the central theoretical issue of the iterative refinement algorithms, which are discussed in this paper, is similar to that of Schwarz type algorithms namely the design and study of algorithms for which the rate of convergence is independent of the number of subproblems, i.e. the number of refinement levels as well as and the mesh sizes.

In section 2, we introduce the finite element problems on composite meshes, certain projections and our algorithms in their basic form.

In Section 3, we consider a multiplicative algorithm for the multi level case and show that its rate of convergence is optimal.

In Section 4, we introduce several multilevel additive algorithms and provide a number of bounds. The principal result is that one of these methods has a rate of convergence which is independent of the number of refinement levels, as well as the mesh sizes.

2. Composite Finite Element Problems and Basic Iterative Methods. We consider linear, self adjoint, elliptic problems discretized by finite element methods on a bounded Lipschitz region Ω in R^n. To simplify the presentation, we assume that the differential operator is the Laplacian and that we use continuous, piecewise linear finite elements. However, almost all of our results can be extended immediately to general conforming finite element approximations of any self adjoint elliptic problem which can be formulated as a minimization problem. The continuous and discrete problems are of the form

$$a(u,v) = f(v), \ \forall \ v \in V,$$

and

$$a(u_h, v_h) = f(v_h), \ \forall \ v_h \in V^h, \tag{1}$$

respectively. The spaces V and V^h are defined in the next few paragraphs. The bilinear form is defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \ dx \ .$$

This form defines a semi-norm $|u|_{H^1} = (a(u,u))^{1/2}$ in $H^1(\Omega)$.

To simplify the presentation, we assume that we have a zero Dirichlet boundary condition on $\partial\Omega$, the boundary of Ω. The space V is thus $H_0^1(\Omega)$. We note that we equally well could have considered an inhomogeneous Dirichlet problem. The space V^h is defined on a composite triangulation, which is possibly the result of a large number of successive refinements. The triangulation of Ω is given in the following way.

We first introduce a relatively coarse triangulation of Ω, also denoted by Ω_1, and denote the corresponding space of finite element functions by V^{h_1}. We can think of this space as having a relatively uniform (or uniform) mesh size h_1. Let Ω_2 be an area where we wish to increase the resolution. We do so by subdividing the elements and introducing an additional finite element space V^{h_2}. We assure that the resulting composite space $V^{h_1} + V^{h_2}$ is conforming by having the functions of V^{h_2} vanish on $\partial\Omega_2$. We repeat this process by selecting a subregion Ω_3 of Ω_2 and introducing a further refinement of the mesh and finite element space etc.. We denote the the resulting nested subregions and subspaces by Ω_i and V^{h_i} respectively. Throughout, we have $\Omega_i \subset \Omega_{i-1}$ and $V^{h_{i-1}} \cap H_0^1(\Omega_i) \subset V^{h_i} \subset H_0^1(\Omega_i)$, $i = 2, \ldots, k$. The composite finite element space on the repeatedly refined mesh, is

$$V^h = V^{h_1} + V^{h_2} + \ldots + V^{h_k} \ .$$

We assume that all the elements are shape regular in the sense that there is a uniform bound on h_K/ρ_K. Here h_K and ρ_K are the diameter and the radius of the largest inscribed sphere of any element K, respectively. Our bounds in the theory developed below also depend on the shape of the subregions Ω_i. Thus in order for our proofs to work, we cannot allow the sets $\Omega_{i-1} \setminus \Omega_i$ to become arbitrarily thin in comparison with the diameter of Ω_{i-1}. We also assume that the area of any triangle on level i can be bounded by *const.* q^{j-i}, $j < i$ times the area of the triangle on level j of which it is a part. Here q is a constant < 1.

The finite element problem is defined by equation (1) and the corresponding stiffness matrix can conveniently be computed by using a process of subassembly. Introducing subscripts to indicate the domain of integration, we write

$$a(u,v) = a_{\Omega_1\setminus\Omega_2}(u,v) + a_{\Omega_2\setminus\Omega_3}(u,v) + \ldots + a_{\Omega_k}(u,v) .$$

The stiffness matrices corresponding to the regions $\Omega_i \setminus \Omega_{i+1}$, $i \leq k-1$, and Ω_k are computed by working with basis functions related to the mesh size h_i . The quadratic form corresponding to the composite stiffness matrix is the sum of the quadratic forms corresponding to $\Omega_i \setminus \Omega_{i+1}$ and Ω_k. In our algorithms, we use solvers for the same elliptic problem on the subregions Ω_i , $i = 1, ..., k$, and the relatively uniform meshes corresponding to h_i . The corresponding stiffness matrices can in fact be obtained at a small extra cost during the assembly of the composite mesh model. When we refine a finite element model locally, the modified stiffness matrix is obtained by replacing the quadratic form associated with the subregion in question by the one corresponding to the refined model on the same subregion. It is therefore relatively easy to design a method which systematically generates the stiffness matrices for all the standard problems necessary while the stiffness matrix of the composite model is computed.

The fundamental building blocks of our algorithms are the P_j^i, $i \leq j$, the projections onto the spaces $V^{h_i} \bigcap H_0^1(\Omega_j)$. We note that if $j > i$, then we solve a problem on Ω_j with a coarser mesh than if V^{h_j} were used. The projection P_j^i, $i \leq j$, is defined, in terms of the unique element of $V^{h_i} \cap H_0^1(\Omega_j)$, which satisfies

$$a(P_j^i v_h, \phi_h) = a(v_h, \phi_h), \ \forall \phi_h \in V^{h_i}. \tag{2}$$

We now introduce the multiplicative and additive algorithms. Since no further effort is involved, we develop a framework which is also useful in other contexts such as the study of algorithms of Schwarz type; cf. Dryja [4], Dryja and Widlund [5] and Lions [9]. We begin by considering the case of two subspaces V_1 and V_2 and the multiplicative (sequential) algorithms. In a first fractional step, we find a correction $\delta_1 u^n \in V_1$ of the current approximation u^n by solving

$$a(\delta_1 u^n, v) = f(v) - a(u^n, v) = a(u^* - u^n, v), \ \forall \ v \in V_1.$$

Here $u^* \in V_1 + V_2$ is the solution of the given problem. The calculation of a second correction $\delta_2 u^n \in V_2$ completes the $(n+1)$th step.

$$a(\delta_2 u^n, v) = f(v) - a(u^n + \delta_1 u^n, v) = a(u^* - (u^n + \delta_1 u^n), v), \ \forall v \in V_2 .$$

As shown in Lions [9], it is easy to see that

$$\begin{aligned} \delta_1 u^n &= P_1(u^* - u^n) \\ \delta_2 u^n &= P_2(u^* - u^n - \delta_1 u^n) = P_2(I - P_1)(u^* - u^n) \end{aligned}$$

and thus the error propagates as

$$u^{n+1} - u^* = (I - P_2)(I - P_1)(u^n - u^*).$$

Here P_1 and P_2 are the orthogonal projections associated with the bilinear form $a(\cdot,\cdot)$ and V_1 and V_2, respectively.

We can thus view this algorithm as a simple iterative method for solving

$$(P_1 + P_2 - P_2 P_1) u_h = g_h,$$

with an appropriate right hand side g_h. We note that this operator is a polynomial of degree two and thus not ideal for parallel computing, since two sequential steps are involved. If we use more than two subspaces and therefore more projections this effect is further pronounced. The basic idea behind the additive form of the algorithm is to work with a simplest possible polynomial in the projections. With k subspaces we thus solve the equation

$$Pu_h = (P_1 + P_2 + \ldots + P_k)u_h = g_h' , \tag{3}$$

by an iterative method. If we can show that the operator P is symmetric and positive definite, the iterative method of choice is the conjugate gradient method at least on a computer with conventional architecture. We must also make sure that this equation has the same solution as equation (1), i.e. we must find the correct right hand side. Since by equation (1), we have

$$a(u_h, \phi_h) = f(\phi_h) ,$$

we can construct the right-hand side g_h' by solving this equation restricted to all the different subspaces and adding the results. It is similarly possible to apply the operator P of equation (3) to any given element of V^h by applying each projection P_i once. Most of the work , in particular that which involves the individual projections, can be carried out in parallel.

It is well known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to $\sqrt{\kappa}$, where κ is the condition number of P; see e.g. Golub and Van Loan [7]. We therefore need to establish that the operator P of equation (3) is not only invertible but that satisfactory upper and lower bounds on its eigenvalues can be obtained.

We end this section by discussing a symmetric version of the multiplicative algorithm. The projections generally do not commute and if we wish to have a symmetric expression in the operators P_i, which allows us to accelerate the convergence by the standard conjugate gradient method, we have to use additional fractional steps. In the case of two subspaces, we can solve the first problem again. Since $(I - P_2)^2 = (I - P_2)$ we can write the resulting operator as

$$I - (I - P_1)(I - P_2)(I - P_1) = P_1 + P_2 - P_1P_2 - P_2P_1 + P_1P_2P_1 = I - T_2T_2^*,$$

where $T_2 = (I - P_1)(I - P_2)$. The corresponding operator for k subspaces involves $2k - 1$ fractional steps and has the form

$$I - T_kT_k^*, \text{ where } T_k = (I - P_1)(I - P_2)\cdots(I - P_k).$$

The error propagation operator for the basic multiplicative algorithm is T_k^*, while the convergence rate of the symmetrized algorithm can be bounded in terms of the condition number of $I - T_kT_k^*$. We note that a bound on the spectral radius of T_k is obtained immediately from the spectral bounds on the symmetric operator.

3. The Multiplicative Algorithm. In this section, we establish the following result. We note that our analysis resembles that of Bramble et al. [2] in the case of $k = 2$.

Theorem 1. *The symmetrized, multiplicative iterative refinement algorithm based on the projections P_i^i, $i \leq k$, has a condition number which is independent of k and the number of degrees of freedom. The spectral radius of the basic multiplicative algorithm is bounded by a constant which is uniformly less than one.*

We begin by designing a preconditioner $b(u, v)$ for the finite element problem on the space

$$V^h = V^{h_1} + \cdots + V^{h_k}.$$

We then show that this quadratic form can be bounded uniformly from above and below by the quadratic form of the composite finite element problem. Finally we establish that we can work practically with this preconditioner and that the error propagation matrix is the same as that of the symmetrized multiplicative algorithm, introduced in the previous section. The algorithms are thus the same.

In addition to the projections P_i^i, we also use another family of operators, H_j^i, $i = j-1, j$, defined by

$$H_j^i v_h(x) \in V^{h_1} + \cdots + V^{h_i},$$
$$H_j^i v_h(x) = v_h(x), \quad x \in \Omega \setminus \Omega_j,$$
$$a(H_j^i v_h, w_h) = 0, \quad \forall w_h \in V^{h_i} \cap H_0^1(\Omega_j).$$

We can call $H_j^i v_h$ the h_i–harmonic extension of v_h to Ω_j, since it is the solution in V^{h_i} of a discrete Dirichlet problem with zero right hand side and with boundary data on $\partial\Omega_j$ given by v_h.

In order to prepare for the work that remains, we formulate three lemmas.

Lemma 1. $H_{i-1}^{i-2} H_i^{i-1} = H_{i-1}^{i-2}$, $3 \leq i \leq k$.

The proof follows immediately from the definition.

Lemma 2. $a(H_i^{i-1}u_h, H_i^{i-1}v_h) = a(H_i^{i-1}u_h, v_h)$, $\forall u_h \in V^h$, $\forall v_h \in V^{h_{i-1}}$, $2 \leq i \leq k$.

Proof: From the definition of H_i^{i-1} follows that $H_i^{i-1}v_h - v_h \in V^{h_{i-1}} \cap H_0^1(\Omega_i)$. Since $H_i^{i-1}u_h$ is h_{i-1}–harmonic on Ω_i the result follows by the orthogonality inherent in the definition of h_{i-1}–harmonic functions.

The following lemma is a consequence of the extension theorem given in Widlund [15].

Lemma 3. *There exists a constant C which is independent of u_h and the mesh sizes, such that for $i = j-1, j$,*

$$a_{\Omega_j}(H_j^i u_h, H_j^i u_h) \leq C a_{\Omega_{j-1}\setminus\Omega_j}(u_h, u_h), \ \forall u_h \in V^h.$$

We will not discuss the proof of this lemma. We note that the constant C necessarily blows up if we let the area of $\Omega_{j-1} \setminus \Omega_j$ shrink to zero, keeping Ω_{j-1} fixed. Such situations are not of particular interest in our applications.

Trivially, we can write $u_h = P_k^k u_h + (I - P_k^k)u_h$. It is easy to see that the second term equals $H_k^k u_h$. The two terms are orthogonal in the sense of the bilinear form and thus

$$\begin{aligned} a(u_h, v_h) &= a(P_k^k u_h, P_k^k v_h) + a((I - P_k^k)u_h, (I - P_k^k)v_h) \\ &= a(P_k^k u_h, P_k^k v_h) + a(H_k^k u_h, H_k^k v_h) \ . \end{aligned}$$

We now introduce a preconditioner, which in the case k=2 is the final one but which in the general case is only a first step of our construction.

$$a(P_k^k u_h, P_k^k v_h) + a(H_k^{k-1} u_h, H_k^{k-1} v_h) \ .$$

The h_k–harmonic function on Ω_k has thus been replaced by the h_{k-1}–harmonic function with the same boundary values. Since the latter space of discrete harmonic functions is smaller, it is easy to see that the preconditioner is bounded from below by the original quadratic form

$$a(u_h, u_h) \leq a(P_k^k u_h, P_k^k u_h) + a(H_k^{k-1} u_h, H_k^{k-1} u_h) \ .$$

To find a bound from above, we have to show that the energy attributable to the h_{k-1}-harmonic function can be bounded by that of the h_k–harmonic function. This follows directly from Lemma 3.

When we now turn to the case of $k > 2$, we repeatedly replace certain discrete harmonic functions by others. In each step, we replace the current last term of the preconditioner by two. We begin by considering the identity

$$H_k^{k-1}u_h = P_{k-1}^{k-1}H_k^{k-1}u_h + (I - P_{k-1}^{k-1})H_k^{k-1}u_h .$$

It is easy to see that the second term equals $H_{k-1}^{k-1}H_k^{k-1}u_h$. As in the first step, we replace this last term by another, namely $H_{k-1}^{k-2}H_k^{k-1}u_h$, and we obtain a preconditioner with three terms:

$$a(P_k^k u_h, P_k^k v_h) + a(P_{k-1}^{k-1}H_k^{k-1}u_h, P_{k-1}^{k-1}H_k^{k-1}v_h) + \\ a(H_{k-1}^{k-2}H_k^{k-1}u_h, H_{k-1}^{k-2}H_k^{k-1}v_h)$$

We simplify this expression, by using Lemma 1 and replace the third term of the preconditioner by $a(H_{k-1}^{k-2}u_h, H_{k-1}^{k-2}v_h)$. By repeating the process just outlined, we arrive at the following preconditioner:

$$\begin{aligned} b(u_h, v_h) = a(P_k^k u_h, P_k^k v_h) + a(P_{k-1}^{k-1}H_k^{k-1}u_h, P_{k-1}^{k-1}H_k^{k-1}v_h) \\ + \cdots a(P_2^2 H_3^2 u_h, P_2^2 H_3^2 v_h) + a(H_2^1 u_h, H_2^1 v_h). \end{aligned} \tag{4}$$

We can now complete our proof of the optimality of the preconditioner. By using the definitions of the operators P_i^i and H_j^i, we find that

$$\begin{aligned} a(P_k^k u_h, P_k^k u_h) &= a_{\Omega_k}(P_k^k u_h, P_k^k u_h) \\ &= a_{\Omega_k}(u_h, u_h) - a_{\Omega_k}(H_k^k u_h, H_k^k u_h), \end{aligned}$$

and, for $3 \le i \le k$,

$$\begin{aligned} a(P_{i-1}^{i-1}H_i^{i-1}u_h, P_{i-1}^{i-1}H_i^{i-1}u_h) &= a_{\Omega_{i-1}}(P_{i-1}^{i-1}H_i^{i-1}u_h, P_{i-1}^{i-1}H_i^{i-1}u_h) \\ &= a_{\Omega_{i-1}}(H_i^{i-1}u_h, H_i^{i-1}u_h) - a_{\Omega_{i-1}}(H_{i-1}^{i-1}H_i^{i-1}u_h, H_{i-1}^{i-1}H_i^{i-1}u_h) \\ &= a_{\Omega_{i-1}}(H_i^{i-1}u_h, H_i^{i-1}u_h) - a_{\Omega_{i-1}}(H_{i-1}^{i-1}u_h, H_{i-1}^{i-1}u_h). \end{aligned}$$

In the last step, we have used Lemma 1.

By using the definition of the H_j^i operators, we can rewrite the preconditioner as

$$\begin{aligned} b(u_h, v_h) &= a_{\Omega_k}(u_h, u_h) - a_{\Omega_k}(H_k^k u_h, H_k^k u_h) \\ &+ a_{\Omega_{k-1}\setminus\Omega_k}(u_h, u_h) + a_{\Omega_k}(H_k^{k-1}u_h, H_k^{k-1}u_h) - a_{\Omega_{k-1}}(H_{k-1}^{k-1}u_h, H_{k-1}^{k-1}u_h) \\ &+ a_{\Omega_{k-2}\setminus\Omega_{k-1}}(u_h, u_h) + a_{\Omega_{k-1}}(H_{k-1}^{k-2}u_h, H_{k-1}^{k-2}u_h) - a_{\Omega_{k-2}}(H_{k-2}^{k-2}u_h, H_{k-2}^{k-2}u_h) \\ &+ \cdots a_{\Omega_1\setminus\Omega_2}(u_h, u_h) + a_{\Omega_2}(H_2^1 u_h, H_2^1 u_h) \end{aligned}$$

This quadratic form can be written as the sum of $a_\Omega(u_h, u_h)$ and a number of terms of the form

$$a_{\Omega_i}(H_i^{i-1}u_h, H_i^{i-1}u_h) - a_{\Omega_i}(H_i^i u_h, H_i^i u_h)$$

Since, on Ω_i, $H_i^{i-1}u_h$ and $H_i^i u_h$ are h_{i-1}–harmonic and h_i–harmonic functions, respectively, with the same boundary values on $\partial\Omega_i$, it is easy to see that these terms are positive. This shows that the preconditioner is bounded from below by $a_\Omega(u_h, u_h)$. To get an upper bound, we only have to estimate the positive terms. By Lemma 3, the energy attributable to the subregion Ω_i, of the h_{i-1}–harmonic function $H_i^{i-1}u_h$ can be estimated by a constant times $a_{\Omega_{i-1}\setminus\Omega_i}(u_h, u_h)$. The proof of the upper bound is completed by summing over i.

What remains is to establish that the use of this preconditioner leads to a series of problems which are directly related to the projections P_i^i in particular the symmetric multiplicative algorithm given in section 2. For the unaccelerated algorithm, the equation that we have to solve is of the form

$$b(\delta u_h, v_h) = a(e_h, v_h) ,$$

where e_h is the error before the current step and δu_h the next correction. We first choose test functions v_h in the subspace V^{h_k}. All terms of the preconditioner, except the first, vanish for such test functions. Therefore the result of this first fractional step equals $P_k^k \delta u_h = P_k^k e_h$. We choose $V^{h_{k-1}}$ as our second space of test functions. All the terms of the preconditioner except the first two vanish and the first is already known and can therefore be moved over to the right hand side. A straight forward calculation and Lemma 2 show that the new right hand side is equal to $a((1 - P_k^k)e_h, v_h)$. In this second fractional step, we obtain $P_{k-1}^{k-1} H_k^{k-1} \delta u_h = P_{k-1}^{k-1}(1 - P_k^k)e_h$. Proceeding in this manner, we compute, in the $k-$th fractional step, $H_2^1 \delta u_h = P_1^1(I - P_2^2) \cdots (I - P_k^k)e_h$. At this time, the values of δu_h are available on $\Omega_1 \setminus \Omega_2$. We can use its boundary values on $\partial\Omega_2$ as Dirichlet values and solve a problem in V^{h_2} to obtain the values of δu_h in $\Omega_2 \setminus \Omega_3$. Proceeding in this manner, using a total of $(2k - 1)$ fractional steps, we finally obtain δu_h everywhere. A calculation shows that the error propagation operator corresponding to the whole step is equal to

$$(I - P_k^k)(I - P_{k-1}^{k-1}) \cdots (I - P_1^1)(I - P_2^2) \cdots (I - P_k^k) .$$

This shows that the method defined by the preconditioner indeed is the same as that of the symmetrized multiplicative algorithm defined in section 2.

4. Some Additive Algorithms. We recall, that in the so called additive algorithms, we solve equation (3) by the conjugate gradient or some other standard iterative algorithm. The different algorithms can simply be defined by specifying the subspaces V_i or alternatively the projections P_i. Before we discuss specific algorithms, we make some remarks on estimating the eigenvalues of P from above and below.

The upper bound on the spectrum is obtained by bounding

$$a(Pv_h, v_h) = a(P_1 v_h, v_h) + a(P_2 v_h, v_h) + \cdots + a(P_k v_h, v_h) ,$$

from above in terms of $a(v_h, v_h)$. We can use Schwarz' inequality and the fact that P_i is a projection to prove that each term is bounded by $a(v_h, v_h)$ and thus the spectrum of P is bounded from above by k. Our goal, however, is to establish a uniform bound on the condition number. This can be done if the terms are orthogonal or almost orthogonal; cf. discussion below.

A lemma from Lions [9] provides a method for obtaining lower bounds. Since the proof of his result is quite short, we include it in this paper.

Lemma 4. *Let $u_h = \sum_{i=1}^k u_{h,i}$, where $u_{h,i} \in V_i$, be a partition of an element of V^h and assume further that $\sum_{i=1}^k a(u_{h,i}, u_{h,i}) \leq C_0^2 a(u_h, u_h)$, $\forall u_h \in V^h$. Then $\lambda_{min}(P) \geq C_0^{-2}$.*

Proof: By elementary properties of symmetric projections and the representation of u_h as a sum, we find that

$$a(u_h, u_h) = \sum_{i=1}^k a(u_h, u_{h,i}) = \sum_{i=1}^k a(u_h, P_i u_{h,i}) = \sum_{i=1}^k a(P_i u_h, u_{h,i}) .$$

Therefore,

$$a(u_h, u_h) \le (\sum_{i=1}^{k} a(P_i u_h, P_i u_h))^{1/2} (\sum_{i=1}^{k} a(u_{h,i}, u_{h,i}))^{1/2} .$$

By the assumption of the lemma

$$a(u_h, u_h) \le C_0^2 \sum_{i=1}^{k} a(P_i u_h, P_i u_h) = C_0^2 \sum_{i=1}^{k} a(P_i u_h, u_h) = C_0^2 a(P u_h, u_h),$$

and the lemma is established.

We consider three different algorithms and distinguish between them by using a superscript 1, 2 or 3. The most natural algorithm amounts to using the projections $P_i^{(1)} = P_i^i$, where the projection operators P_i^i have been defined in equation (2). The condition number of this algorithm grows linearly with k. We have already remarked that the eigenvalues of P always are bounded from above by k. This bound is attained if $V^{h_1} \cap H_0^1(\Omega_k)$ is not empty, i.e. when the mesh size is fine enough. Any such function belongs to V^{h_i}, $i = 1, 2, \dots, k$, and is exactly reproduced by each of the projection operators. It is thus an eigenfunction with eigenvalue k. Similarly, any function which belongs to $V^{h_1} \cap H_0^1(\Omega_1 \setminus \Omega_2)$ is an eigenfunction with eigenvalue 1. We will show later that all the eigenvalues are bounded from below by a constant, which completes the proof that the condition number of $P^{(1)}$ is of order k.

A very promising method, for which to our knowledge the optimality has only been established in a quite special model case, cf. [10], is based on using $P_i^{(2)} = P_i^i - P_{i+1}^i$, $i \le k-1$ and $P_k^{(2)} = P_k^k$ as the basic projections in equation (3). It easy to show that these differences of projections are projections and that the composite finite element space V^h is the direct sum of the corresponding subspaces. A difficulty experienced when trying to establish that the eigenvalues of $P^{(2)}$ are bounded uniformly from below, by using Lions' lemma, is related to this complete lack of flexibility in representing a given element of V^h as a sum of elements of the subspaces. So far we have only been able to prove that $C_0^2 \le const.\ \ell$.

We can, however, prove the following result.

Theorem 2. *The eigenvalues of $P^{(2)}$ are uniformly bounded from above by a constant.*

Before we turn to the proof proper, we observe that the unbalance of the first method, which resulted in the amplification of certain functions by a factor k, is no longer possible. By regrouping terms, we also see that

$$P^{(2)} = P_1^1 + (P_2^2 - P_2^1) + \dots + (P_k^k - P_k^{k-1}) . \tag{5}$$

All the terms, except the first, are similar to multigrid corrections since they each represent the difference between two solutions on the same subregion, using two different mesh sizes.

By using the representation of $P^{(2)}$, given in equation (5), we obtain

$$\begin{aligned} P^{(2)} u_h &= P_1^1 u_h + (P_2^2 - P_2^1) u_h + \dots + (P_k^k - P_k^{k-1}) u_h \\ &= u_1 + u_2 + \dots + u_k. \end{aligned}$$

We show that these terms are increasingly orthogonal:

$$a(u_\ell, u_m) \le const.\ q_1^{|\ell - m|} (a(u_\ell, u_\ell))^{1/2} (a(u_m, u_m))^{1/2},$$

where $q_1 < 1$, uniformly. This is a so-called strengthened Cauchy inequality. Without loss of generality, we assume that $\ell < m$. It is easy to show that $a(u_m, v_h) = 0$, $\forall v_h \in$

$V^{h_{m-1}} \cap H_0^1(\Omega_m)$, i.e. u_m is h_{m-1}–harmonic on Ω_m. Let $u_\ell = \sum \alpha_i \phi_i^{h_\ell}$ where the $\phi_i^{h_\ell}$ are basis functions in V^{h_ℓ}. Any such basis function is orthogonal to u_m if its support does not intersect $\partial\Omega_m$ since then either its support belongs to $\Omega_\ell \setminus \Omega_m$ or $\phi_i^{h_\ell} \in V^{h_{m-1}} \cap H_0^1(\Omega_m)$. Thus the only contributions to $a(u_m, u_\ell)$ originate from the triangles of V^{h_ℓ} which intersect $\partial\Omega_m$. Consider one such triangle T. The restriction of u_ℓ to T is a linear function and thus has a constant gradient. It can also be written as a linear combination of basis functions in $V^{h_{m-1}}$. The support of most of these do not intersect $\partial\Omega_m$ and therefore

$$\begin{aligned} a_T(u_m, u_\ell) &= a_{T\cap S}(u_m, u_\ell) \le (a_T(u_m, u_m))^{1/2}(a_{T\cap S}(u_\ell, u_\ell))^{1/2} \\ &\le Const.\ q_1^{|m-\ell|} a_T(u_{m,m})^{1/2} a_T(u_\ell, u_\ell)^{1/2} . \end{aligned}$$

Here S denotes the union of the small triangles that are next to $\partial\Omega_m$. We also use our assumptions on the triangulations, given in section 2, and the fact that the gradient of u_ℓ is constant on T and that therefore its contribution to the quadratic form is directly proportional to the area of integration. The proof of the strengthened Cauchy inequality is completed by summing over the triangles of V^{h_ℓ}.

We now note that

$$a(P^{(2)}u_h, P^{(2)}u_h) = \sum_{i,j=1}^{k} a(u_i, u_j) \le U^T A U.$$

Here

$$A = const. \begin{pmatrix} 1 & q_1 & q_1^2 & \cdots & q_1^{k-1} \\ q_1 & 1 & q_1 & \cdots & q_1^{k-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q_1^{k-1} & q_1^{k-2} & q_1^{k-3} & \cdots & 1 \end{pmatrix}$$

and

$$U^T = (a(u_1, u_1)^{1/2}, (a(u_2, u_2)^{1/2}, \ldots, a(u_k, u_k)^{1/2}).$$

It is easy to show that the Euclidean norm of A is uniformly bounded. Thus

$$a(P^{(2)}u, P^{(2)}u) \le |A|_{\ell_2} \sum_{i=1}^{k} a(u_i, u_i) \le const. \sum_{i=1}^{k} a(u_i, u_i).$$

To complete the proof, we note that

$$\begin{aligned} a(u_i, u_i) &= a(P_i^i u_h - P_i^{i-1} u_h, P_i^i u_h - P_i^{i-1} u_h) \\ &= a(P_i^i u_h - P_i^{i-1} u_h, P_i^i u_h), \end{aligned}$$

since $P_i^i u_h - P_i^{i-1} u_h$ is h_{i-1}–harmonic. By using elementary properties of the projections, we find that $a(u_i, u_i) = a(u_i, u_h)$. Therefore

$$\sum_{i=1}^{k} a(u_i, u_i) = \sum_{i=1}^{k} a(u_i, u_h) = a(P^{(2)}u_h, u_h).$$

Thus

$$a(P^{(2)}u_h, P^{(2)}u_h) \le const.\ a(P^{(2)}u_h, u_h),$$

from which the upper bound on $P^{(2)}$ follows.

If we modify our projections, we arrive at an algorithm for which we have a proof of optimality.

Theorem 3. *The additive method defined by the projections $P_i^{(3)} = P_i^i - P_{i+2}^i$, $i \leq k-2$, $P_{k-1}^{(3)} = P_{k-1}^{k-1}$ and $P_k^{(3)} = P_k^k$ has a condition number which is independent of k and the number of degrees of freedom.*

A uniform upper bound is obtained by using Theorem 2 and the following formula.

$$\begin{aligned} P^{(3)} &= P_1^1 - P_3^1 + P_2^2 - P_4^2 + \cdots + P_{k-1}^{k-1} + P_k^k \\ &= P_1^1 - P_2^1 + P_2^2 - P_3^2 + \cdots + P_k^k \\ &\quad + P_2^1 - P_3^1 + P_3^2 - P_4^2 + \cdots + P_{k-1}^{k-1} \\ &= P^{(2)} + \widetilde{P}^{(2)} \end{aligned}$$

By Theorem 2, there is an upper bound for $P^{(2)}$ and the same result also provides a uniform upper bound for $\widetilde{P}^{(2)}$, which is an operator corresponding to a composite finite element problem using $k-1$ levels.

From this argument and the fact that $\widetilde{P}^{(2)}$ is positive definite, it follows that the condition number of $P^{(2)}$ cannot be much better than that of $P^{(3)}$. A lower bound of the spectrum of $P^{(3)}$ is given by a partitioning formula and Lions' Lemma. Before we give the details, we note that this argument also provides a lower bound for the spectrum of $P^{(1)}$ since the subspaces related to the projections of that method include those of $P^{(3)}$.

We partition an arbitrary $u_h \in V^h$ as

$$u_h = \sum_{i=1}^{k} u_{h,i}, \qquad u_{h,i} \in V_i^{(3)} \ ,$$

where $V_i^{(3)}$ is the range of $P_i^{(3)}$. The formulas for the $u_{h,i}$ are given by

$$u_{h,1} = \begin{cases} u_h & \text{on} \quad \Omega_1 \setminus \Omega_2 \\ h_1 - harmonic & \text{on} \quad \Omega_2 \setminus \Omega_3 \\ 0 & \text{on} \quad \Omega_3 \end{cases}$$

and for $2 \leq i \leq k-2$, and

$$u_{h,i} = \begin{cases} u_h - u_{h,i-1} & \text{on} \quad \Omega_i \setminus \Omega_{i+1} \\ h_i - harmonic & \text{on} \quad \Omega_{i+1} \setminus \Omega_{i+2} \\ 0 & \text{on} \quad \Omega_{i+2} \end{cases}$$

Since the construction of $u_{h,h-1}$ and $u_{h,h}$ is straightforward, we do not provide details. We note that on each region $\Omega_i \setminus \Omega_{i+1}$ only two of the terms differ from zero and that $u_{h,i} = u_h$ on $\partial\Omega_{i+1}$. By applying the projection $P_i^{(3)}$ to $u_{h,i}$, we find that $u_{h,i} \in V_i^{(3)}$. The proof is completed by showing that

$$a(u_{h,i}, u_{h,i}) \leq const.\ a_{\Omega_i \setminus \Omega_{i+1}}(u_h, u_h).$$

This follows from a variant of Lemma 3 and elementary considerations. The proof of the lower bound is completed by summing over i.

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