CHAPTER 5

Parallel Algorithms for Solving Partial Differential Equations

T. Lu*
T. M. Shih†
C. B. Liem†

Abstract. In §1 to §5, two synchronized parallel algorithms for the solution of boundary value problems of partial differential equations are proved. Algorithm 1 is based on the minimum modulus principle; therefore, it can be applied to nonlinear PDE's. At such cases, a linearization step should be done before each iteration. The proof of the convergence uses the iterative method of groupwise projection. Algorithm 2 is based on the discrete maximum principle. A synchronized parallel algorithm for the Dirichlet problem of linear equations satisfying the uniformly elliptic condition is given in §6.

§1 The parallel algorithm 1

The boundary value problem of a partial differential equation, in general, can be written as

$$\begin{cases} Lu = f, & \text{in } \Omega \\ lu = g, & \text{on } \partial\Omega \end{cases} \qquad (1)$$

Ω is a bounded region with boundary $\partial\Omega$. L is a differential operator and l is a boundary operator. We may apply either the finite difference or the finite element method to (1) and obtain an algebraic system

*Institute of Mathematical Sciences, Academia Sinica, Chengdu, China.

† Department of Mathematical Studies, Hong Kong Polytechnic, Hong Kong.

$$L^h u^h(P_j) = \sum_{i \in I} C_{ji} u^h(P_i) = f_j, \quad j \in I \tag{2}$$

where I is the set of indices of all grid points in Ω_h and Ω_h is the set of all grid points inside Ω.

We define the discrete neighbourhood

$$N_j \equiv N(P_j) \equiv \{P_i;\ C_{ji} \neq 0\}.$$

Then (2) becomes

$$L^h u^h(P_j) = \sum_{P_i \in N_j} C_{ji} u^h(P_i) = f_j, \quad j \in I \tag{3}$$

In order to solve (3) by a parallel algorithm, we divide Ω_h into m subsets $\Omega_h^1, \ldots, \Omega_h^m$, $\Omega_h = \bigcup_{i=1}^m \Omega_h^i$, where some of the subsets may be overlapping. In order to reduce the waiting time, every Ω_h^i, $i = 1, \ldots, m$ should contain nearly the same number of grid points.

Define the discrete neighbourhood of Ω_h^i as follows

$$N(\Omega_h^i) = \bigcup_{P \in \Omega_h^i} N(P).$$

P_j is called a k-multiple point, denoted by $P_j \in \pi_k$, if there exists at most k subsets $\Omega_h^{i_1}, \ldots, \Omega_h^{i_k}$ such that $P_j \in \bigcap_{s=1}^k N(\Omega_h^{i_s})$.

The procedure of the parallel algorithm 1 is as follows:

1° Choose a tolerance $\varepsilon > 0$ and an initial $u_0 = \{u_0(P_j), j \in I\}$. Set $0 \Rightarrow n$.

2° Compute parallelly for each Ω_h^i, $i=1, \ldots, m$ the coefficient C_{js} of the discrete system (for non-linear case, a linearization process is needed) and the residuals

$$\bar{f}_j^i = f_j - \sum_{P_s \in N_j} C_{js} u_n^i(P_s), \quad j \in I_i$$

$$F^i = \max_{j \in I_i} |\bar{f}_j^i|, \quad i = 1, \ldots, m.$$

3° If $F = \max_{1 \le i \le m} F^i \le \varepsilon$, stop the process and output u_n, otherwise proceed to the next step.

4° Set equations for the correction Δu_n^i in each Ω_h^i, $i = 1, \ldots, m$

$$A_i: \sum_{P_s \varepsilon N_j} C_{js} \quad u_n^i(P_s) = \bar{f}_j^i, \qquad j \varepsilon I_i.$$

5° Find the minimum modulus solutions of A_i, $i = 1, \ldots, m$ parallelly

$$\Delta u_n^i = C_i^+ \bar{f}^i$$

where C_i^+ is the Moore-Penrose generalized inverse matrix of the coefficient matrix of A_i.

6° If $P_j \varepsilon \pi_k$, then there exists $i_1, \ldots, i_k$ such that

$$P_j \varepsilon / \bigcap_{s=1}^{k} N(\bar{\Omega}_h^{i_s})$$

and define

$$\Delta u_n(P_j) = \frac{1}{k} \sum_{s=1}^{k} \Delta u_n^{i_s}(P_j), \qquad \text{for } j \varepsilon I \tag{4}$$

7° Set $u_n + \omega \Delta u_n \Rightarrow u_n$, $n+1 \Rightarrow n$ and go to 2°, $0 < \omega < 2$.

§2 The parallel algorithm 2

Algorithm 2 is based on the resulting algebraic system is linear and satisfies the discrete maximum principle. For convenience, we consider the following Dirichlet problem

$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{5}$$

The discrete system in the entire region is

$$\begin{cases} L^h u_h = f^h, & \text{in } \Omega_h \\ u_h = g^h, & \text{on } \partial\Omega_h \end{cases} \tag{6}$$

Divide Ω_h into m subsets: $\Omega_h = \bigcup_{i=1}^{m} \Omega_h^i$, Ω_h^i can be overlapping.

$\partial\Omega_h^i = N(\Omega_h^i) \setminus \Omega_h^i$ is the discrete boundary of Ω_h^i.

Let u* be the unique solution of (6). (6) is equivalent to the system

$$\begin{cases} L^h u_h^i = f^h, \text{ in } \Omega_h^i \\ u_h^i = u^*, \qquad \text{on } \partial\Omega_h^i \end{cases} \tag{7}$$

$i = 1, \ldots, m.$

Since u* is unknown, (7) can only be solved by iterative methods. The 1st, 2nd and 3rd step in the procedure are the same as those for algorithm 1, the other steps are as follows:

4° Solve

$$\begin{cases} L^h u^i_{n+1} = f^h, & \text{in } \Omega^i_h \\ u^i_{n+1} = u_n, & \text{on } \partial\Omega^i_h \end{cases} \tag{8}$$

for i = 1, ..., m parallelly.

5° If $P_j \in \pi_k$, then there exist $i_1, \ldots, i_k$ such that

$$P_j \in \bigcap_{s=1}^{k} N(\Omega^{i_s}_h)$$

Set

$$u_{n+1}(P_j) = \frac{1}{k} \sum_{s=1}^{k} u^{i_s}_{n+1}(P_j); \quad j \in I .$$

6° Set $n+1 \Rightarrow n$ and go to 2°.

§3 The iterative method of groupwise projection for linear systems.

The parallel algorithm 1 introduced in §1 is based on the iterative method of groupwise projection for linear systems. The method was first established by S. Kaczmarz [2]; its further development can be found in [1], [3] and [4]. In those papers the discussion was confined to the case of one equation in one group. To deal with our problem, we shall extend the method to the case of many equations contained in one group.

Consider the linear system

$$(a_j, x) = a_{j1} x_1 + \ldots + a_{j\ell} x_\ell = b_j, \; j = 1, \ldots, \ell \tag{9}$$

and assume it has a unique solution x*.

Divide the set of indices $I = \{1, 2, \ldots, \ell\}$ into m subsets:

$I = \bigcup_{i=1}^{m} I_i$, where different subsets can be overlapping. Then the system (9) is divided into m groups:

$$G_i: (a_j, x) = b_j, \quad j \in I_i \tag{10}$$

where i = 1, ..., m.

For each i, G_i has at least a solution x*, in general, the solution is not unique. It is well known that the minimum modulus solution of G_i exists and is unique. This solution is denoted by $E_i x$, where E_i is the projection onto the subspace H_i = span $\{a_j;\ j \in I_i\}$.

The iterative method of groupwise projection is as follows:

Choose a relaxation factor ω $(0 < \omega < 2)$ and an initial approximation $x^0 = \{x^0(P_j),\ j \in I\}$, then the process of getting x^{k+1} from x^k can be proceed as follows:

$$x^k_{(s+1)} = x^k_{(s)} + \omega \Delta x^k_{(s)}, \qquad s = 1, \ldots, 2m$$

$$x^k_{(s)} = \{x^k_{(s)}(P_j),\ j \in I\} \text{ and } x^k_{(1)} = x^k,$$

where the correction $\Delta x^k_{(s)}$ is the minimum modulus solution of the system

$$(a_j, \Delta x^k_{(s)}) = b_j - (a_j, x^k_{(s)}),\ j \in I_i, i = \min(s, 2m+1-s)$$

Obviously, $\Delta x^k_{(s)}(P_j)$ is defined only for $j \in I_i$. We extend it to all $j \in I$ by simply setting $\Delta x^k_{(s)}(P_j) = 0$ for $j \in I \setminus I_i$. Finally, we set

$$x^{k+1} = x^k_{(2m+1)} \ .$$

Now we are going to prove the convergence. In fact, the exact correction value of $x^k_{(s)}$ is $x^* - x^k_{(s)}$ and the minimum modulus solution $\Delta x^k_{(s)}$ is the projection of $x^* - x^k_{(s)}$ on the subspace H_i, i.e., $\Delta x^k_{(s)} = E_i(x^* - x^k_{(s)})$.

Hence, from $x^k_{(s+1)} = x^k_{(s)} + \omega\, \Delta x^k_{(s)}$, we have

$$x^* - x^k_{(s+1)} = (I - \omega E_i)(x^* - x^k_{(s)})$$

Let $Q_i = I - \omega E_i$. Then

$$x^* - x^{k+1} = Q_1 \ldots Q_m\, Q_m \ldots Q_1(x^* - x^k)$$

$$= (Q\, Q^*)^{k+1}(x^* - x^o) \qquad (11)$$

where $Q = Q_1 \ldots Q_m$. It is known, by direct computation, that $\|Q_i\| \le 1$, $i = 1, \ldots, m$ and hence $\|Q\| \le 1$. The equality holds only if there exists a vector y, $\|y\| = 1$ such that $Qy = y$. This means that y is orthogonal to all a_j, $j \in I$ and hence $y = 0$. The contradiction shows that $\|Q\| = r < 1$ and

$$\|x^* - x^k\| \leq \|Q\|^{2k} \|x^* - x^0\| = r^{2k} \|x^* - x^0\|.$$

It follows that $x^k \to x^*$ as $k \to \infty$.

The iterative method of groupwise projection when apply to non-linear systems, the convergence will also follow.

§4 The proof of algorithm 1

In the following, we shall give the proof of linear problems. Non-linear problems can be proved in the similar way.

Let

$$\sum_{P_s \in N_j} C_{js} u_s = f_j, \ j \in I, \ u_s = u^h(P_s) \qquad (12)$$

be the discrete system defined in the entire set Ω_h and

$$A_i: \quad \sum_{P_s \in N_j} C_{js} u_s^i = f_j, \ j \in I_i \qquad (13)$$

$$(i = 1, \ldots, m)$$

be the discrete system defined in the ith subset Ω_h^i.

When we solve A_i by using the parallel algorithm 1, we take in account that $u_s^{i_1}$ and $u_s^{i_2}$ $(i_1 \neq i_2)$ are independent. From this point of view, we have assumed that A_{i_1} and A_{i_2} have no unknowns in common. As a compensation to this assumption, we add the following extra restrictions to A_i,

$$B_s: \begin{array}{l} u_s^{i_1} - u_s^{i_2} = 0 \\ \cdots \quad \cdots \\ u_s^{i_1} - u_s^{i_k} = 0 \end{array} \qquad \text{for } P_s \in \pi_k, \ k \geq 2 \qquad (14)$$

Then, the equations A_i, $i = 1, \ldots, m$ together with the equations $B_s, P_s \in \pi_k$, $k \geq 2$ are equivalent to the equations (12).

We name A_i, $i = 1, \ldots, m$ as group 1 and B_s, $P_s \in \pi_k$, $k \geq 2$ as group 2. Since A_{i_1} and A_{i_2} $(i_1 \neq i_2)$ have no unknowns in common, the minimum modulus solution of group 1 simply is the union of the minimum modulus solutions of A_i, $i = 1, \ldots, m$ and the latter can be found parallelly.

Let $u_0 = \{u_0(P_j), \ j \in I\}$ be an initial approximation to $u^* = \{u^h(P_j), \ j \in I\}$, which is the solution of (12). Let $\Delta u_n^i = \{u_n^i(P_j), \ j \in I_i\}$, $i = 1, \ldots, m$ be the correction of u_n^i

obtained from the minimun modulus solution of group 1 and let $\delta u_n^i(P_s)$, $P_s \in \pi_k$, $k \geq 2$ be the correction of $u_n^i + \Delta u_n^i$ obtained from the minimum modulus solution of group 2. Noting that when $s_1 \neq s_2$, B_{s_1} and B_{s_2} have no unknowns in common, we may find the minimum modulus solutions of $B_s, P_s \in \pi_k$, $k \geq 2$ parallelly. Substituting $u_n^i + \Delta u_n^i + \delta u_n^i$ into B_s, we have

$$C_s: \quad \begin{aligned} \delta u_n^{i_1}(P_s) - \delta u_n^{i_2}(P_s) &= -(\Delta u_n^{i_1}(P_s) - \Delta u_n^{i_2}(P_s)) \\ &\cdots \\ \delta u_n^{i_1}(P_s) - \delta u_n^{i_k}(P_s) &= -(\Delta u_n^{i_1}(P_s) - \Delta u_n^{i_k}(P_s)) \end{aligned}$$

The minimum modulus solution of C_s is

$$\delta u_n^{i_j}(P_s) = \frac{1}{k}\left(\sum_{j=1}^{k} \Delta u_n^{i_j}(P_s)\right) - \Delta u_n^{i_j}(P_s)$$

Finally, we have

$$u_{n+1}(P_s) = u_n(P_s) + \frac{1}{k}\left(\sum_{j=1}^{k} \Delta u_n^{i_j}(P_s)\right), \quad P_s \in \pi_k, \quad k \geq 1$$

Let E_1, E_2 be the projections defined by group 1 and group 2 repectively. Let $Q_i = I - \omega E_i$, $i = 1, 2$ and $Q = Q_1 Q_2$. According to the proof given in §3, we conclude that

$$\|u^* - u_n\| = \|(Q\, Q^*)^n (u^* - u_0)\| \leq r^{2n} \|u^* - u_0\|$$

where $\quad r = \|Q\| = \|(I - \omega E_1)(I - E_2)\| < 1.$

§5 The proof of the algorithm 2

We know, from (7) and (8), that $u_{n+1}^i - u^*$ satisfies

$$\begin{cases} L^h(u_{n+1}^i - u^*) = 0, & \text{in } \Omega_h^i \\ u_{n+1}^i - u^* = u_n - u^*, & \text{on } \partial\Omega_h^i \setminus \partial\Omega_h \\ u_{n+1}^i - u^* = 0, & \text{on } \partial\Omega_h^i \cap \partial\Omega_h \end{cases} \tag{15}$$

where $u_{n+1}^i = \{u_{n+1}^i(P);\ P \in N(\Omega_h^i)\}$, $\partial\Omega_h^i \cap \partial\Omega_h \neq 0$.

Let $u_{n+1} = \{\bar{u}_{n+1}^i(P);\ P \in N(\Omega_h^i),\ i = 1, \ldots, m\}$, where $\bar{u}_{n+1}^i(P)$ is defined as follows: if $P \in \pi_k$, there exist $i_1, \ldots, i_k$ such that $P \in \bigcap_{j=1}^{k} N(\Omega^{i_j})$ and $\bar{u}^i_{n+1}(P) = \frac{1}{k}\sum_{j=1}^{k} u^{i_j}_{n+1}$

It can be shown, by using the discrete maximum principle, that the sequence $\|u^* - u_n\|_\infty$ is strictly monotone decreasing, i.e., if $\|u^* - u_n\|_\infty \neq 0$, then $\|u^* - u_{n+1}\|_\infty < \|u^* - u_n\|_\infty$.

Hence $\{u_{n+1} - u^*\}$ has a convergent subsequence. For convenience, we still write the subsequence as $u_{n+1} - u^*$ and let its limit be $v - u^*$. Let S be an operator such that

$$u_{n+1} - u^* = S(u_n - u^*)$$

then we have

$$v - u^* = S(v - u^*) \tag{16}$$

and

$$\|v - u^*\|_\infty = \|S(v - u^*)\|_\infty < \|v - u^*\|_\infty \tag{17}$$

It proves that $v = u^*$.

We have proved that the sequence $\{\|u_n - u^*\|_\infty\}$ has a limit and there is a subsequence of $\{\|u - u^*\|_\infty\}$ has limit zero, hence $\|u - u^*\|_\infty \to 0$, as $n \to \infty$.

§6 A synchronized parallel algorithm for solving Dirichlet problems of second order elliptic PDE.

Consider the following Dirichlet problem

$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{18}$$

where $\Omega \in R^2$ is a bounded open set, L is a linear operator satisfying the uniformly elliptic condition.

Let $\Omega = \bigcup_{i=1}^{m} \Omega_i$, $\partial\Omega_i \cap \partial\Omega \neq 0$, and $\partial\pi_k$ either coincide with $\partial\Omega$ or is an arc with endpoints on $\partial\Omega$.

The algorithm is as follows:

1^o Choose an initial u^0 with $u^0|\partial\Omega = g$

2^o Solve parallelly

$$\begin{cases} Lu^i_{n+1} = f, & \text{in } \Omega i \\ u^i_{n+1} = u^n, & \text{on } \partial\Omega i \setminus \partial\Omega \\ u^i_{n+1} = g, & \text{on } \partial\Omega i \cap \partial\Omega \end{cases} \tag{19}$$

3^o $u^{n+1} = \frac{1}{K}\sum_{j=1}^{m} u^{i_j}_{n+1}$

__Theorem.__ If (19) has a bounded solution for each i, i = 1, 2, ...m, then either there exists a $q \in (0,1)$, such that

$$\|u^* - u^{n+1}\|_\infty \leq q \|u^* - u^n\|_\infty$$

or the above procedure will converge in finite steps. We state the following lemma without proof.

__Lemma__

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_1 \cup \Gamma_2 \\ u = g, & \text{on } \Gamma_3 \end{cases}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

If $\widehat{MN}$ is a smooth curve in Ω with $M \in \Gamma_1$, $N \in \Gamma_2$, then there exists a constant $q \in (0,1)$ independent of g such that

$$|u(P)| \leq qQ, \text{ for } P \in \widehat{MN}.$$

where $Q = \max_{\Gamma_3} |g|$.

__Proof of Theorem__

Suppose that u* is the solution of (18), then

$$L(u^* - u^i_{n+1}) = 0, \qquad \text{in } \Omega_i$$

$$u^* - u^i_{n+1} = u^* - u^n, \qquad \text{on } \partial\Omega_i \setminus \partial\Omega$$

$$u^* - u^i_{n+1} = 0, \qquad \text{on } \partial\Omega_i \cap \partial\Omega$$

If $\|u^* - u^{n+1}\|_\infty = 0$, the iteration converges in finite steps.

If $\|u^* - u^{n+1}\|_\infty \neq 0$, there exists $P \in \overline{\Omega}$ such that

$$|u^*(P) - u^{n+1}(P)| = \|u^* - u^{n+1}\|_\infty \neq 0 \qquad (20)$$

__Case I__

If $P \in \pi_1$ there exist Ω_{i_0} and $P \in \overline{\Omega}_{i_0}$ such that

$u^{n+1}(P) = u^{i_0}_{n+1}(P)$, hence $u^* - u^{i_0}_{n+1}$ is a constant on $\overline{\Omega}_{i_0}$. Since $\partial\Omega_{i_0} \cap \partial\Omega \neq 0$, we have $\|u^* - u^{n+1}\|_\infty = 0$. This contradicts (20).

__Case II__

If $P \in \pi_k, k \geq 2$. Since $L(u^* - u^{n+1}) = 0$, for $P \in \mathring{\pi}_k$. From the maximum principle, $P \in \partial\pi_k$. We may assume that $P \in \widehat{MN} \subset \partial\pi_k$ with M, N $\in \partial\Omega$. From our assumption, evidently there is i_1, $\widehat{MN} \subset \Omega_{i_1}$.

From Lemma, there exists $q \in (0,1)$ such that

$$|u^* - u_{n+1}^{i_1}| \le q \max_{Q \in \partial\Omega_{i_1}} |u^*(Q) - u_{n+1}^{i_1}(Q)|$$

and

$$|u^*(P)-u^{n+1}(P)| \le \frac{1}{k}|u^*(P)-u_{n+1}^{i_1}(p)| + \frac{1}{k}\sum_{j=2}^{k}|u^*(P)-u_{n+1}^{i_j}(P)|$$

$$\le \frac{q}{k} \max_{Q \in \partial\Omega_{i_1}} |u^*(Q) - u_{n+1}^{i_1}(Q)| + \frac{1}{k}\sum_{j=2}^{k} \max_{Q \in \partial\Omega_{i_j}} |u^*(Q) - u_{n+1}^{i_j}(Q)|$$

$$\le (\frac{q}{k} + \frac{k-1}{k}) \max_{Q \in \partial\Omega_{i_j}} |u^*(Q) - u_{n+1}^{i_j}(Q)|$$

$$= (\frac{q}{k} + \frac{k-1}{k}) \max_{Q \in \partial\Omega_{i_j}} |u^*(Q) - u^n(Q)|$$

$$\le (\frac{q}{k} + \frac{k-1}{k})\|u^* - u^n\|_\infty < \bar{q} \; \|u^* - u^n\|_\infty , \text{ where } \bar{q} = \frac{q}{k} + \frac{k-1}{k} < 1.$$

REFERENCES

[1] A. BJORCK AND T. ELFVING, Accelerated projection method for computation psendoinverse solution of systems of linear equations, BIT 19 (1979), pp. 143 - 163.

[2] S. KACZMARZ, Angenaherte Auflosung von Systemen Linear Gleichungen, Bull. Internat. Acad. polon. Science et Lettres (1937), pp. 355 - 357.

[3] J. Q. LIU, Convergence of SOR projection method for nonlinear systems, J. Sys. Sci. & Math. Sci., 1(1981), pp. 77 - 79, China (in English).

[4] T. LU AND Q. LIN, A projection iterative method for solving integral equations of the first kind and algebraic systems, Math. Numer. Sinica, 2 (1984), pp. 113 - 120 (in Chinese).