# Remarks on Spectral Equivalence of Certain Discrete Operators

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**Abstract** We consider the Neumann-Dirichlet preconditioner for the discrete Laplacian in the unit square. We show that the capacitance matrix C is equal to the identity even for problems with discontinuous coefficients. In the case of many Neumann-Dirichlet strips this is no longer true if the strips are extremely thin. The conditioning of C in this case is significantly better when the Neumann strips correspond to regions with larger coefficients.

**1. Introduction** It is well known that all uniformly elliptic operators L defined on $\Omega$ with the same boundary conditions are *spectrally equivalent* [5]:

$$c_1(L_1x,x) \le (L_2x,x) \le c_2(L_1x,x),$$

where $c_1$, $c_2$ are positive constants, and (x,y) is a proper inner product. Similar relations hold for the descretized form of these operators:

$$a_1x^TAx \le x^TBx \le a_2x^TAx, \quad \forall x\in \Re^n, \quad \forall n,$$

where A, B are nxn symmetric positive definite matrices, and $a_1$, $a_2$ are positive constants independent of n. These inequalities imply that the ratio of extreme eigenvalues of $AB^{-1}$ is bounded by $a_2/a_1$, called the *spectral equivalence bound* [1]. Thus, we could use one discrete elliptic operator as an efficient preconditioner for another one. In particular, the discrete Laplacian would be an excellent candidate for such a preconditioner. Let us have a closer look at some questions that can be posed: How large is the bound for $\kappa(AB^{-1})$? What is the effect of changing the boundary conditions? And finally, how all this relates to domain decomposition?

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**1.1 Changing boundary conditions** Let us consider two nxn matrices, A and B, that represent the one dimensional -Laplacian on $\Omega = [0,1]$ with the Dirichlet and Neumann boundary condition at the left end of the interval, and with the Dirichlet condition at the other end, respectively. They differ only in the (1,1) element:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

It has been shown by Hald, see [7, p.457], that $AB^{-1}$ is a rank one modification of an identity, $AB^{-1} = I + uv^T$, where $u^T=(1,0,\ldots,0)$ and $v^T=(n,n-1,\ldots,1)$. Moreover, only two singular values of $AB^{-1}$ differ from 1. These two coincide with the eigenvalues of $I_2 + (v+\alpha u,u)^T(u,v+\alpha u)$, where $\alpha = v^Tv/2 \approx n^3/6$. Its characteristic polynomial is $\lambda^2-2(1+\alpha+\gamma)\lambda+(1+\gamma^2+2\gamma)=0$, where $\gamma=u^Tv=n$. Thus, the smallest singular value of $AB^{-1}$ is equal to about $\sigma_{min} \approx 3/n$, the largest $\sigma_{max} \approx n^3/3$, and $\kappa(AB^{-1}) = \sqrt{\sigma_{max} / \sigma_{min}} \approx n^2/3$. Consequently, A and B are far from being spectrally equivalent, although they represent the same operator (-Laplacian), albeit with different boundary conditions.

**2. Neumann-Dirichlet preconditioner (two subdomains)** Let us now consider the same - Laplacian on $\Omega = [0,1]$ with the Dirichlet boundary conditions at both end points, represented by the nxn matrix A. Let us impose artificial inner boundary conditions at x=0.5 such that we have the Dirichlet condition to the left of it, and the Neumann condition to the right. The resulting matrix B has the form:

$$B = \begin{pmatrix} 2 & -1 & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & -1 & 2 & -1 & & & \\ & & 0 & 2 & -2 & & \\ & & & -1 & 2 & -1 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & -1 & 2 \end{pmatrix}$$

Using the same technique as above, we can show that $AB^{-1} = I + uv^T$, where $u^T=(0,\ldots,0,1,0,\ldots,0)$, i.e., $u_i = \delta_{ik}$, $v^T = 1/k\ (-1,\ldots-(k-1),0,k-1,\ldots,1)$, and $k = (n+1)/2$. Since now $\gamma = u^Tv = 0$, the characteristic polynomial of $I_2 + (v+\alpha u,u)^T(u,v+\alpha u)$ is simplified to $\lambda^2 - 2(1+\alpha)\lambda + 1 = 0$. Here, the new $\alpha = v^Tv/2 \approx n/3$. Thus, $\sigma_{min} \approx 3/n$, similarly as before, but $\sigma_{max} \approx n/3$. As a result, we obtain a much more favorable ratio, and $\kappa(AB^{-1}) = \sqrt{\sigma_{max} / \sigma_{min}} \approx 2(\alpha+1) \approx n/3$. Yet again, A and B fail to be spectrally equivalent. Numerical experiments confirm this analysis, see Table 1.

| # points | 5 | 15 | 25 | 35 | 45 |
|---|---|---|---|---|---|
| $\kappa(AB^{-1})$ in **1.1** | 11.08 | 79.49 | 214.50 | 416.16 | 684.50 |
| $\kappa(AB^{-1})$ in **2.** | 2.75 | 6.21 | 9.59 | 12.94 | 16.29 |

Table 1. Condition numbers $\kappa(AB^{-1})$ as a function of the number of grid points.

In a two dimensional analog of our example, $n^2$x $n^2$ matrix $\mathbb{A}$, represents the - Laplacian on a unit square, $\Omega=[0,1]x[0,1]$ with the Dirichlet boundary conditions at $\partial\Omega$, and $\mathbb{B}$ represents the same operator with added Neumann-Dirichlet conditions at the artificial interface (x;y=0.5):

$$\mathbb{A} = \begin{pmatrix} T & -I & & & \\ -I & T & -I & & \\ & \cdot & \cdot & \cdot & \\ & & -I & T & -I \\ & & & -I & T \end{pmatrix}, \qquad T = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix} \quad (1')$$

$$\mathbb{B} = \begin{pmatrix} T & -I & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & -I & T & -I & & & \\ & & 0 & T & -2I & & \\ & & & -I & T & -I & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & -I & T \end{pmatrix} \quad (1'')$$

Here, $\mathbb{A}\mathbb{B}^{-1} = I + UV^T$, where the rectangular $n^2$xn matrices U and V have the form $U^T=(0,I_n,0)$, $V^T=(-X^T,0_n,X^T)$, and mxn matrix X will be defined further, m=n(n-1)/2. As an example, for the simplest case, n=3, we have

$$X = 1/56 \begin{pmatrix} 15 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

As before, we can conclude that there are only 2n singular values of $\mathbb{A}\mathbb{B}^{-1}$ that differ from 1.

| $N=n^2$ | $\lambda_{min}$ | $\lambda_{max}$ | cond # |
|---|---|---|---|
| 9 | .76 | 1.31 | 1.72 |
| 25 | .66 | 1.51 | 2.28 |
| 49 | .60 | 1.68 | 2.82 |
| 81 | .55 | 1.83 | 3.35 |

Table 2. Extreme eigenvalues and the condition number of $A\mathbb{B}^{-1}$ in 2D.

Table 2 demonstrates the results of numerical experiments. Here, the condition number grows linearly with $\sqrt{N}$ (approximately as $\sqrt{N}/4+1$), yet, once more, $\mathbb{A}$ and $\mathbb{B}$ fail to be spectrally equivalent.

It should be noted, that application of a special symmetrizer to $\mathbb{B}$, see [8], results in a matrix $B$ that *is* spectrally equivalent to $\mathbb{A}$. More precisely, only $n$ eigenvalues of $\mathbb{A}B^{-1}$ are equal to 2, while all the rest are 1.

**3. Capacitance matrix** The method of domain decomposition often can be considered as a process in a *subspace*, see [6]. This amounts to performing the main iteration with the capacitance matrix C of the form $C = S^TAB^{-1}S$, where $S^T$ is a restriction operator $S^T=(I_p,0)$, and $p$ is the number of grid points on the separator, $p \ll n$. Note, that for our one dimensional examples in sections 1.1 and 2., $p$ equals to 1 and the capacitance $C = S^T(I + uv^T)S$ is equal to $n+1$ and $1$, respectively.

In general, we can write matrices $\mathbb{A}$ and $\mathbb{B}$ in a 2 by 2 block form:

$$\mathbb{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad (2)$$

It has been shown, see [3] and also [4], that if $A_{11}$ is invertible then $\mathbb{A}\mathbb{B}^{-1}$ has the form:

$$\mathbb{A}\mathbb{B}^{-1} = \begin{pmatrix} I & 0 \\ Z & C \end{pmatrix} \qquad (3)$$

where $C = C_1C_2^{-1}$, $C_1 \equiv (S^T\mathbb{A}S)^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$,
$C_2 \equiv (S^T\mathbb{B}S)^{-1} = B_{22} - B_{21}A_{11}^{-1}A_{12}$, and $Z = (A_{21} - CB_{21})A_{11}^{-1}$.

$C_1$ and $C_2$ are called Schur complements of $\mathbb{A}$ and $\mathbb{B}$, respectively, and thus it is appropriate to call C the Schur complement of $\mathbb{A}\mathbb{B}^{-1}$.

Matrix $\mathbb{B}$ defined by (1") can be used efficiently as the so-called Neumann-Dirichlet preconditioner for $\mathbb{A}$, see [2]. We want to show not only that $\mathbb{A}$ and $\mathbb{B}$ are spectrally equivalent in a *subspace* :

$$a_1 x^T S^T \mathbb{A}\, Sx \le x^T S^T \mathbb{B}\, Sx \le a_2 x^T S^T \mathbb{A}\, Sx, \qquad \forall x \in \Re^n, \quad \forall n$$

but also find the proportionality constants $a_1$, $a_2$.

When we reorder $\mathbb{A}$ and $\mathbb{B}$ from (1) into the block form (2) we obtain:

$$\mathbb{A} = \begin{pmatrix}
T & -I & & & & & & & & & 0 \\
-I & T & -I & & & & & & & & 0 \\
 & \cdot & \cdot & \cdot & & & & & & & \cdot \\
 & & -I & T & -I & & & & & & 0 \\
 & & & -I & T & & & & & & -I \\
 & & & & & T & -I & & & & -I \\
 & & & & & -I & T & -I & & & 0 \\
 & & & & & & \cdot & \cdot & \cdot & & \cdot \\
 & & & & & & & -I & T & -I & 0 \\
 & & & & & & & & -I & T & 0 \\
0 & 0 & 0 & 0 & -I & -I & 0 & 0 & 0 & 0 & T
\end{pmatrix}$$

where matrix $A_{11}$ is $(n^2-n)x(n^2-n)$, $A_{22}=T$ is nxn, and $A_{12}=A_{21}^T$ is $(n^2n)xn$.

Matrix $\mathbb{B}$ differs from $\mathbb{A}$ only in the (21) block as $B_{22} = A_{22} = T$:

$$B_{21} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2I \quad 0 \quad 0 \quad 0 \quad 0).$$

$A_{11}$ is a 2x2 block diagonal matrix with two identical diagonal blocks. As a consequence, so is its inverse. Let us denote $A_{11}^{-1} = \text{diag}(G,G)$, where the mxm matrix G represents the discrete Green's function for $\Delta_n$ on a half of the unit square with Dirichlet boundary conditions. The action of $A_{21}$ on $A_{11}^{-1}A_{12}$ is identical to that of $B_{21}$ on $A_{11}^{-1}A_{12}$ and is equal to $2S^TGS$, where the mxn matrix S is defined by $S^T = (0,\ldots,0,I)^T$. Therefore, $C_1 = C_2 = T - 2S^TGS$, and finally we get $C = C_1C_2^{-1} = I$.

We have thus shown that C, the Schur complement of $AB^{-1}$for the case of the Neumann-Dirichlet preconditioner, is equal to the identity. Consequently, in a subspace, $\mathbb{B}$ is an excellent preconditioner for $\mathbb{A}$.

Additional item. Matrix $Z = (A_{21} - CB_{21})A_{11}^{-1} = (0,\ldots,0,-I,I,0,\ldots,0)\ \text{diag}(G,G) = (X,-X)$, where the nxm matrix $X = -S^TG$. Thus, $\mathbb{A}\mathbb{B}^{-1} = I + SZ^T$.

**4. Many Neumann-Dirichlet strips** In the case of many Neumann-Dirichlet strips the situation is somewhat different: the capacitance C is not equal to the unity matrix any more. Actually, it is not even symmetric. Nevertheless, since $C_1$ and $C_2$ remain symmetric, all eigenvalues of C are real, as the following argument shows.

$$C\phi = C_1C_2^{-1}\phi = \lambda\phi, \quad C_2^{-1/2}C_1C_2^{-1/2}\varphi = \lambda\varphi, \qquad \text{where } C_2^{-1/2}\phi = \varphi.$$

Table 3 demonstrates the dependence of the number of grid points across and along each strip on the condition number of C in the case of four strips. For very narrow strips, $\kappa(C)$ is not independent of the grid size h. In the extreme case of only one inner grid point across the strip, $\kappa(C)$ grows roughly linearly with the number of grids in the other direction, i.e., with the inverse of the grid size, 1/h. On the other hand, when the number of grid points across the strip grows even slightly, $\kappa(C)$ rapidly decays to one.

| | k=1 | k=2 | k=3 | k=4 | k=5 | k=7 | k=9 | k=11 |
|---|---|---|---|---|---|---|---|---|
| m=1 | 1.226 | 1.520 | 1.926 | 2.431 | 3.030 | 4.510 | 6.375 | 8.631 |
| m=2 | 1.056 | 1.171 | 1.353 | 1.590 | 1.874 | 2.570 | - | - |
| m=3 | 1.015 | 1.062 | 1.153 | 1.283 | 1.446 | - | | |
| m=4 | 1.004 | 1.023 | 1.070 | 1.145 | - | | | |

Table 3. Condition number of the capacitance matrix in the case of four strips, d=4. Here, k and m are the number of inner grid points along and across the strip, respectively.

In the simplest example with four strips, when m=k=1, we have

$$C_1 = 1/4 \begin{pmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{pmatrix} \qquad C_2 = 1/4 \begin{pmatrix} 14 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

$$C = C_1 C_2^{-1} = 1/96 \begin{pmatrix} 97 & 7 & 0 \\ 7 & 97 & -48/7 \\ -1 & -7 & 96 \end{pmatrix}$$

with the eigenvalues equal to 1.1130, 1.0000 and 0.9078, which results in $\kappa(C)=1.226$. We note that out of total k(d-1) eigenvalues of C only k of them are exactly equal to 1, the rest are unequally clustered around the value of 1. For example, for k=5 and m=3 we have $\lambda_{max}=1.22$, $\lambda_{min}=0.85$, and the remaining 13 eigenvalues are between 0.97 and 1.03.

| | d=8 | | | | | d=16 | |
|---|---|---|---|---|---|---|---|
| | k=1 | k=2 | k=3 | k=4 | k=5 | k=1 | k=2 |
| m=1 | 1.305 | 1.723 | 2.331 | 3.117 | 4.076 | 1.326 | 1.780 |
| m=2 | 1.074 | 1.229 | 1.483 | - | - | 1.078 | - |
| m=3 | 1.019 | 1.082 | - | - | - | - | - |
| m=4 | 1.005 | 1.031 | - | - | - | - | - |

Table 4. Condition number of the capacitance matrix in the case of d=8 and d=16 strips. Here, k and m are the number of inner grid points along and across the strip, respectively.

In the case of eight strips, see Table 4, the values of $\kappa(C)$ do not change much. A further subdivision to 16 strips produces an increase in $\kappa(C)$ of only a few percent even for the extreme case of only one inner grid point across the strip.

**5. Discontinuous coefficients** Let us now, instead of the Laplace equation consider problems with self-adjoint operators div(k(x)gradu) in $\Omega$ and Dirichlet boundary conditions at $\partial\Omega$. Here, the diffusion function k(x) is discontinuous at the interface across $\Omega$, and constant in each of the subdomains. As before, we impose the Neumann-Dirichlet condition at this interface.

Let us consider one dimensional problems with self-adjoint operators:

$$-\operatorname{div}(k(x)\operatorname{grad}u) = f(x) \quad \text{on } \Omega = [0,1],$$

with Dirichlet boundary conditions at $\partial\Omega$. We shall use a standard uniform *staggered grid* approximation that gives rise to a symmetric matrix representation of A:

$$-k_{i-1/2}u_{i-1} + (k_{i-1/2} + k_{i+1/2})u_i - k_{i+1/2}u_{i+1} = h^2 f_i \qquad \text{for } i=1,\dots,n.$$

For $k_1(x)=1$ in the Dirichlet strip and $k_2(x)=10$ in the Neumann strip this approximation results in the following matrices A and B:

$$A = \begin{pmatrix} 2 & -1 & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 11 & -10 & & \\ & & & -10 & 20 & -10 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & -10 & 20 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & -1 & 2 & -1 & & & \\ & & 0 & 11 & -11 & & \\ & & & -10 & 20 & -10 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & -10 & 20 \end{pmatrix}.$$

The resulting matrix $AB^{-1}$ is better conditioned than in the case $k_1(x)=k_2(x)$ reported in section 2 as the numerical data in Table 5 indicate. Here, $\kappa(AB^{-1})$ is about n/6.

| # points | 5 | 15 | 25 | 35 | 45 |
|---|---|---|---|---|---|
| $\kappa(AB^{-1})$ in 2. | 2.75 | 6.21 | 9.59 | 12.94 | 16.29 |
| $\kappa(AB^{-1})$ in 5. | 2.08 | 3.96 | 5.71 | 7.43 | 9.14 |

Table 5. Condition numbers $\kappa(AB^{-1})$ as a function of the number of grid points.

In a two dimensional analog of our example, we use the following staggered grid scheme:

$$-(k_{i+1/2,j-1/2}+k_{i+1/2,j+1/2})u_{i+1,j}-(k_{i-1/2,j-1/2}+k_{i+1/2,j-1/2})u_{i,j-1}-(k_{i-1/2,j+1/2}+k_{i+1/2,j+1/2})u_{i,j+1}$$

$$-(k_{i-1/2,j-1/2}+k_{i-1/2,j+1/2})u_{i-1,j}+2(k_{i-1/2,j-1/2}+k_{i-1/2,j+1/2}+k_{i+1/2,j-1/2}+k_{i+1/2,j+1/2})u_{i,j}=2h^2 f_{i,j}$$

$$\text{for } i,j=1,\dots,n$$

The matrix $\mathbb{A}$ that arises from this discretization has a form similar to that in section 3.:

$$\mathbb{A} = \begin{pmatrix} T & -I & & & & & & & & & 0 \\ -I & T & -I & & & & & & & & 0 \\ & \cdot & \cdot & \cdot & & & & & & & \cdot \\ & & -I & T & -I & & & & & & 0 \\ & & & -I & T & & & & & & -I \\ & & & & & cT & -cI & & & & -cI \\ & & & & & -cI & cT & -cI & & & 0 \\ & & & & & & \cdot & \cdot & \cdot & & \cdot \\ & & & & & & & -cI & cT & -cI & 0 \\ & & & & & & & & -cI & cT & 0 \\ 0 & 0 & 0 & 0 & -I & -cI & 0 & 0 & 0 & 0 & sT \end{pmatrix},$$

where the constant $c=k_2(x)/k_1(x)$ and $s=(c+1)/2$.

As before, matrix $\mathbb{B}$ differs from $\mathbb{A}$ only in the (21) block :

$$B_{21} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2sI \quad 0 \quad 0 \quad 0 \quad 0).$$

**Theorem** Let -div(k(x)gradu)=f in $\Omega=(0,1)^2$ with Dirichlet boundary conditions at $\partial\Omega$. Let the diffusion function k(x) be discontinuous at the interface across that halves $\Omega$, and constant in each of the subdomains. Let the Neumann-Dirichlet conditions be imposed at this interface. Then the capacitance matrix C is equal to the identity.

**Proof** We have $A_{11}^{-1}$ = diag(G,1/cG), where the mxm matrix G represents, as before, the discrete Green's function for $\Delta_n$ on the half of the unit square with Dirichlet boundary conditions. Here, as before, S is defined by $S^T = (0,\ldots,0,I)^T$. Then

$$A_{11}^{-1} A_{12} = \begin{pmatrix} G & 0 \\ 0 & 1/cG \end{pmatrix} \begin{pmatrix} S \\ cS \end{pmatrix} = \begin{pmatrix} GS \\ GS \end{pmatrix}$$

$$A_{21} A_{11}^{-1} A_{12} = (S^T, cS^T) \begin{pmatrix} GS \\ GS \end{pmatrix} = (c+1)S^T GS.$$

$$B_{21} A_{11}^{-1} A_{12} = (0, (c+1)S^T) \begin{pmatrix} GS \\ GS \end{pmatrix} = (c+1)S^T GS.$$

Thus the action of $A_{21}$ on $A_{11}^{-1}A_{12}$ is identical to that of $B_{21}$ on $A_{11}^{-1}A_{12}$ and is equal to $(c+1)S^TGS$. Therefore, $C_1 = C_2 = sT - 2S^TGS$, and finally we get $C = C_1C_2^{-1} = I$. Thus, C is equal to the identity, even in the case of discontinuous coefficients.

In the case of many Neumann-Dirichlet strips the results are anti-intuitive: the conditioning of the capacitance matrix C is significantly better for problems with discontinuous coefficients than with continuous ones, as the following table indicates.

| | k=1 | k=2 | k=3 | k=4 | k=5 | k=7 | k=9 | k=11 |
|---|---|---|---|---|---|---|---|---|
| m=1 | 1.038 | 1.081 | 1.134 | 1.194 | 1.261 | 1.420 | 1.612 | 1.842 |
| m=2 | 1.010 | 1.029 | 1.057 | 1.090 | 1.127 | 1.210 | - | - |
| m=3 | 1.003 | 1.011 | 1.026 | 1.047 | 1.071 | - | | |
| m=4 | 1.001 | 1.004 | 1.012 | 1.025 | - | | | |

Table 6. Condition number of the capacitance matrix for the problem with discontinuous coefficients ($c=k_2/k_1=10$) in the case of four strips, d=4. Here, k is the number of inner grid points along the strip, and m is the number of grid points across the strip.

To gain some insight, let us examine the simplest example with four strips, when m=k=1.

$$B = \left(\begin{array}{cccc|ccc} 4 & & & & -1 & & \\ & 40 & & & -10 & -10 & \\ & & 4 & & & -1 & \\ & & & 40 & & & -10 \\ \hline & -11 & & & 22 & & \\ & -11 & & & & 22 & \\ & & & -11 & & & \cdot 22 \end{array}\right)$$

while A is symmetric ($A_{21}=A_{12}^T$). In accordance with (3), we obtain

$$C_1 = 1/4\begin{pmatrix} 77 & -10 & 0 \\ -10 & 77 & -1 \\ 0 & -1 & 77 \end{pmatrix}, \qquad C_2 = 1/4\begin{pmatrix} 77 & -11 & 0 \\ -11 & 77 & 0 \\ 0 & 0 & 77 \end{pmatrix},$$

$$C = C_1 C_2^{-1} = 1/528\begin{pmatrix} 529 & 7 & 0 \\ 7 & 529 & -48/7 \\ -1 & -7 & 528 \end{pmatrix}$$

with the $\lambda_i(C)$ equal to 1.0205, 1.0000 and 0.9832, which results in $\kappa(C)=1.038$. Note that matrix $C_2$ is the same as the one for continuous coefficients, scaled by $(c+1)/2=11/2$. Increasing the number of strips above 4 does not change much the conditioning of C:

| | k=1 | k=2 | k=3 | k=4 | k=5 | k=1 | k=2 |
|---|---|---|---|---|---|---|---|
| m=1 | 1.050 | 1.108 | 1.182 | 1.271 | 1.374 | 1.053 | 1.115 |
| m=2 | 1.013 | 1.038 | 1.076 | - | - | - | - |

Table 7. Condition number of the capacitance matrix for the problem with discontinuous coefficients ($c=k_2/k_1=10$) in the case of d=8 strips. Here, k and m are the number of inner grid points along tand across the strip, respectively.

Let us now investigate the influence of the ratio of the diffusion coefficients on the condition number of matrix C.

| $c=k_2/k_1$ | 0.001 | 0.01 | 0.1 | 0.3 | 1.0 | 3. | 10. | 100. | 1000. |
|---|---|---|---|---|---|---|---|---|---|
| κ(C) (m=k=1) | 1.502 | 1.497 | 1.448 | 1.368 | 1.226 | 1.107 | 1.038 | 1.004 | 1.0004 |
| κ(C) (m=k=3) | 1.329 | 1.326 | 1.295 | 1.245 | 1.153 | 1.074 | 1.026 | 1.003 | 1.0003 |

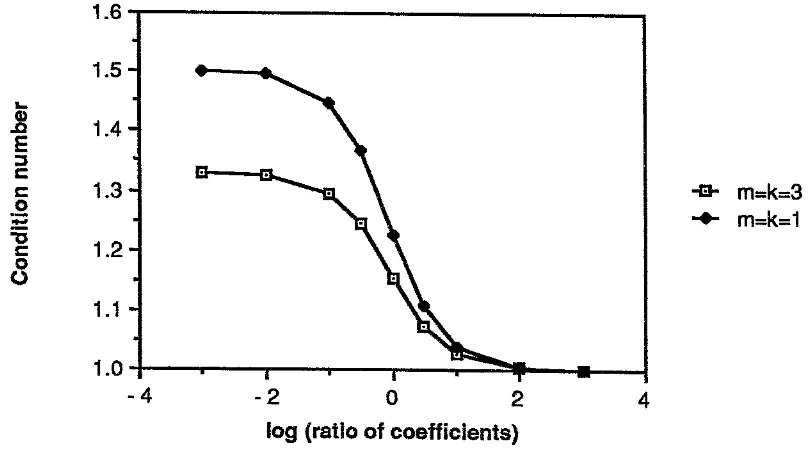
Table 8. Condition number of the capacitance matrix for the problem with discontinuous coefficients $k_2$ and $k_1$ in the case of d=4 strips. Here, k and m are the number of inner grid points along tand across the strip, respectively.

Results in Table 8 strongly indicate that as c, the ratio of the coefficients, grows, the condition number approaches the value of 1.0, and κ(C) approaches a somewhat larger value as c decreases, see Fig.1. Coefficient $k_1$ corresponds here to the Dirichlet and $k_2$ to the Neumann strips. Let us investigate the problem for the simplest case when k=m=1. Using the formulas in (3) we arrive at the following representation

$$C_1 = \frac{1}{4}\begin{pmatrix} 7(c+1) & -c & 0 \\ -c & 7(c+1) & -1 \\ 0 & -1 & 7(c+1) \end{pmatrix}, \qquad C_2 = \frac{1}{4}\begin{pmatrix} 7(c+1) & -(c+1) & 0 \\ -(c+1) & 7(c+1) & 0 \\ 0 & -1 & 7(c+1) \end{pmatrix},$$

$$C = C_1 C_2^{-1} = I + 1/48(c+1)R, \qquad \text{where } R = \begin{pmatrix} 1 & 7 & 0 \\ 7 & 1 & -48/7 \\ -1 & -7 & 0 \end{pmatrix}$$

Fig.1



Thus the eigenvalues of C are $\lambda(C) = 1 + 1/48(c+1)\ \lambda(R)$. It is easy to verify that the eigenvalues of R are 0 (R is singular), and $1 \pm \sqrt{97}$. Consequently, the condition number of C is $\kappa(C) = (48(c+1) + 1 + \sqrt{97}) / (48(c+1) + 1 - \sqrt{97})$. This formula gives us $\kappa(C)=1.226$ for c=1, and $\kappa(C)=1.503$ for c=0 (the limiting case).

Examining $C_1$ and $C_2$ we see that for large values of c these matrices are "similar", in contrast to the situation when c are small (in the limit, for c=0, the zeros appear in the "wrong" positions). This strongly suggests that *in the preconditioner $\mathbb{B}$ the Neumann strips should correspond to the regions with larger diffusion coefficients.*

Finally, we investigate the influence of the number of inner grid points along the strip on the condition number when the strips are *wrongly* placed, i.e., c<1.

| k | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|---|---|
| κ(C) | 1.448 | 2.137 | 3.273 | 4.950 | 7.278 | 14.33 | 25.21 | 40.44 |

Table 9. Condition number of the capacitance matrix for the problem with discontinuous coefficients ($c=k_2/k_1=0.1$) in the case of d=4 strips and m=1. Here, the k and m are the number of inner grid points along tand across the strip, respectively.

Clearly, for c<1, in this worst case of extremely thin strips (m=1), $\kappa(C)$ grows almost quadratically with k, the number of inner grid points along the strip.

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