CHAPTER 32

Two Domain Decomposition Techniques for Stokes Problems*

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Abstract. I will develop two domain decomposition techniques for Stokes problems in this talk. The first uses a reformulation of the saddle point system developed in [4] and reduces the derivation of domain decomposition algorithms for Stokes to the definition of domain decomposition preconditioners for second order problems. The second applies domain decomposition directly to Stokes and gives rise to a saddle point system for the velocity nodes on the subdomain boundaries and the mean values of the pressure on the subdomains. This system is solved iteratively.

1. Introduction. In this talk, I will discuss two domain decomposition techniques for the iterative solution of the discrete systems which arise in finite element approximation to Stokes problems. Specifically, we consider the velocity-pressure formulation of the Stokes equations where the divergence constraint is treated by a Lagrange multiplier technique and the pressure variable corresponds to the multiplier. The discrete systems which arise are of the saddle point type.

In Section 2, we review some properties of saddle point systems and discuss a reformulation of the saddle point system developed in [4]. This reformulation provides a framework for the development of iterative methods for saddle point problems. The rate of convergence of the resulting iterative methods can be estimated in terms of a corresponding 'inf-sup' condition. Moreover, preconditioning can be incorporated into the scheme.

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Section 3 defines the model Stokes problem and gives the corresponding weak formulation. The finite element approximation is then defined in terms of the weak formulation.

Section 4 defines the first domain decomposition technique for Stokes. Using the reformulation of Section 2, the task of developing rapidly convergent algorithms for the full saddle point problem is reduced to the development of effective preconditioners for second order problems. Domain decomposition algorithms for Stokes result from the use of standard domain decomposition preconditioners developed earlier in, for example, [5,6,7,8,9].

In Section 5, we develop iterative algorithms for Stokes problems by directly applying domain decomposition to the discrete Stokes systems. We develop iterative algorithms for the solution of the original Stokes system which require the solution of discrete Stokes problems on subdomains at each iterative step. The work in [11] provided insight for the development of this technique.

To present the ideas most clearly, I will only consider the simplest applications and approximation techniques. Many generalizations are possible and will be addressed elsewhere.

2. Iterative methods for saddle-point systems. We consider two techniques used to develop iterative methods for saddle point systems in this section. The first technique is well known and the second was developed in [4]. We include this discussion for completeness and continuity of exposition since the techniques will be used extensively in later sections of this paper.

Let H^1 and H^2 be Hilbert spaces and consider the problem

$$M\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.1}$$

where $X, f \in H^1$ and $Y, g \in H^2$. We study operators M of the form,

$$M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}. \tag{2.2}$$

We assume that A is a positive definite, symmetric operator on H^1 with a bounded inverse and that B and B^* are adjoints with respect to the inner products in H^1 and H^2. We further assume that $A^{-1}B$ and $B^*A^{-1}B$ are bounded. We shall use the notation $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the inner products and norms on H^1 and H^2.

Saddle point problems of the form (2.1) arise in many applications. For example, such systems must be solved for finite element Lagrange multiplier approximations to Dirichlet and interface problems [2,3], velocity-pressure formulations of the equations of Stokes and elasticity [10], and mixed finite element methods [15].

Applying block Gaussian elimination to (2.1) implies that the solution of (2.1) satisfies

$$\begin{pmatrix} A & B \\ 0 & B^*A^{-1}B \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} f \\ B^*A^{-1}f - g \end{pmatrix}. \tag{2.3}$$

Thus, (2.1) is solvable if and only if $B^*A^{-1}B$ is invertible (A, B and B^* need not be bounded). But $B^*A^{-1}B$ is symmetric and non-negative. Hence, $B^*A^{-1}B$ is solvable if and only if it is definite. A straightforward computation gives

$$(B^*A^{-1}Bu, u) = \sup_{\theta \in H^1} \frac{(Bu, \theta)^2}{(A\theta, \theta)} \tag{2.4}$$

and hence solvability of (2.1) will follow if we can verify

$$\sup_{\theta \in H^1} \frac{(Bu, \theta)^2}{(A\theta, \theta)} \geq c_0 \|u\|^2 \tag{2.5}$$

holds for some positive constant c_0. Inequality (2.5) is equivalent to the classical L-B-B (Ladyzhenskaya-Babuška-Brezzi) condition. In addition to being a sufficient condition for the solvability of (2.1), the constant c_0 in (2.5) will be an ingredient in determining convergence rates for the iterative methods to be subsequently discussed.

By (2.3), we see that the solution of (2.1) can be computed by by first solving

$$B^*A^{-1}BY = B^*A^{-1}f - g \tag{2.6}$$

and then back solving (2.3) for X, i.e. $X = A^{-1}(f - BY)$. For our applications, $B^*A^{-1}B$ is a full matrix and expensive to compute. One alternative is to iteratively solve (2.6), e.g. apply conjugate gradient iteration. The rate of convergence for this iteration is related to the condition number K of $B^*A^{-1}B$. From the above discussion, we clearly have that $K \leq c_1/c_0$ where c_0 satisfies (2.5) and c_1 satisfies the reverse inequality,

$$\sup_{\theta \in H^1} \frac{(Bu, \theta)^2}{(A\theta, \theta)} \leq c_1 \|u\|^2 . \tag{2.7}$$

One gets a rapidly convergent algorithm for the computation of Y if the condition number K is not too large. This is the first iterative technique for solving (2.1) to be considered.

One problem with the iterative technique just developed is that it requires the evaluation of the action of A^{-1} at each step in the iteration. In many applications, the action of A^{-1} is more expensive to compute than that of a suitable preconditioner. The next technique was developed and analysed in [4] and leads to a rapidly convergent algorithm for solving (2.1) which utilizes the preconditioner for A^{-1} without requiring the computation of the action of A^{-1}.

Let A_0 be a good preconditioner for A^{-1}. This means that the evaluation of the action of A_0^{-1} is much more economical than that of A^{-1} and that A_0 satisfies inequalities of the form

$$\alpha_0(Av, v) \leq (A_0 v, v) \leq \alpha_1(Av, v) \qquad \text{for all } v \in H^1$$

with α_1/α_0 not too large. By scaling A^{-1}, we may assume that $\alpha_1 < 1$. Again, applying block matrix manipulations to (2.1) gives

$$\begin{pmatrix} A_0^{-1}A & A_0^{-1}B \\ B^*A_0^{-1}(A-A_0) & B^*A_0^{-1}B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A_0^{-1}f \\ B^*A_0^{-1}f - g \end{pmatrix} \tag{2.8}$$

which we rewrite

$$\tilde{M} \begin{pmatrix} X \\ Y \end{pmatrix} = \tilde{f} \tag{2.9}$$

with the obvious definitions of $\tilde{M}$ and $\tilde{f}$. It is straightforward to see that $\tilde{M}$ is a symmetric operator in the inner product

$$\left[\begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} W \\ X \end{pmatrix} \right] = ((A - A_0)U, W) + (V, X). \tag{2.10}$$

Moreover, it was shown in [4] that $\tilde{M}$ is positive definite in this inner product and is well conditioned provided that c_1/c_0 and α_1/α_0 are not too large. The second iterative technique for solving (2.1) applies conjugate gradient in the $[\cdot,\cdot]$ inner product to the reformulated (well conditioned) problem (2.8).

3. The model Stokes problem. In this section, we describe the model Stokes problem and its finite element discretization. Let Ω be a domain in N dimensional Euclidean space for $N = 2$ or $N = 3$. The velocity-pressure formulation of the steady-state Stokes problem is: Find $\mathbf{u}$ and P satisfying

$$\begin{aligned} -\Delta \mathbf{u} - \nabla P &= \mathbf{F} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= 0 \text{ on } \partial\Omega, \\ \int_\Omega P &= 0. \end{aligned} \tag{3.1}$$

Here, $\mathbf{u}$ is a vector valued function defined on Ω and P is a scale valued function defined on Ω. The first equation is, of course, a vector equality at each $x \in \Omega$ and Δ denotes the componentwise Laplace operator.

We restrict ourselves to the model problem (3.1) for simplicity. Applications to problems with variable coefficients and the equations of linear elasticity are similar.

We consider a weak formulation of problem (3.1). Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product and $\|\cdot\|$ the denote the corresponding norm applied either to scalar or vector functions. Let $H_0^1(\Omega)$ be the Sobolev space of functions defined on Ω which vanish (in an appropriate sense) on $\partial\Omega$ and which along with their first derivatives are square integrable on Ω. Define $\mathbf{H} \equiv H_0^1(\Omega) \times H_0^1(\Omega)$ and let $\|\cdot\|_1$ denote the corresponding norm. Let $\Pi = L^2(\Omega)$ and $\Pi/1$ denote the functions in Π with zero mean value on Ω. Multiplying (3.1) by functions in $\mathbf{H}$ and Π and integrating by parts when appropriate, it is easy to see that the solution $(\mathbf{u}, P)$ satisfies

$$\begin{aligned} D(\mathbf{u},\mathbf{v}) + (P, \nabla\cdot\mathbf{v}) &= (\mathbf{F},\mathbf{v}) \qquad \text{for all } \mathbf{v} \in \mathbf{H}, \\ (\nabla\cdot\mathbf{u}, q) &= 0 \qquad \text{for all } q \in \Pi/1. \end{aligned} \tag{3.2}$$

Here, D is the Dirichlet form defined by

$$D(\mathbf{w},\mathbf{v}) \equiv \sum_{i=1}^{N} \int_\Omega \nabla w_i \cdot \nabla v_i \, dx.$$

Clearly, (3.2) is of the form of (2.1). The corresponding operator A is unbounded but has a bounded inverse. Moreover, it is well known that the corresponding inf-sup condition:

$$\sup_{\theta\in\mathbf{H}} \frac{(p,\nabla\cdot\theta)^2}{A(\theta,\theta)} \ge C_0 \|p\|^2 \qquad \text{for all } p \in \Pi/1 \tag{3.3}$$

holds for some positive constant C_0. It then follows that there is a unique solution $(\mathbf{u},P)$ in $\mathbf{H}\times\Pi/1$ to (3.2).

To approximately solve (3.2), we introduce a collection of pairs of approximation subspaces $\mathbf{H}_h \subset \mathbf{H}$ and $\Pi_h \subset \Pi$ indexed by h in the interval $0 < h < 1$. We will assume that the inf-sup condition holds for the pair of spaces; i.e. we assume that there is a constant c_0 which does not depend upon h such that

$$\sup_{\theta\in\mathbf{H}_h} \frac{(p,\nabla\cdot\theta)^2}{A(\theta,\theta)} \ge c_0 \|p\|^2 \qquad \text{for all } p \in \Pi_h/1. \tag{3.4}$$

Many subspace pairs satisfying (3.4) have been studied and their approximation properties are well known [10,14,16].

The approximations to the functions $(\mathbf{u},P)$ are defined by replacing the spaces in (3.2) by their discrete counterparts. Specifically, the approximations are defined as the functions $\mathbf{u}_h \in \mathbf{H}_h$ and $P_h \in \Pi_h/1$ satisfying

$$\begin{aligned} D(\mathbf{u}_h,\mathbf{v}) + (P_h,\nabla\cdot\mathbf{v}) &= (\mathbf{F},\mathbf{v}) \qquad \text{for all } \mathbf{v}\in\mathbf{H}_h,\\ (\nabla\cdot\mathbf{u}_h,q) &= 0 \qquad \text{for all } q\in\Pi_h/1. \end{aligned} \tag{3.5}$$

Existence and uniqueness for the solution of (3.5) follows from (3.4) and the discussion in Section 2.

We conclude this section with an example of a pair of approximation subspaces. For simplicity of exposition, we shall only describe these spaces when Ω is the unit square. Generalizations to certain more complex domains are possible.

Let $n > 0$ be given. We start by breaking the square into $2n \times 2n$ subsquares and define $h = 1/2n$ (see Figure 3.1). Let $x_i \equiv ih$ and $y_j = jh$ for $i,j = 1,\dots,2n$. We partition the subsquares into pairs of triangles using one of the subsquares diagonals (for example, the diagonal going from the bottom right corner to the upper left corner of the subsquare). Let H_h be the collection of functions which vanish on the boundary of the square and are piecewise linear and continuous on this triangulation. The subspace $\mathbf{H}_h$ is defined to be $H_h \times H_h$.

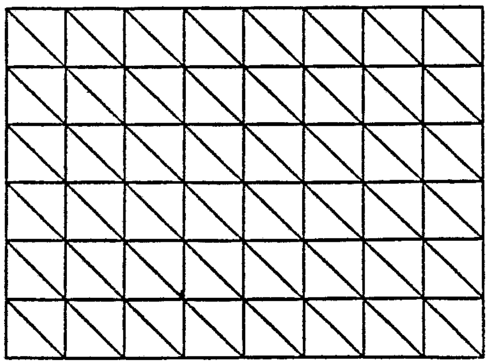

Figure 3.1: *The triangular mesh.*

To define the space Π_h, we first consider the space $\tilde{\Pi}_h$ which is defined to be the space of functions which are piecewise constant on the subsquares (see Figure 3.2). It is interesting to note [12] that the subspace pair $\{\mathbf{H}_h, \tilde{\Pi}_h/1\}$ is not stable in L^2, i.e. the inf-sup condition fails to hold for the subspace pair. To get a stable pair, we shall consider a somewhat smaller subspace of $\tilde{\Pi}_h$. Let θ_{kl} for $k, l = 1, \ldots, 2n$ be the function which is one on the subsquare $[x_{k-1}, x_k] \times [y_{l-1}, y_l]$ and vanishes elsewhere. We define the functions $\phi_{i,j} \in \tilde{\Pi}_h$, for $i, j = 1, \ldots, n$, by (see also, Figure 3.2)

$$\phi_{ij} \equiv \theta_{2i-1,2j-1} - \theta_{2i,2j-1} - \theta_{2i-1,2j} + \theta_{2i,2j}. \tag{3.6}$$

We then define Π_h by

$$\Pi_h \equiv \{Q \in \tilde{\Pi}_h | (Q, \phi_{ij}) = 0 \text{ for } i, j = 1, \ldots, n\}.$$

An estimate of the form of (3.4) holds with c_0 independent of h for the subspace pair $\{\mathbf{H}_h, \Pi_h\}$ [12]. Furthermore, the exclusion of the functions of the form (3.6) does not result in a change in the order of approximation for the space (we obviously still have the subspace of constants on the mesh of size $2h$).

REMARK: The exclusion of functions of the form (3.6) poses no difficulty in practice. In fact, it only affects the definition of the corresponding B^* in a trivial way. By definition, $B^*\mathrm{v} \equiv Q$ where $Q \in \Pi_h$ solves

$$(Q, R) = (\nabla \cdot \mathrm{v}, R) \qquad \text{for all } R \in \Pi_h/1.$$

It is easy to see that Q is the L^2 orthogonal projection (into $\Pi_h/1$) of the function $\tilde{Q} \in \tilde{\Pi}_h$ satisfying

$$(\tilde{Q}, R) = (\nabla \cdot \mathrm{v}, R) \qquad \text{for all } R \in \tilde{\Pi}_h. \tag{3.7}$$

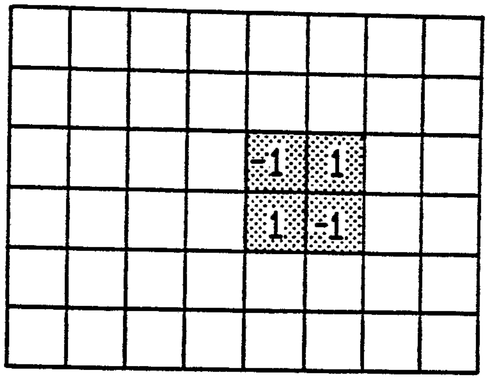

Figure 3.2. *The rectangular mesh used for* $\tilde{\Pi}_h$*; the support (shaded) and values for a typical* ϕ_{ij}.

This projection is a trivial local operation since the supports of the functions $\{\phi_{ij}\}$ are disjoint. Furthermore, the computation of $\tilde{Q}$ is straightforward since the gram matrix for (3.7) is diagonal (with the obvious choice of basis).

4. The first domain decomposition technique for Stokes. We consider the direct application of the second iterative technique of Section 2 to the saddle point problem corresponding to the Stokes discretization. As we shall see, all that is required is effective preconditioners for second order problems. Thus, domain decomposition algorithms for Stokes result from standard domain decomposition preconditioners for second order problems.

Let us introduce some operator notation. Let $A : \mathbf{H}_h \mapsto \mathbf{H}_h$ be defined by

$$(Av, w) = D(v, w) \qquad \text{for all } w \in \mathbf{H}_h. \tag{4.1}$$

Clearly, (4.1) defines a symmetric positive definite operator on $\mathbf{H}_h$. We define $B : \Pi_h/1 \mapsto \mathbf{H}_h$ by

$$(Bp, w) = (p, \nabla \cdot w) \qquad \text{for all } w \in \mathbf{H}_h.$$

Its adjoint, $B^* : \mathbf{H}_h \mapsto \Pi_h/1$ is then defined by

$$(B^* w, q) = (\nabla \cdot w, q) \qquad \text{for all } q \in \Pi_h/1,$$

and is nothing more than the divergence followed by L^2 projection into $\Pi_h/1$. The discrete solution pair (u_h, P_h) satisfies (2.1).

The operator A involves two componentwise operators corresponding to the standard discrete Dirichlet operator on H_h. Consequently, it can be preconditioned componentwise by domain decomposition preconditioners for the Dirichlet problem. Results concerning the development of domain decomposition preconditioners for the general second order problems have been given in [5,6,7,8,9]. One example, described in [6], develops a domain decomposition preconditioner for the second order problem in R^2 and gives rise to a problem which, even though not well conditioned, has a condition number growth bounded by $c(1+\ln^2(d/h))$. Here, d is roughly the size of the subdomains. In this case, the condition number of (2.8) also will grow like $c(1+\ln^2(d/h))$.

5. A direct domain decomposition approach. In this section, we shall directly apply domain decomposition to the Stokes problem. We shall develop algorithms for solving the discrete system (3.5), which only require the solution of smaller discrete Stokes systems on the subdomains and some type of reduced system. In this case, the reduced system will involve the values of $\mathbf{u}_h$ on the boundary of the subdomains and the mean value of the pressure on the subdomains.

We assume that $\bar{\Omega}$ has been partitioned into a number of subdomains $\bar{\Omega}=\cup_{i=1}^m \bar{\Omega}_i$. We require that the boundary of the subdomains ($\Gamma \equiv \cup_{i=1}^m \partial\Omega_i$) align with the mesh in $\mathbf{H}_h$ and Π_h. We then define

$$\mathbf{H}_h^i = \{\phi \in \mathbf{H}_h | \text{support}(\phi) \subset \Omega_i\} \tag{5.1}$$

and

$$\Pi_h^i = \{\phi \in \Pi_h | \text{support}(\phi) \subset \Omega_i\}. \tag{5.2}$$

We shall assume that the inf-sup condition holds for each subspace pair, i.e.

$$\sup_{\theta \in \mathbf{H}_h^i} \frac{(q, \nabla\cdot\theta)^2}{A(\theta,\theta)} \geq c_0 \|q\|_{\Omega_i}^2 \qquad \text{for all } q \in \Pi_h^i/1 \tag{5.3}$$

and that the function which is one on Ω_i and vanishes in the remainder of Ω is an element in Π_h. Note that, since the functions in $\mathbf{H}_h$ are continuous, the subspace pair $(\mathbf{H}_h^i, \Pi_h^i)$ can be used to approximate the Stokes problem with zero boundary conditions on the subdomains.

Because of (5.3), local Stokes problems on the subdomains are solvable. The first step is to solve these local problems and reduce the problem to one which implicitly involves fewer degrees of freedom. To do this, we let $(\mathbf{v}_h^i, Q_h^i)$ be the solution of

$$\begin{aligned} A(\mathbf{v}_h^i, \mathbf{w}) + (Q_h^i, \nabla\cdot\mathbf{w}) &= (\mathbf{F},\mathbf{w}) \qquad \text{for all } \mathbf{w} \in \mathbf{H}_h^i, \\ (\nabla\cdot\mathbf{v}_h^i, q) &= 0 \qquad \text{for all } q \in \Pi_h^i/1. \end{aligned} \tag{5.4}$$

We set $\mathbf{v}_h = \sum \mathbf{v}_h^i$, $Q_h = \sum Q_h^i$ and define $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$ and $R_h = P_h - Q_h$. Then, $\mathbf{w}_h$ and R_h satisfy

$$\begin{aligned} A(\mathbf{w}_h, \mathbf{v}) + (R_h, \nabla\cdot\mathbf{v}) &= F(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in \mathbf{H}_h, \\ (\nabla\cdot\mathbf{w}_h, q) &= G(q) \qquad \text{for all } q \in \Pi_h/1. \end{aligned} \tag{5.5}$$

The functionals F and G vanish for functions in $\mathbf{H}_h^i$ and $\Pi_h^i/1$ respectively. Thus, the functions $\mathbf{w}_h$ and R_h lie in a subspace of $\mathbf{H}_h \times \Pi_h/1$ with significantly lower dimension. We shall parameterize this subspace and then derive equations for the parameters corresponding to the solution $\mathbf{w}_h$ and R_h.

We shall parameterize the solution $(\mathbf{w}_h, R_h)$ in terms of parameters $\sigma \in \mathbf{H}(\Gamma)$ and $\lambda \in \Pi_0$ where

$$\mathbf{H}(\Gamma) \equiv \{\phi|_\Gamma, \phi \in \mathbf{H}_h\}$$

and

$$\Pi_0 \equiv \{\phi \in \Pi_h/1 \text{ such that } \phi \text{ is constant on } \Omega_i \text{ for each } i\}.$$

To do this, we define the operators $S : \mathbf{H}(\Gamma) \mapsto \Pi_h$ and $T : \mathbf{H}(\Gamma) \mapsto \mathbf{H}_h$ satisfying the following:

(1) $S(\gamma)|_{\Omega_i} \in \Pi_h^i/1$,
(2) $T(\gamma)|_\Gamma = \gamma$,
(3) $D(T(\gamma), \phi) + (S(\gamma), \nabla \cdot \phi) = 0 \quad$ for all $\phi \in \mathbf{H}_h^i$,
(4) $(\nabla \cdot T(\gamma), q) = 0 \quad$ for all $q \in \Pi_h^i/1$.

It is not difficult to show that the above conditions uniquely define S and T. Moreover, if $\sigma = \mathbf{w}_h|_\Gamma$ and $\lambda \in \Pi_0$ is the function which has the same mean values on the subdomains as R_h then

$$\mathbf{w}_h = T(\sigma) \text{ and } R_h = S(\sigma) + \lambda. \tag{5.6}$$

Thus, (5.6) gives a parameterization of $\mathbf{w}_h$ and R_h in terms of the parameters (σ, λ) in $\mathbf{H}(\Gamma) \times \Pi_0$. Note that given a value of γ, the evaluation of $S(\gamma)$ and $T(\gamma)$ essentially only involves the solution of discrete Stokes problems on the subdomains.

We next give equations for the determination of σ and λ satisfying (5.6). To do this, we define a quadratic form $E : (\mathbf{H}(\Gamma) \times \Pi_0)^2 \mapsto R^1$ given by

$$\begin{aligned} E((\gamma_1, \delta_1), (\gamma_2, \delta_2)) = D(T(\gamma_1), T(\gamma_2)) &+ (\delta_1, \nabla \cdot T(\gamma_2)) \\ &+ (\nabla \cdot T(\gamma_1), \delta_2). \end{aligned} \tag{5.7}$$

It is not difficult to see that, given local bases, $\{\phi_i\}$ for $\mathbf{H}(\Gamma)$ and $\{\psi_i\}$ for Π_0, we can compute the data $\tilde{F}$ satisfying

$$E((\sigma, \lambda), (\phi_i, \psi_j)) = \tilde{F}(\phi_i, \psi_j) \tag{5.8}$$

from the functionals F and G in (5.5). This can be carried out only using a few local operations per basis function, without knowing σ, λ and without computing $T(\phi_i)$.

From the definition of E, it is clear that (5.8) gives rise to a symmetric indefinite system of the form (2.1) which can be used to compute σ, λ satisfying (5.6). The form $D(T(\gamma_1), T(\gamma_2))$ corresponds to the operator A in (2.1). The form $(\delta_1, \nabla \cdot T(\gamma_2))$ corresponds to B, etc. Stability properties for the above system are given in the following theorem which will be proven in a subsequent paper under reasonable assumptions on the domain subdivision.

THEOREM. *There are positive constants $\alpha_0, \alpha_1, c_0, c_1$, independent of d and h, such that*

$$(5.9) \qquad \alpha_0 D(T(\gamma), T(\gamma)) \leq \sum_{i=1}^{m} |\gamma|^2_{1/2,\partial\Omega_i} \leq \alpha_1 D(T(\gamma), T(\gamma)),$$

and

$$(5.10) \qquad c_0 \|\delta\|^2 \leq \sup_{\gamma \in \mathbf{H}(\Gamma)} \frac{(\delta, \nabla \cdot T(\gamma))^2}{D(T(\gamma), T(\gamma))} \leq c_1 \|\delta\|^2.$$

where $|\cdot|_{1/2,\partial\Omega_i}$ *denotes the Sobolev semi-norm of order 1/2 on* $\partial\Omega_i$ *(c.f. [13]).*

Inequalities (5.10) imply that the 'inf-sup' condition corresponding to form E on the subspace pair $(\mathbf{H}(\Gamma), \Pi_0)$ is well conditioned independently of h. The boundary form $D(T(\gamma), T(\gamma))$ is not well conditioned but is equivalent to a sum of seminorms on the boundaries of the subdomains. The corresponding form,

$$\ll \gamma_1, \gamma_2 \gg_{1/2} \equiv \sum_{i=1}^{m} < \gamma_1, \gamma_2 >_{1/2,\partial\Omega_i}$$

has been well studied in the development of domain decomposition preconditioners for second order problems. In fact, each domain decomposition technique developed in [1,5,6,7,8,9] gives rise to a computationally effective domain decomposition preconditioner for $\ll \cdot, \cdot \gg_{1/2}$. Thus, we can solve (5.8) by using the second iterative technique of Section 2, with preconditioner A_0 corresponding componentwise to the boundary part of a second order method developed in [1,5,6,7,8,9]. For example, we can use the technique presented in [6]. This means the preconditioner for the boundary velocities will involve inverting the $l_0^{1/2}$ operator on the edge segments and the solution of a coarse grid problem with the number of unknowns equal to the number of 'cross-points' in the subdomain subdivision. The resulting symmetric positive definite reformulation of (5.8) will have a condition number bounded by $C(1 + \ln^2(d/h))$.

It is possible to implement the above technique in such a way that each Stokes subdomain problem need be solved only once per step in the iterative algorithm for the solution of σ, λ. Once these parameters are solved to satisfactory accuracy, $\mathbf{w}_h$ and R^h can be computed with one more set of subdomain solves.

6. Conclusion. We have provided two domain decomposition techniques for solving Stokes problems. The technique of Section 4 applies an algebraic reformulation to the discrete Stokes equations and consequently can utilizes any available domain decomposition preconditioner for the second order problem. The technique of Section 5 applies domain decomposition directly to the discrete Stokes equations.

Although the technique of Section 5 is theoretically interesting, we feel that the one described in Section 4 will probably lead to the most flexible and computationally effective algorithms. The first reason for this is that, with the second order preconditioners, one has a much greater flexibility in the form of the subproblems which are to be solved. In contrast, since the method of Section 5 is a dimension reduction technique, the given Stokes problem must be solved on the subdomains. In addition, there are a large number of techniques available for developing 'fast-solvers' for second order problems while there are few (if any) fast Stokes-solvers available.

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