CHAPTER 20

Wavefront Elimination and Renormalization*

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Abstract. Recently, a new class of optimal fast solver for the model problem –wavefront elimination ($\mathcal{WE}$)– has been developed using template operators. The complexity of this type of new algorithms is only one fraction of the cost of the multigrid method applied to the same problem. These algorithms have potential for an efficient parallel implementation and also as preconditioning operators for general elliptic problems.

A more interesting fact is that this algorithm shows some conceptual connection with two new theories – renormalization in physics and fractals in mathematics. In this paper, we will demonstrate these rather interesting relations between the wavefront elimination, the renormalization theory and fractals.

Key Words. Wavefront elimination ($\mathcal{WE}$), renormalization, fractal, template operator ($\mathcal{TO}$), fast solver.

1. Introduction. The study of fast solvers is always an important part of research in domain decomposition [5]. Using template operator ($\mathcal{TO}$) - a new structure of the linear operator in finite dimensional space [6] - some optimal fast solvers are presented here. This result has answered an open question, namely, can a direct approach for solving the model problem achieve an optimal complexity? In particular, the complexity of our new algorithm, called wavefront elimination ($\mathcal{WE}$), is even better than the complexity of the multi-grid method for solving the same problem. Different from a traditional approach, there is more than one discrete $\mathcal{TO}$ which is used on the same grid point. The combination of these different $\mathcal{TO}$'s makes the sparsity of the consequent $\mathcal{TO}$'s during the $\mathcal{WE}$ process on the same grid point possible. A more interesting fact is that this new algorithm exhibits some amusing conceptual relationships with two new important theories - renormalization in physics and fractals in mathematics. Renormalization theory was discovered by K.G. Wilson at Cornell[7]. The basic idea of this theory can be understood as a successive thinning out of the degree of freedom in the partition function. The N-particle problem is transformed into an N'-particle

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problem with $N' < N$, whereby the temperature T and the magnetic field H may also have to be renormalized. The wavefront elimination process can be viewed as a renormalization process from different scales. During the elimination process, the template operators display a sequence of self-similar structure on the different scales. In the backward solution process of the $\mathcal{W}\mathcal{E}$, we see a procedure surprisingly similar to the recursively detailing a fractal. These interesting facts contribute another convincing example of a fundamental principle which organizes a whole universe. As we know this principle has been fascinatingly demonstrated by fractal geometry and renormalization theory. As we will see in this paper this principle has also provided us with guidance for designing a whole new class of algorithm, for the numerical solution of P.D.E's.

2. Template Operators. In [6] the template operator was first introduced for identifying the problems for which the Schwarz Splittings are most suitable. In the same paper it was also successfully applied to obtain a "good" splitting when the Schwarz approach is used. The unique features of this structure are as follows. First, in a template operator the artificial sequential constraints in the matrix structure are removed. The original topological frame of the continuous problem from which the discrete operator is derived is well preserved. Second, in particular, the locality of the operator and the proximity of the variables are also maintained in this new structure. Therefore, many physical phenomena which are related to the topology of the solution region can be easily presented in this form. That is the key to the successful application of $\mathcal{TO}$ in the study of Schwarz Splittings. A more important feature is that the $\mathcal{TO}$ provides us a "graphical structure" to think in pictures.

> *In graphical representation, natural processes can be comprehended in their full complexity by intuition. New ideas and associations are stimulated, and the creative potential of all those who think in pictures is awakened.*
>
> —The Beauty of Fractals

In this paper, we will show how the $\mathcal{TO}$ can provide us a "graphical" structure for designing a new class of fast solver.

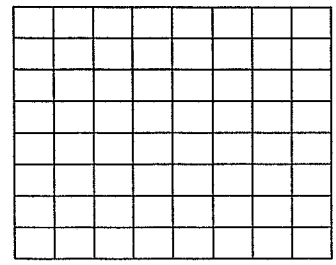
In [6] the template operator is presented as a structure which is mathematically equivalent to the matrix form, in other words, we can find a one-to-one mapping between a template operator and a matrix. Here a similar idea is used, but the formality is different though we can present the following algorithm in terms of the form we used in [6] or even in terms of matrices. In order to display the interesting connection between our fast algorithm and its physical interpretations, we will adapt the definition of the template operator in this paper for this new context[1].

Consider the Dirichlet problem

$$\begin{cases} \triangle u(x,y) = f(x,y) \\ u(x,y)|_{\Gamma_\Omega} = g(x,y) \end{cases}$$

where Ω is a unit square in the (x,y) plane. Let us lay an equally spaced mesh on this square and let the mesh size be $h = \frac{1}{n+1}$. Here we always assume $n = 2^L - 1$. The following figure shows a grid with $n = 2^3 - 1$.

[1] Since we are mainly discussing the Dirichlet problem of the 2-dimensional Poisson equation on a unit square region in this paper, the pictures of templates are mostly demonstrated on a square region with equally spaced mesh. But the same idea can be directly applied to any irregular region and irregular mesh such as finite element triangulations (see [6]).



The traditional numerical technique for this problem is first to discretize the differential operator on every interior node, then to form a matrix equation and finally to solve this matrix equation by some efficient algorithm. Here a different approach is used. There is no explicit system of linear equations involved. Each node can have as many operators as needed. These operators all represent some local physical principles under some conditions. In order to preserve the locality of these operators and the topological relations of the variables, a new structure of the discrete operators is introduced as follows: we define a template $\mathcal{T}$ as:

$$\mathcal{T} = \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \cdots & b_{0,n} & b_{0,n+1} \\ b_{1,0} & O_{1,1} & O_{1,2} & \cdots & O_{1,n} & b_{1,n+1} \\ b_{2,0} & O_{2,1} & O_{2,2} & \cdots & O_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b_{n,0} & O_{n,1} & O_{n,2} & \cdots & O_{n,n} & b_{n,n+1} \\ b_{n+1,0} & b_{n+1,1} & b_{n+1,2} & \cdots & b_{n+1,n} & b_{n+1,n+1} \end{bmatrix}$$

where b_{ij} represent the boundary nodes while O_{ij} represent the interior nodes. A template vector

$$U = \begin{bmatrix} g_{0,0} & g_{0,1} & g_{0,2} & \cdots & g_{0,n} & g_{0,n+1} \\ g_{1,0} & x_{1,1} & x_{1,2} & \cdots & x_{1,n} & g_{1,n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_{n,0} & x_{n,1} & x_{n,2} & \cdots & x_{n,n} & g_{n,n+1} \\ g_{n+1,0} & g_{n+1,1} & g_{n+1,2} & \cdots & g_{n+1,n} & g_{n+1,n+1} \end{bmatrix}$$

can be defined on this template, where g_{ij} are the boundary values on the boundary nodes b_{ij}, which are known, and x_{ij} are the values of the unknown $u(x, y)$ of the continuous problem at node O_{ij} [2].

Given a finite difference approximation, for example a five-point stencil, a template operator Δ_{ij} for node O_{ij} can be defined as:

$$\Delta_{ij} = \frac{1}{h^2} \begin{bmatrix} & & & \overset{j}{\downarrow} & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & -1 & 4 & -1 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

[2] For simplicity, we only define one function value on each node. There is no difficulty in generalizing this definition to include the cases where both the function value and its derivatives are needed on each node. In [6], for a generalized version of $\mathcal{T}$ even a state vector can be defined on each node.

We will also only discuss the finite difference approximation in this paper for the same reason.

where $i = 1, \cdots n$; $j = 1, \cdots n$ and 4 is located in position (i, j). Note that the values of x_{ij} in the template vector U are sampled from the true solution of the continuous Poisson equation.

The operation of Δ_{ij} on U is to multiply the elements of Δ_{ij} with the corresponding elements of U, and the summation of these products is the result of this operation. Thus we have

$$\Delta_{ij} \circ U = \frac{1}{h^2}[-x_{i-1,j} - x_{i+1,j} - x_{i,j-1} - x_{i,j+1} + 4x_{ij}].$$

The Poisson operator at O_{ij} can be written in terms of the τ_O

$$\Delta_{ij} \circ U = f_{ij} + \tau_{ij}, \tag{1}$$

where $f_{ij} = f(ih, jh)$ and τ_{ij} is the truncation term for the five-point stencil. It is known that $\tau_{ij} \approx O(h^2)$.

Since most of the elements in Δ_{ij} are zeros, a compact notation for Δ_{ij} is convenient. Let

$$\Delta(i, j, 1, \frac{1}{h^2}) = \frac{1}{h^2} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,1}$$

denote Δ_{ij}, where i, j is the center position of the template operator; 1 denotes the increment of the indices and $\frac{1}{h^2}$ is the constant factor of the operator. Then (1) can be rewritten as:

$$\begin{aligned} \Delta(i, j, 1, \tfrac{1}{h^2}) \circ U &= \frac{1}{h^2} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,1} \circ U \\ &= \frac{1}{h^2}(-x_{i-1,j} - x_{i+1,j} - x_{i,j-1} - x_{i,j+1} + 4x_{ij}) \\ &= f_{ij} + \tau_{ij} \end{aligned}$$

The compact form for the template operator derived from a skewed five point stencil at node O_{ij} is:

$$\tilde{\Delta}_{ij} = \tilde{\Delta}(i, j, 1, \frac{1}{2h^2}) = \frac{1}{2h^2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{i,j,1}.$$

Then the same Poisson operator at O_{ij} can also be expressed by $\tilde{\Delta}_{ij}$ as:

$$\begin{aligned} \tilde{\Delta}_{ij} \circ U &= \frac{1}{2h^2}(-x_{i-1,j-1} - x_{i+1,j-1} - x_{i-1,j+1} - x_{i+1,j+1} + 4x_{ij}) \\ &= f_{ij} + \tilde{\tau}_{ij} \end{aligned}$$

where $\tilde{\tau}_{ij}$ is the truncation term for the skewed five point stencil at node O_{ij}. Similarly, we have the template operators derived from a nine-point and a skewed nine-point stencil at node O_{ij}:

$$\Delta'_{ij} = \Delta'(i, j, 1, \frac{1}{6h^2}) = \frac{1}{6h^2} \begin{bmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{bmatrix}_{i,j,1}$$

$$\Delta''_{ij} = \Delta''(i,j,1,\frac{1}{12h^2}) = \frac{1}{12h^2}\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 \\ -1 & 0 & 20 & 0 & -1 \\ 0 & -4 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}_{i,j,1}$$

The last two operators have an accuracy of $O(h^4)$. Note, if the template operator derived from the five-point stencil at node O_{ij} uses mesh size $2h$, it can be written as

$$\Delta(i,j,2,\frac{1}{4h^2}) \circ U = \frac{1}{4h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2}$$

$$\begin{aligned} \Delta(i,j,2,\tfrac{1}{4h^2}) \circ U &= \frac{1}{4h^2}(-x_{i-2,j} - x_{i+2,j} - x_{i,j-2} - x_{i,j+2} + 4x_{ij}) \\ &= f_{ij} + O[(2h)^2] \end{aligned} \tag{2}$$

As we mentioned above, in a traditional approach, only one template operator at each node is used. But in our wavefront elimination process, more than one, even several different template operators at each node are used simultaneously. This is one key difference between the $\mathcal{WE}$ and conventional methods.

3. Wavefront Eliminations. The influencing and influenced wavefronts for each node in a template operator are introduced in [6]. These concepts characterize the propagation of the influences between nodes. Here the concept of wavefronts is used in a different context, namely to characterize the change of the non-zero pattern during the elimination process.

The first wavefront of any node O_{ij} in a template operator is defined here as the set of nodes where the elements of the template operator Δ_{ij} are non-zero except the node O_{ij} itself. Since we will use two or even more different template operators at each node O_{ij}, the wavefront will certainly be referred to the corresponding template operator at O_{ij}. For example, the first wavefront W_{ij} for Δ_{ij} is

$$W_{ij} := \{O_{i,j-1}, O_{i,j+1}, O_{i+1,j}, O_{i-1,j}\},$$

while the first wavefront $\widetilde{W}_{ij}$ for $\tilde{\Delta}_{ij}$ is

$$\widetilde{W}_{ij} := \{O_{i-1,j-1}, O_{i-1,j+1}, O_{i+1,j-1}, O_{i+1,j+1}\}.$$

Similarly denote W'_{ij} as the wavefront for Δ'_{ij} and W''_{ij} for Δ''_{ij}, where

$$W'_{ij} := W_{ij} \cup \widetilde{W}_{ij},$$

and

$$W''_{ij} := \widetilde{W}_{ij} \cup \{O_{i-2,j}, O_{i+2,j}, O_{i,j+2}, O_{i,j-2}\}.$$

There are many ways of performing wavefront eliminations. It is actually a whole class of direct approaches for the numerical solution of elliptic P.D.E's. To illustrate the basic idea of the wavefront eliminations let us introduce the simplest version of wavefront eliminations below. It is easy to see from the following description of the new algorithms: if we change the combination of the template operators and chose

the different wavefronts, we may obtain many different algorithms. They all have an optimal complexity though the constants for each approach will be slightly different. The simplest case of a wavefront elimination only uses two kinds of template operators at each node O_{ij}, namely Δ_{ij} and $\tilde{\Delta}_{ij}$.

Let us define the addition $\Delta_1+\Delta_2$ of two template operators Δ_1 and Δ_2 to be a new template operator such that the elements of $\Delta_1+\Delta_2$ are the sum of the corresponding elements of Δ_1 and Δ_2. Likewise, define the scalar product $\alpha\Delta$ be a new template operator such that the elements of $\alpha\Delta$ are the products of α with the corresponding elements of Δ.

One step of the wavefront elimination is to eliminate the non-zero elements of a template operator, say Δ_{ij}, by the template operators at the nodes of its first wavefront. Let

$$\Delta_{ij} \circ U = f_{ij} + \tau_{ij}, \tag{3}$$
$$\Delta_{i-1,j} \circ U = f_{i-1,j} + \tau_{i-1,j}, \tag{4}$$
$$\Delta_{i+1,j} \circ U = f_{i+1,j} + \tau_{i+1,j}, \tag{5}$$
$$\Delta_{i,j-1} \circ U = f_{i,j-1} + \tau_{i,j-1}, \tag{6}$$
$$\Delta_{i,j+1} \circ U = f_{i,j+1} + \tau_{i,j+1}, \tag{7}$$

where $\tau_{ij}, \tau_{i+1,j}, \cdots$ are the truncation terms.

It is known that $\tau_{ij} \approx O(h^2), i, j = 1, \cdots, n$. Then one step of the wavefront elimination for Δ_{ij} can be written as

$$\begin{aligned}\Delta_{ij} &+\frac{1}{4}(\Delta_{i-1,j} + \Delta_{i+1,j} + \Delta_{i,j-1} + \Delta_{i,j+1}) \\ &= \frac{1}{4h^2}\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & 12 & 0 & -1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}_{i,j,1}\end{aligned}$$

To simplify the notation, let

$$\begin{aligned}\mathcal{W}_1(y_{ij}) &= y_{ij} + \frac{1}{4}(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1}), \\ \mathcal{W}_2(y_{ij}) &= y_{ij} + \frac{1}{4}(y_{i-1,j-1} + y_{i+1,j-1} + y_{i+1,j+1} + y_{i-1,j+1})\end{aligned}$$

where y_{ij} is defined on node O_{ij}. Note that y can be a template operator or a grid function. Denote

$$\Delta^{(0.5)}(i,j,1,\frac{1}{4h^2}) = \frac{1}{4h^2}\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & 12 & 0 & -1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}_{i,j,1}.$$

Then we have

$$\mathcal{W}_1(\Delta_{ij}) = \Delta^{(0.5)}(i,j,1,\frac{1}{4h^2}).$$

If the wavefront elimination is carried out on the equations (3) – (7) we will have

$$\mathcal{W}_1(\Delta_{ij}) \circ U = \mathcal{W}_1(f_{ij}) + \mathcal{W}_1(\tau_{ij}).$$

Comparing the non-zero pattern of $\mathcal{W}_1(\Delta_{ij})$ with $\tilde{\Delta}_{ij}$, it is easy to see that in one more step of elimination the non-zero elements of $\mathcal{W}_1(\Delta_{ij})$ at position $(i+1, j+1)$, $(i+1, j-1)$, $(i-1, j+1)$, $(i-1, j-1)$ can be annihilated as follows:

$$\Delta^{(0.5)}(i,j,1,\tfrac{1}{4h^2}) - \tilde{\Delta}_{ij} = \frac{1}{4h^2}\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}_{i,j,1} \tag{8}$$

$$= \frac{1}{4h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2} \tag{9}$$

$$= \Delta(i,j,2,\frac{1}{(2h)^2}) \tag{10}$$

To summarize the above steps, we have

$$\mathcal{W}_1(\Delta_{ij}) - \tilde{\Delta}_{ij} = \Delta(i,j,2,\frac{1}{(2h)^2}).$$

Let

$$\begin{aligned} \Delta_{ij}^{(2)} &= \Delta(i,j,2,\frac{1}{(2h)^2}), \\ f_{ij}^{(1)} &= f_{ij}, \\ \tilde{f}_{ij}^{(1)} &= \tilde{f}_{ij}, \\ \tau_{ij}^{(1)} &= \tau_{ij}, \\ \tilde{\tau}_{ij}^{(1)} &= \tilde{\tau}_{ij}, \\ f_{ij}^{(2)} &= \mathcal{W}_1(f_{ij}) - \tilde{f}_{ij}^{(1)}, \\ \tau_{ij}^{(2)} &= \mathcal{W}_1(\tau_{ij}) - \tilde{\tau}_{ij}^{(1)}. \end{aligned}$$

We have

$$\begin{aligned} \Delta_{ij}^{(2)} \circ U &= \frac{1}{(2h)^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2} \circ U \\ &= \frac{1}{4h^2}(-x_{i-2,j} - x_{i+2,j} - x_{i,j-2} - x_{i,j+2} + 4x_{ij}) \\ &= f_{ij}^{(2)} + \tau_{ij}^{(2)}. \end{aligned}$$

Applying a similar procedure to $\tilde{\Delta}_{ij}$, we will have

$$\begin{aligned} \tilde{\Delta}_{ij}^{(2)} &= \mathcal{W}_2(\tilde{\Delta}_{ij}) - \Delta_{ij}^{(2)} \\ &= \tilde{\Delta}(i,j,2,\frac{1}{8h^2}) \\ &= \frac{1}{8h^2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{i,j,2}, \end{aligned}$$

and

$$\tilde{\Delta}_{ij}^{(2)} \circ U = \tilde{f}_{ij}^{(2)} + \tilde{\tau}_{ij}^{(2)},$$

where

$$\begin{aligned} \tilde{f}_{ij}^{(2)} &= \mathcal{W}_2(f_{ij}^{(1)}) - f_{ij}^{(2)}, \\ \tilde{\tau}_{ij}^{(2)} &= \mathcal{W}_2(\tau_{ij}^{(1)}) - \tau_{ij}^{(2)}. \end{aligned}$$

This is one complete step of a wavefront elimination. Here a new set of template operators at node O_{ij} is obtained:

$$\begin{aligned} \Delta_{ij}^{(2)} \circ U &= \frac{1}{(2h)^2} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2} \circ U = f_{ij}^{(2)} + \tau_{ij}^{(2)}, \\ \tilde{\Delta}_{ij}^{(2)} \circ U &= \frac{1}{8h^2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{i,j,2} \circ U = \tilde{f}_{ij}^{(2)} + \tilde{\tau}_{ij}^{(2)}. \end{aligned}$$

The increment of the index in these operators is 2 now. If we recursively proceed with this process we will obtain

$$\begin{aligned} \Delta_{ij}^{(k)} \circ U &= \frac{1}{(2^{k-1}h)^2} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2^{k-1}} \circ U = f_{ij}^{(k)} + \tau_{ij}^{(k)}, \\ \tilde{\Delta}_{ij}^{(k)} \circ U &= \frac{1}{2(2^{k-1}h)^2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{i,j,2^{k-1}} \circ U = \tilde{f}_{ij}^{(k)} + \tilde{\tau}_{ij}^{(k)}. \end{aligned}$$

Here a sequence of self-similar template operators is generated for the same node at different stages of the elimination. If we call them W–template operators [3], they look exactly the same as an ordinary template operator which is derived from a large mesh size. The key difference here is the right hand side of the equation derived from wavefront elimination – it is derived from a *renormalization* process which we will discuss in the next section. The right hand side of the traditional template operator is simply the function value of the source term at the given node. Notice that we have always kept the truncation term with the template operator during the above discussion. The motivation is to let both template operators $\Delta_{ij}^{(k)}$ and $\tilde{\Delta}_{ij}^{(k)}$ operate on the same U. Furthermore, the analysis of the error propagation during the elimination process can be easily shown by the growth of τ_{ij}^{k} and $\tilde{\tau}_{ij}^{(k)}$. So far the rigorous error analysis has not been completed but the following informal discussion provides us with a very rough picture. From the definition of the wavefront elimination process, we have the recurrence relations

$$\begin{aligned} \tau_{ij}^{(k)} &= \mathcal{W}_1(\tau_{ij}^{(k-1)}) - \tilde{\tau}_{ij}^{(k-1)}, \\ \tilde{\tau}_{ij}^{(k)} &= \mathcal{W}_2(\tilde{\tau}_{ij}^{(k-1)}) - \tau_{ij}^{(k)}. \end{aligned}$$

[3] W - stands for wavefront

Because the numbers of the elements in $\tau_{ij}^{(k)}$ and $\tilde{\tau}_{ij}^{(k)}$ are reduced by a factor of 4 at each step, we can not apply some traditional techniques for the recurrence relations to this problem. But from this relation we may observe a rough estimate of the growth factor of $\tilde{\tau}_{ij}^{(k)}$ and $\tilde{\tau}_{ij}^{(k)}$. As we know that the eigenvalues of the operator $\mathcal{W}_1$ are:

$$\lambda_{ij} = 1 + \frac{1}{2}(\cos\frac{i\pi}{2^{L-k}} + \cos\frac{j\pi}{2^{L-k}}), \tag{11}$$

$$i = 1, 2, \cdots, (2^{L-k} - 1), \tag{12}$$

$$j = 1, 2, \cdots, (2^{L-k} - 1). \tag{13}$$

Thus the maximum growth factor for $\mathcal{W}_1(\tau_{ij})$ is less than 2. After subtracting $\tilde{\tau}_{ij}^{(k-1)}$ from $\mathcal{W}_i(\tau_{ij}^{(k-1)})$, the rough estimate of the growth factor of $\tau_{ij}^{(k)}$ is less than 1 or close to 1. We may obtain a similar rough estimate for $\tilde{\tau}_{ij}^{(k)}$. Preliminary numerical tests agree with this estimate but the result is by no means concrete.

Since the increment of the index is doubled at each step of the elimination, the number of the nodes involved in the elimination process is reduced by a factor of 4. A compact description of the above discussion is as follows:

1. Let $k = 1$

$$\begin{aligned} f_{ij}^{(1)} &= f((i-1)h, (j-1)h) \\ \tilde{f}_{ij}^{(1)} &= f((i-1)h, (j-1)h) \\ & i = 1, \cdots, 2^L - 1; \\ & j = 1, \cdots, 2^L - 1; \end{aligned}$$

2. For nodes $p = 2^k i$, $q = 2^k j$ where $i = 1, \cdots, 2^{L-k} - 1$, $j = 1, \cdots, 2^{L-k} - 1$.

$$\begin{aligned} f_{pq}^{(k+1)} &= \mathcal{W}_1(f_{pq}^{(k)}) - \tilde{f}_{pq}^{(k)}, \\ \tilde{f}_{pq}^{(k+1)} &= \mathcal{W}_2(\tilde{f}_{pq}^{(k)}) - f_{pq}{}^{(k+1)}. \end{aligned}$$

3. k=k+1, if $k < L$ go to step 2.

This is a forward elimination process. After $L-1$ steps of wavefront eliminations, the template operator $\Delta_{ii}^{(L-1)}$ for the center node O_{ii} where $i = 2^{L-1} - 1$ has reached the boundary of the solution region, namely, the increment of the index in $\Delta_{ii}^{(L-1)}$ is 2^{L-1}:

$$\Delta_{ii}^{(L-1)} \circ U = \frac{1}{(2^{L-1}h)^2} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2^{L-1}} \circ U = f_{ii}^{(L-1)}.$$

Since the boundary values are known, we can obtain the result of x_{ii} directly from this operator. Then a backwards solution process can be started. The complexity of the forward process is $\frac{10}{3}N$ where $N = n^2 = (2^L - 1)^2$.

The backward process has the same complexity as the forward process. We will need another $3N$ operations to obtain the value of x_{ij} at the finest level. Therefore the complexity of the whole process is approximately $10N$. As we mentioned above, for each node O_{ij} there are many different template operators with different accuracies.

Due to the symmetry of these operators, many combinations of these operators can be chosen to perform the wavefront elimination. A more detailed discussion will be presented in a future report. The other algorithms also have an optimal complexity but the constants will be slightly bigger. The idea of using this algorithm as a preconditioner for a general elliptic equation is also considered. We are also studying the possibility of directly generalizing these algorithms to more general problems. Another interesting application of this algorithm is in combination with other efficient algorithms. For example, three or four steps of wavefront elimination are first executed then an other efficient algorithm is applied to solve the reduced system. This strategy can be used in case the truncation term grows during the elimination. If three steps of wavefront elimination are applied, the size of the reduced system is only $\frac{1}{64}$th of the original one. A great saving in computational cost can be achieved.

Some other generalizations can also be derived from this algorithm. First, the application of the $\mathcal{WE}$ to problems on an irregular solution region is considered. Suppose a solution of the Dirichlet problem on an irregular region Ω is wanted:

$$\begin{cases} \triangle U(x,y) = f(x,y) \\ U(x,y)|_{\Gamma_\Omega} = g(x,y) \end{cases}$$

We can enclose Ω by a larger rectangular region Ω' and let

$$f'(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \Omega \\ 0 & \text{if } (x,y) \in \Omega' - \Omega \end{cases}$$

The wavefront elimination can be applied to the problem

$$\begin{cases} \triangle \tilde{U}(x,y) = f'(x,y) \\ \tilde{U}(x,y)|_{\Gamma_\Omega} = 0. \end{cases}$$

Then apply an optimal algorithm due to Greengard and Rakhlin [1], [4] to the Dirichlet problem of the Laplace equation

$$\begin{cases} \triangle \tilde{U}(x,y) = 0 \\ \tilde{U}(x,y)|_{\Gamma_\Omega} = g(x,y) - \tilde{U}(x,y). \end{cases}$$

It is easy to verify that $\tilde{U} - U'$ is the solution of the original problem and the complexity of this procedure is also optimal. The generalization of this idea to a three dimensional problem is also considered. Unfortunately, there is no optimal algorithm in analogy with the two-dimensional version. The best result we can obtain so far has a complexity of $O(N \log n)$, where $N = n^3$. How to find a renormalization mapping for three dimensional problems is a challenging open problem.

From the above discussion, we can see that the study of the new algorithms does create a challenging new direction of research for fast solvers. It is more important than the optimum of the complexity per se.

4. Renormalization Theory and Wavefront Eliminations. In the last section a new class of algorithm $\mathcal{WE}$ was introduced. During the elimination process a sequence of self-similar W-template operators is derived on the same node. Let's see how the process can be interpreted in terms of physics. Imagine the solution U to be temperature, therefore the right hand side of the Poisson equation ought to be the heat source. It is not difficult to see that Δ_{ij} or $\tilde{\Delta}_{ij}$ can be interpreted as a conservation

relation between heat source and heat flux around the small cell where node O_{ij} is located. After one step of wavefront elimination, a new W-template operator

$$\Delta_{ij}^{(2)} \circ U = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_{i,j,2} \circ U = f_{ij}^{(2)}$$

where

$$f_{ij}^{(2)} = \mathcal{W}_1(f_{ij}^{(1)}) - \tilde{f}_{ij}^{(1)}, \tag{14}$$

is obtained.

Imagine this process as the result of zooming our observation position a little further away from the solution plane. This new equation can also be viewed as another conservation relation for node O_{ij} except the scale is different. In order to make the conservation valid without changing the solution then the heat source has to be renormalized. The mapping (14) actually can be viewed as the renormalization transformation. If we compare this process with K.G. Wilson's renormalization theory for the magnet, there are surprising similarities between the two completely different subjects. In his theory, the same magnet of given temperature, when viewed on different scales, looks as if it were at different temperature [4]. Consider a magnet of N atoms with inter-atomic distance a and temperature T. On a coarse scale where the elementary block is taken to have sidelength $a' = b \cdot a$ and comprises b^3 atoms, the magnet looks like one with $N' = N/b^3$ atoms but with another *renormalized* temperature T'. The relation $T' = R_b(T)$ is called renormalization transformation. In a recent development of this theory, it can also be derived that the pattern of fluctuation at the critical value of temperature is self-similar. This basic idea eventually led to quantitative results and explained the physics of phase transitions in a satisfying way. L. P. Kadanoff first discovered the scaling law in 1966 [2], but it was K. G. Wilson who finally surmounted the difficulties and developed the method of renormalization into a technical instrument that has proven its worth in innumerable applications. It is not surprising that renormalization theory has recently led to fractal phase boundary. The book "The Beauty of Fractals" [3] cites a dictum from V. F. Weisskopf,

> *There's a fog of events and suddenly you see a connection. It expresses a complex of human concern that goes deeply to you that connects things that were always in you that were never put together before.*

From the sequence of hierarchical self-similar W-template operators [5] we can also sense the principle which Mandelbrot discovered that organizes a whole universe of self-similar structure. If the forward process of the wave-front elimination is equivalent to a renormalization of the heat source term, then the backward solution process is surprisingly similar to recursively detailing the solution when our "camera" is zooming into the fine scales. This interesting connection between wavefront elimination, renormalization theory and fractals has inspired us to generalize the wavefront elimination to

[4] In wavefront elimination, the same temperature were produced from a different heat source if the temperature field is viewed from different scales

[5] It is also worth-while to mention that template operators provide us a right structure to exhibit their self-similarity and renormalization relations?

It would be very difficult to observe this fact from the structure of a matrix

other kinds of elliptic equations. Some preliminary study shows that this is a promising new direction for research into fast algorithms.

Before we conclude this section, it is also very interesting to compare wavefront elimination with the multi-grid method. It is not difficult to see that one basic idea behind the two very different approaches is the same, namely, both are using the idea of hierarchical computation in order to achieve the optimal complexity. But to achieve the same goal, the approaches they use are two extreme examples. In the wavefront elimination a renormalization of the source term, in other words the right hand side, is used to march from the fine grid level to coarse while in multi-grid the error of the solution is projected from a fine grid to a coarser one. In the wavefront elimination case we are able to complete the solution process in one scan while the latter need a few scans.

5. Conclusion. A whole new class of fast solvers is briefly discussed in this paper. A great many interesting open problems remain to be studied in this area. In particular, the discussion of the error analysis in section 3 is informal. Even though there appears to be no instability problem in our preliminary numerical tests, more studies are needed for a concrete result regarding the error analysis of the $\mathcal{WE}$.

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