Acceleration of Domain Decomposition Algorithms for Mixed Finite Elements by Multi-Level Methods

R. Glowinski*
W. Kinton**
M.F. Wheeler*

Abstract. In this paper we consider the numerical solution of elliptic partial differential equations by a combination of domain decomposition algorithms, mixed finite element methods and multi-level procedures. The multi-level procedures are used to accelerate convergence of the algorithm which iteratively adjusts the matching conditions at the interfaces of the subdomains. Numerical results are included in this paper which exhibit improvements in convergence by applying this multi-level approach, compared to more traditional iterative methods.

*Department of Mathematics, University of Houston, Houston, Texas 77004, U.S.A.

**Department of Mathematical Sciences, Rice University, Houston, Texas 77251, U.S.A.

0. Introduction. In [1] Glowinski and Wheeler defined domain decomposition algorithms for solving mixed finite element approximations of elliptic problems with non-constant coefficients. A key result in [1] was the formulation of the matching conditions at the interfaces of the subdomains as variational problems defined over convenient trace space. These new problems were solved by conjugate gradient algorithms using simple preconditioners resulting in a $0(h^{-.5})$ number of iterations to achieve convergence. In this paper we shall discuss a procedure for accelerating the convergence of the above algorithms which is essentially based on a multi-level technique acting on the trace space associated to the interfaces.

In Section 1, we shall give some examples of elliptic problems originating from flow in porous media. Compared to more traditional solution methods the algorithm described in this paper have been quite successful as we shall demonstrate in Section 4. In Section 2 which follows closely [1] we shall recall the mixed variational formulation of elliptic problems, the mixed finite element approximations and the associated domain decomposition methods. In Section 3 we shall discuss a multilevel method to speed up convergence of the domain decomposition algorithms discussed in Section 2. Results of numerical experiments will be discussed in Section 4. Finally some mesh refinement methods well suited for domain decomposition and mixed finite element methods will be discussed in Section 5.

1. Motivation for Robust Elliptic Solvers.

In our first example we consider the pressure equation which arises from *miscible displacements* in porous media. The equation has the form

(1.1) $$u = -A \text{ grad } p \text{ in } \Omega,$$

(1.2) $$\nabla \cdot u = q \text{ in } \Omega,$$

(1.3) $$u \cdot \nu = 0 \text{ on } \partial\Omega,$$

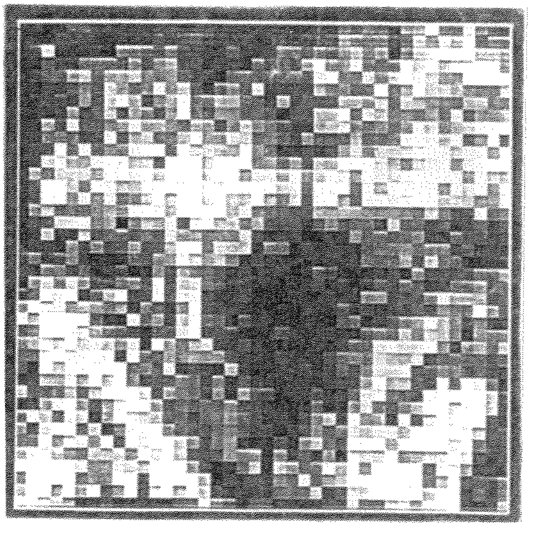
where

$$A = k(x, y)/\mu(c).$$

In this problem Ω is the flow region, u is the Darcy velocity, p is the pressure, q is a source or sinks term, k is the permeability of the porous media, μ is the viscosity of the concentration c of the fluid which is flowing through the porous media. In this example we use a permeability field and a form of the viscosity which has been previously obtained from laboratory experiments. In Figure 1.1, a visualization of A is shown. In this case we have

$$\max A = .810\times 10^{-2} \text{ and } \min A = .282\times 10^{-3},$$

implying that (1.1)-(1.3) is badly conditioned. However, as it will be seen with more detail in Section 4, we have been able to solve this problem, using domain decomposition, in less than 10 iterations.



Variation of coefficient A

Figure 1.1

2. Mixed Formulation of Elliptic Problems · Associated Finite Element Approximation and Domain Decomposition.

2.1 The Model Problem.

We consider on $\Omega \subset \mathbb{R}^n$ the following Neumann problem

$$(2.1) \qquad \begin{cases} -\nabla \cdot A \nabla p = f \quad in\ \Omega, \\ \\ A \nabla p \cdot \nu = g \quad on\ \partial\Omega (= \Gamma), \end{cases}$$

where ν is the outward normal vector. We assume the compatibility condition

$$(2.2) \qquad \int_\Omega f\, dx + \int_\Gamma g\, d\Gamma = 0.$$

Our formalism is motivated from flow in porous media where (2.1) is the pressure equation, but the method to be described applies to other branches of science and engineering. Also we have been considering the pure Neumann problem since it is the one occurring most frequently in applications. In fact, it is also the most difficult case.

2.2 A Mixed Variational Formulation of Problem (2.1)

Define u by

$$(2.3) \qquad u = -A \nabla p.$$

We then have

$$(2.4) \qquad \nabla \cdot u - f = 0,$$

and

(2.5) $\nabla p = -A^{-1}u.$

Multiplying (2.4) and (2.5) by q and v respectively, we obtain

(2.6) $$\int_\Omega (\nabla\cdot u - f) q\,dx = 0, \ \forall\ q \epsilon L^2(\Omega),$$

and

(2.7) $$\int_\Omega A^{-1}\, u\cdot v\ dx - \int_\Omega p\nabla\cdot v\ dx = 0, \ \forall v \epsilon V_o,$$

where

(2.8) $V_o = \{v \mid v \epsilon H(\Omega, \text{div}),\ v\cdot\nu = 0 \ \textit{on}\ \Gamma\}.$

Here

(2.9) $H(\Omega; \text{div}) = \{\ v\ \epsilon (L^2(\Omega))^n \ \textit{and}\ \text{div}\ v \epsilon L^2(\Omega)\}.$

Suppose $f \epsilon L^2(\Omega)$, $g \epsilon\ H^{-\frac{1}{2}}(\Gamma)$ and A is symmetric such that $A \epsilon (L^\infty(\Omega))^{n\times n}$ and

$$A(x)\xi\cdot\xi \geq \alpha|\xi|^2, \ \forall \xi \epsilon \mathbb{R}^n, \text{ a. e. on } \Omega,$$

with α a positive constant.

If (2.2) holds then (2.1) has a unique solution on $H^1(\Omega)/\mathbb{R}$ implying the uniqueness of u. An alternative formulation of (2.1) is given by

Find $p \epsilon L^2(\Omega)$, $u \epsilon H(\Omega; \text{div})$, *such that*

$$u\cdot\nu + g=0 \text{ on } \Gamma,$$

$$\int_\Omega (\nabla\cdot u-f)\, q\, dx=0, \quad \forall\ q\epsilon L^2(\Omega), \tag{2.10}$$

$$\int_\Omega A^{-1}\, u\cdot v\, dx-\int_\Omega p\, \nabla\cdot v\, dx=0, \quad \forall\ v\epsilon V_o.$$

2.3 Finite Element Approximation of Problem (2.10).

We denote by W^h and V^h finite dimensional subspaces of $L^2(\Omega)$ and $H(\Omega;\text{ div})$, respectively. In addition we set $V_o^h=V^h\cap V_o$. We shall assume that div $V^h\subset W^h$.

It is natural then to approximate problem (2.1), using its mixed equivalent formulation, by

Find $p_h\epsilon W^h$, $u_h\epsilon V^h$ *satisfying*

$$\int_\Gamma (u_h\cdot\nu+g)v\cdot\nu\, d\Gamma=0, \quad \forall v\epsilon V^h,$$

$$\int_\Omega (\nabla\cdot u_h-f)q\, dx=0, \quad \forall\ q\epsilon W^h, \tag{2.11}$$

$$\int_\Omega A^{-1}u_h\cdot v dx-\int_\Omega p_h\, \nabla\cdot v\, dx=0, \quad \forall\ v\epsilon V_o^h.$$

Examples of particular finite element spaces for which (2.11) is well posed and for which lim $u_h\rightarrow u$ and lim $p_h\rightarrow p$ can be found in [2]. Additional convergence results including error estimates can be found in [3, 4].

2.4 Domain Decomposition Method for Problem (2.1), (2.11).

We follow here the notation and methodology developed in [1]. Considering first the continuous problem whose formula is much simpler we suppose that Ω has been decomposed in two subdomains Ω_1 and Ω_2. Figures 2.1a and 2.1b show such domain decompositions and corresponding notation.

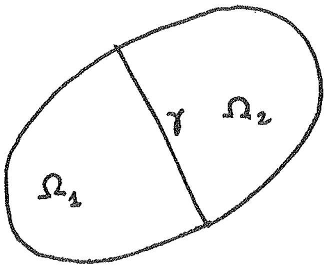


Figure 2.1a

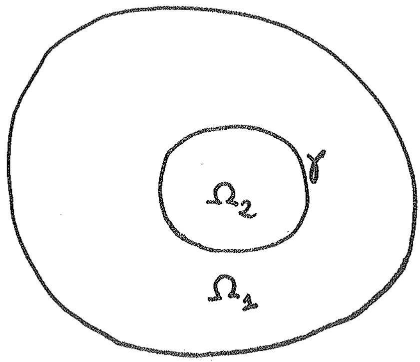


Figure 2.1b

If we denote by $\{p_i, u_i\}$ the restriction of $\{p, u\}$ to Ω_i there is equivalence between (2.10) and

$$(2.12)\qquad \begin{cases} \int_{\Omega_i} (\nabla\cdot u_i - f)\, q_i dx = 0, \quad \forall\ q_i \epsilon L^2(\Omega_i), \\ \int_{\Omega_i} \left(A^{-1}\, u_i\cdot v_i\ -\ p_i \nabla\cdot v_i\right) dx = 0, \quad \forall v_i \epsilon V_{io},\ i=1,\ 2, \end{cases}$$

$$(2.13)\qquad u_i\cdot\nu_i + g = 0 \quad on\ \Gamma\cap\partial\Omega_i, \quad i=1,\ 2,$$

$$(2.14)\qquad \sum_{i=1}^{2} u_i\cdot\nu_i = 0 \quad on\ \gamma,$$

$$(2.15)\qquad \sum_{i=1}^{2} \int_{\Omega_i} \left(A^{-1}u_i\cdot v - p_i\nabla\cdot v\right)\, dx = 0, \ \forall v\epsilon V_o,$$

with

$$V_{io}=\left\{v_i | v_i \epsilon H(\Omega_i, \text{div}),\ v_i \cdot \nu_i = 0 \ \textit{on}\ \partial\Omega_i\right\}.$$

Since $V_o=V_{1o}\oplus V_{2o}\oplus V_{\gamma o}$ (where $V_{\gamma o}$ is a complementary subspace of $V_{1o}\oplus V_{2o}$ in V_o) it follows from (2.12) and (2.15) that (2.15) can be replaced by

$$(2.16) \qquad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot v - p_i \ \nabla\cdot v)\ dx=0, \quad \forall v \epsilon V_{\gamma o}.$$

In addition to (2.12)-(2.16), $\{p_i, u_i\}$ must satisfy the compatibility condition

$$(2.17) \qquad \int_{\Omega_i} f\ dx + \int_{\partial\Omega_i \cap \Gamma} g\ d\Gamma + \int_{\gamma} u_i \cdot \nu_i\ d\gamma = 0.$$

From (2.12)-(2.16), the local solutions satisfy at the interface γ the matching conditions (2.14) and (2.16). From this observation we can generate two classes (at least) of iterative methods for solving problem (2.11) by domain decomposition. In both approaches we assume that one of the matching conditions is satisfied by an appropriate choice of boundary conditions on γ and we try iteratively to satisfy the other matching condition. In this paper we shall concentrate on the case where the balance given by (2.14) is satisfied; we try therefore to verify (2.16).

This leads to the introduction of a variational problem involving functional spaces defined on γ. Precisely such a functional space is $V^o_{\gamma o}$ defined by

$$(2.18) \qquad V^o_{\gamma o}=\{\mu | \mu\ \epsilon V_{\gamma o},\ \int_{\gamma} \mu \cdot \nu\ d\gamma = 0\}.$$

We define next a bilinear form $a(\cdot,\cdot)$ over $V^o_{\gamma o} \times V^o_{\gamma o}$ as follows:

Consider $\mu \epsilon V^o_{\gamma o}$; we associate to μ, $u_i(\mu)$ and $p_i(\mu)$ by solving

(2.19) $$\int_{\Omega_i} \nabla\cdot u_i(\mu)\, v_i dx=0,\quad \forall v_i \epsilon L^2(\Omega_i),$$

(2.20) $$\int_{\Omega_i} (A^{-1} u_i(\mu)\cdot v_i - p_i(\mu)\, \nabla\cdot v_i)dx=0,\quad \forall v_i \epsilon V_{io},$$

(2.21) $$u_i(\mu)\cdot\nu_i=0 \;\; on\; \Gamma\cap\partial\Omega_i,\;\; u_i(\mu)\cdot\nu_i=\mu\cdot\nu_i \;\; on\; \gamma.$$

Since $\int_{\partial\Omega_i} u_i(\mu)\cdot\nu_i\, d\Gamma_i=0$, the above problem is well posed in $H(\Omega_i, div) \times L^2(\Omega_i)/\mathbb{R}$. To insure uniqueness of $p_i(\mu)$ we enforce the conditions

(2.22) $$\int_{\Omega_1} p_1(\mu)\, dx=0\;,\; \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i(\mu)\cdot\Pi - p_i(\mu)\, \nabla\cdot\Pi)\, dx=0$$

where $\Pi\epsilon(V_{\gamma o}-V^o_{\gamma o})$. Finally we define $a(\cdot,\cdot)$ by

(2.23) $$a\left(\mu, \mu'\right)=\sum_{i=1}^{2} \int_{\Omega_i} \left(A^{-1}u_i(\mu)\cdot\mu' - p_i(\mu)\, \nabla\cdot\mu'\right)dx,\; \forall\, \mu'\epsilon V^o_{\gamma o}.$$

It has been shown in [1] that the bilinear form $a(\cdot\,,\,\cdot)$ is symmetric and positive semi-definite over $V^o_{\gamma o}\times V^o_{\gamma o}$. Moreover, it is elliptic for the norm induced by $H(\Omega;\, div)$ over the quotient space $V^o_{\gamma o}/\hat{R}$, where $\hat{R}$ is the equivalence relation defined by $\mu\, \hat{R}\, \mu' \leftrightarrow \left(\mu-\mu'\right)\cdot\nu=0$ on γ.

From the above result we can interpret (2.12)-(2.17) as a *linear* variational problem in $V^o_{\gamma o}$. To formulate this latter problem consider $\lambda_o\, \epsilon H(\Omega;\, div)$ such that

(2.24) $$\lambda_o\cdot\nu+g=0 \;\; on\; \Gamma,$$

(2.25) $$\int_{\Omega_i} f\, dx + \int_{\Gamma\cap\partial\Omega_i} g\, d\Gamma + \int_{\gamma} \lambda_o\cdot\nu_i\, d\gamma=0,\; \forall\; i=1,\, 2;$$

solve then for i=1, 2,

$$\int_{\Omega_i} (\nabla\cdot u_{oi}-f)q_i \, dx=0, \quad \forall q_i \epsilon L^2(\Omega_i), \tag{2.26}$$

$$\int_{\Omega_i} (A^{-1} u_{oi}\cdot v_i - p_{oi}\nabla\cdot v_i) dx=0, \; \forall \; v_i \epsilon V_{io}, \tag{2.27}$$

$$u_{oi}\cdot\nu_i + g=0 \; \textit{on} \; \gamma\cap\partial\Omega_i, \tag{2.28}$$

$$u_{oi}\cdot\nu_i=\lambda_o\cdot\nu_i \; \textit{on} \; \gamma. \tag{2.29}$$

The constants associated to the p_{oi} are adjusted as follows:

$$\int_{\Omega_1} p_{o1} \, dx=0, \tag{2.30}$$

$$\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1}u_{oi}\cdot\Pi - p_{oi} \; \nabla\cdot\Pi) \, dx=0. \tag{2.31}$$

Let us now denote by u_o the element of $H(\Omega; \text{div})$ such that $u_o|_{\Omega_i}=u_{oi}$. If we define $\bar{u}$ by

$$\bar{u}=u-u_o, \tag{2.32}$$

we clearly have $\bar{u}\epsilon V_o$. Denoting $\bar{\lambda}\epsilon V_{\gamma o}$ as the component of $\bar{u}$ in the decomposition $V_o=V_{1o}\oplus V_{2o}\oplus V_{\gamma o}$ we have from (2.17), (2.25), (2.28), (2.29) that

$$\int_{\gamma} \bar{\lambda}\cdot\nu_i \, d\gamma=0, \; \textit{i.e.} \; \bar{\lambda}\epsilon V^o_{\gamma o}; \tag{2.33}$$

define similarly $\bar{p}_i$ by $\bar{p}_i = p_i - p_{oi}$.

We have then

$$\int_{\Omega_i} \nabla \cdot \bar{u}_i q_i dx = 0, \quad \forall\ q_i \epsilon L^2(\Omega_i), \tag{2.34}$$

$$\int_{\Omega_i} (A^{-1}\ \bar{u}_i \cdot v_i - \bar{p}_i\ \nabla \cdot v_i) dx = 0, \quad \forall v_i \epsilon V_{io}, \tag{2.35}$$

$$\bar{u}_i \cdot \nu_i = 0 \quad on\ \partial\Omega_i \cap \Gamma,\ \bar{u}_i \cdot \nu_i = \bar{\lambda} \cdot \nu_i \text{ on } \gamma, \tag{2.36}$$

$$\int_{\Omega_1} \bar{p}_1\ dx = 0, \quad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} \bar{u}_i \cdot \Pi - \bar{p}_i\ \nabla \cdot \Pi) dx = 0. \tag{2.37}$$

It follows from (2.16) that

$$\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot \mu - p_i\ \nabla \cdot \mu)\ dx = 0, \quad \forall\ \mu\ \epsilon V^o_{\gamma o}. \tag{2.38}$$

From the definition of $\bar{u}_i$, p_i and from (2.38) we obtain

$$\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} \bar{u}_i \cdot \mu - \bar{p}_i\ \nabla \cdot \mu)\ dx = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi}\ \nabla \cdot \mu)\ dx, \quad \forall \mu V^o_{\gamma o}. \tag{2.39}$$

It follows from (2.23) and (2.33) that $\bar{\lambda}$ is the unique solution of the linear variational equation

$$\begin{cases} \textit{Find } \bar{\lambda} \epsilon V^o_{\gamma o} \textit{ such that} \\ a(\bar{\lambda}, \mu) = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi}\ \nabla \cdot \mu) dx, \quad \forall\ \mu \epsilon V^o_{\gamma o}. \end{cases} \tag{2.40}$$

In [1], we showed that the variational problem (2.40) can be approximated by a finite dimensional problem of the same nature, obtained by combining the mixed approximation of Section 2.3 with the domain decomposition principle of Section 2.4. In addition, a conjugate gradient method for solving this finite dimensional problem approximating (2.40) was discussed in detail in the above reference.

In the following Section 3, we shall describe multilevel techniques for solving the finite dimensional problem approximating (2.40); it can be seen as a multigrid method operating on the interface γ.

3. Multilevel Solution of Problem (2.40).

3.1. Domain Decomposition of the Discrete Problem.

Following Section 2.3, it is easily shown that the discrete mixed problem (2.11) is equivalent to finding $\{u_{h,i}, p_{h,i}\}$, $i=1, 2$, satisfying

$$(3.1)\qquad \int_{\Omega_i} (\nabla\cdot u_{h,i} - f)\, q_i\, dx = 0, \quad \forall q_i \in W_{h,i},$$

$$(3.2)\qquad \int_{\Omega_i} (A^{-1} u_{h,i}\cdot v_i - p_{h,i} \nabla\cdot v_i)\, dx = 0, \quad \forall v_i \in V_{o\,h,i}$$

$$(3.3)\qquad \int_{\partial\Omega_i \cap \Gamma} (u_{h,i}\cdot\nu + g)\; v\cdot\nu\, d\Gamma = 0, \quad \forall\; v_i \in V_{oh,i},$$

$$(3.4)\qquad \sum_{i=1}^{2} u_{h,i}\cdot\nu_i = 0 \quad on\ \gamma,$$

$$(3.5)\qquad \sum_{i=1}^{2} \int_{\Omega} (A^{-1} u_{h,i}\cdot v - p_{h,i}\; \nabla\cdot v)\, dx = 0, \quad \forall v \in V_{oh},$$

where $V_{oh,i}$ (resp. $W_{h,i}$) is equal to $V_{oh}|_{\Omega_i}$ (resp. $W_h|_{\Omega_i}$). As in the continuous case we associate to γ a complementary subspace $V_{oh,\gamma}$ of $V_{oh,1} \oplus V_{oh,2}$ in V_{oh}; that is

$$V_{oh}=V_{oh,1}\oplus V_{oh,2}\oplus V_{oh,\gamma}.$$

It follows from (3.1) and (3.2) that (3.5) can be replaced by

$$(3.6)\qquad \sum_{i=1}^{2}\int_{\Omega_i}(A^{-1}u_{h,i}\cdot v-p_{h,i}\ \nabla\cdot v)\ dx=0,\quad \forall\ v\epsilon V_{oh,\gamma}.$$

In addition to (3.5) and (3.6) $\{u_{h,i}, p_{h,i}\}$ has to satisfy the compatibility conditions

$$(3.7)\qquad \int_{\Omega_i} f\ dx+\int_{\partial\Omega_i\cap\Gamma} g d\Gamma+\int_{\gamma} u_{h,i}\cdot\nu_i d\gamma=0,\ i=1,\ 2.$$

Finally we decompose $V_{oh,\gamma}$ as the direct sum,

$$(3.8)\qquad V_{oh,\gamma}=V^{o}_{oh,\gamma}\ \oplus\ V^{\Pi}_{oh,\gamma}$$

where

$$(3.9)\qquad V^{o}_{oh,\gamma}=\{z\epsilon V_{oh,\gamma}\ |\ \int_{\gamma} z\cdot\nu\ d\gamma=0\},$$

and

$$(3.10)\qquad V^{\Pi}_{o,h,\gamma}=\{t\Pi,\ t\ \epsilon\ \mathbb{R}\ \textit{and}\ \Pi\epsilon\ V_{oh,\gamma}\ \textit{with}\ \int_{\gamma}\Pi\cdot\nu\ d\gamma\neq 0\}.$$

3.2. Discretization of the Boundary Problem (2.40).

Following the development in Section 2.4, we approximate (2.40) by the following variational problem in $V^{o}_{oh,\gamma} \times V^{o}_{oh,\gamma}$:

$$(3.11) \quad \begin{cases} \textit{Find } \bar{\lambda}_h \in V^{o}_{oh,\gamma} \textit{ such that} \\ a_h(\bar{\lambda}_h,\mu) = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oh,i} \cdot \mu - p_{oh,i} \nabla\cdot\mu) dx, \quad \forall \mu \in V^{o}_{oh,\gamma}, \end{cases}$$

where $\bar{\lambda}_h$, $u_{oh,i}$ and $p_{oh,i}$ are obtained as discrete analogues of $\bar{\lambda}$, u_{oi} and $p_{o,i}$ in Section 2.4 (see [1] for all the details).

3.3. Multilevel Algorithms for Solving Problem (3.11).

3.3.1. Synopsis

We first introduce a discretization parameters h_j to which we associate all the above discrete spaces. For simplicity we denote by Z^j the space $V^{o}_{oh_j,\gamma}$. We assume that the sequence $\{Z^j\}$ satisfies the following inclusion property

$$(3.12) \quad Z^0 \subset Z^1 \subset \ldots \subset Z^J.$$

At level J (the finest level) we wish to solve problem (3.11) with $h = h_J$.

Before defining a multilevel algorithm for solving problem (3.11), we describe in the following Section 3.3.2 the solution of general variational problems by multilevel methods. The application to the specific problem (3.11) will be discussed in Section 3.3.3.

3.3.2. A Multi level Method for Linear Variational Problem in Hilbert Spaces.

Let V be a Hilbert space with $(\cdot\,,\,\cdot)$ as inner product and $\|\cdot\|_V$ the corresponding norm. We consider the following problem

$$(3.13)\qquad \begin{cases} \textit{Find } u\in V \textit{ such that} \\ \\ a(u, v)=L(v), \quad \forall v\in V, \end{cases}$$

where

(1) a: $V\times V\to\mathbb{R}$ is *bilinear*, *continuous* and V-*elliptic*,

(2) L: $V\to\mathbb{R}$ is *linear* and *continuous*.

We consider now a family of finite dimensional subspaces $V^0\subset V^1\subset V^2\subset\ldots\subset V^J\subset V$. The idea here is to approximate (3.13) by

$$(3.14)\qquad \begin{cases} \textit{Find } u^J\in V^J \textit{ such that} \\ \\ a_J(u^J, v)=L_J(v), \ \forall\, v\in V^J, \end{cases}$$

where a_J and L_J are approximations to $a(\cdot,\cdot)$ and L respectively (for those applications associated to mixed finite element approximations, a_J and L_J are never the restrictions of $a(\cdot,\cdot)$ and L to $V\times V$ and V respectively).

The basic principle of all multilevel methods is to solve (3.14) using solutions of problems of the form (3.14) defined on V^j, $j=0, 1, \ldots J-1$. A classical way to handle this is to use a V-cycle multilevel method [5, 6, 7, 8]. For problem (3.14) the V-cycle with J levels takes the following form:

Step 0: *Suppose that* $u_n^J\in V^J$ *is known.*

Step 1: *Starting from* u_n^J, *iterate* ν_J *steps of some iterative method and call the result* u_n^{*J}.

Step 2: *Now for* $j=J-1, \ldots, 1$, *assuming that* u_n^{*j+1} *is known and starting from* 0 *perform* ν_j *steps of some iterative procedure for solving the following variational residual equation*

$$a_j(u_n^j, v) = L_J(v) - \sum_{l=J}^{j+1} a_l(u_n^{*l}, v), \ \forall\ v \in V^j, \qquad (3.15)$$

$$u_n^j \in V^j.$$

Call u_n^{*j} *the result of this smoothing.*

Step 3: *For* $j=0$ *solve exactly the residual equation* (3.15). *Set* $u_n^{po} = u_n^o$.

Step 4: *For* $j=1, 2, \ldots, J$, *assuming* $u_n^{p\,j-1}$ *is known, take* $u_n^{p\,j-1} + u_n^{*j}$ *as an initial condition. Perform* μ_j *steps of some iterative procedure for solving* (3.15). *Call the result* u_n^{pj}.

Step 5: *Take* $u_{n+1}^J = u_n^{pJ}$.

3.3.3 Application of the V-cycle Method to the Solution of Problem (3.11).

Problem (3.11) is a particular case of problem (3.14). Thus, it can be solved by the multilevel method described in Section 3.3.2. Once the basic iterative methods involved in the V-cycle have been specified, thus applying the above multilevel method is canonical.

The numerical results discussed in Section 4 have been obtained using conjugate gradient as a smoother in Steps 1 and 2, taking $\nu_j = 2$. For $j=0$ we also used conjugate gradient to obtain u_n^o. In Step 4 we employed one iteration of steepest descent.

The conjugate gradient algorithm for solving problem (3.11) is described in Section 4 of [1].

4. Numerical Results

In this section we shall present the results of numerical experiments where the mixed element/multi-level domain decomposition methods described in Section 2.3 have been applied to the solution of test problems. The examples considered here include both some standard cases as well as physical problems arising in flow in porous media, such as (1.1)-(1.3) of Section 1. In all our examples, the discrete problem (2.11) approximating the elliptic problem (2.1) has been obtained using for W^h and V^h the Raviart-Thomas mixed finite element spaces. A full description of these elements can be found in [1] and [2]; however for completeness we shall describe these spaces in the following Section 4.1.

4.1 Mixed Finite Element Approximations of Problem (2.1).

Let Ω be the rectangular domain $(0, x_L)\times(0, y_L)$ and let Δ_x: $0=x_0<x_1<\ldots<x_{N_x}=x_L$ and Δ_y: $0=y_0<y_1<\ldots<y_{N_y}=y_L$ define partitions of $[0, x_L]$ and $[0, y_L]$, respectively. For Δ a partition, define the piecewise polynomial space

$$M_s^r(\Delta)=\{v\epsilon C^s([0, L]): \text{ v } \textit{is a polynomial of degree} \le r \textit{ on each subinterval of } \Delta\},$$

where $s=-1$ refers to the discontinuous functions. We define now the following approximations of $L^2(\Omega)$, $H(\Omega; \text{div})$ and V_o respectively

$$W_h^{s,r}=M_s^r(\Delta_x)\otimes M_s^r(\Delta_y),$$

$$V_h^{s,r}=\left[M_{s+1}^{r+1}(\Delta_x)\otimes M_s^r(\Delta_y)\right]\times\left[M_s^r(\Delta_x)\otimes M_{s+1}^{r+1}(\Delta_y)\right],$$

$$V_{h,o}^{s,r}=V_h^{s,r}\cap\left\{v: \ v\cdot\nu=0 \ on \ \partial\Omega\right\},$$

where $h=\max_{i,j}\left\{(x_{i+1}-x_i), (y_{j+1}-y_j)\right\}$. We remark that these spaces satisfy

$$\nabla\cdot v \ \epsilon W_h^{s,r}, \ \forall \ v\epsilon V_h^{s,r} (i.e. \ \nabla\cdot V_h^{s,r}\subset W_h^{s,r}).$$

In our numerical experiments we set $r=1$.

4.2. Solution of Standard Test Problems

Motivated by applications in reservoir engineering we are considering now the following class of test problems:

$$(4.1) \qquad \begin{cases} -\nabla\cdot(A\nabla p)=\delta_{(1,0)}-\delta_{(0,1)}, \\ A\nabla p\cdot\nu=0 \ on \ \partial\Omega, \end{cases}$$

where $\Omega=(0, 1)^2$ and where A is defined by either

(i) $A = A_1 = I$,

or

(ii) $A = A_2 = \frac{1}{1+100(x^2+y^2)} I$,

or

(iii) $A = A_3 = \alpha I$, where $\alpha = 100$ if $0 \leq x \leq .5$ and $\alpha = 1$ if $.5 < x \leq 1$.

The partitionings of Ω used to implement the domain decomposition are those shown in Section 8 of [1]. In particular a (N_x, N_y) decomposition involves a partitioning into $N_x N_y$ rectangular subdomains whose edges are parallel to the coordinate axis.

Table 4.1 depicts the number of multi-level V cycles versus mesh and subdomain partitions:

Coefficient	h^{-1}	(#Subdomains, #V cycles)
A_1	20	(4, 6)
	40	(4, 6); (16, 7)
	80	(4, 9); (16, 8); (64, 7)
A_2	20	(4, 6)
	40	(4, 8); (16, 7)
	80	(4, 10); (16, 8); (64, 7)
A_3	20	(4, 7)
	40	(4, 6); (16, 7)
	80	(4, 10); (16, 8); (64, 7)

Number of Cycles versus Mesh Size and Subdomain Partition for the 3-Level V-Cycle.

Table 4.1

Interestingly the above table applies for the three cases (i)−(iii). We also observe that the number of grid points by subdomain is the same for the three decompositions considered and that the number of V cycles is practically independent of h despite the fact that the dimension of the interface problem is growing like h^{-1}.

To further illustrate the efficiency of the above methods we are providing in Table 4.2 below the dimensions of the various finite element and boundary spaces involved in our combined domain decomposition/mixed finite elements methodology (below, γ is defined by an N × M decomposition).

h^{-1}	Dim W^h	Dim V^h	Dim $V^o_{oh,\gamma}$
20	1600	3120	40 (N + M) − 79 −NM
40	6400	12640	80 (N + M) −159 −NM
80	25600	50800	160 (N + M) −319 −NM

Dimension of the Discrete Spaces

Table 4.2

This insensibility to the smooth or fast variation of coefficient A over Ω is a remarkable property which shows that this methodology has attractive potential for the solution of badly conditioned practical problem such as geostatistics problems arising in porous media ([9, 10].)

The above results represent a substantial improvement in terms of robustness and speedup compared to the results obtained in [1] for the same test problems with the same grids and decompositions.

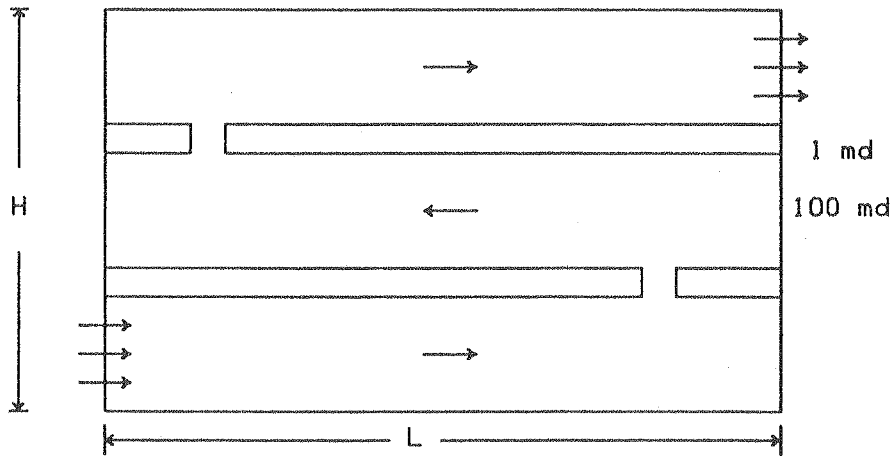
Another interesting property of the above methodology (already observed in [1]) is that the subdomain problems need not be solved exactly. We also observed, concerning the multilevel solution of the matching problem, that one to two V cycles are sufficient in practice to achieve the solution within truncation error; in particular, with $\nu_j = \mu_j = 2$ in the algorithm of Section 3.3.2, the initial residual is reduced by six orders of magnitude in six to seven iterations, the largest reduction taking place in the first V-cycle.

4.3. Solution of Real-Life Test Problems.

To be honest the test cases discussed here are more relevant to [1] since the domain decomposition methodology is exactly the one described in the above reference, i.e. without-yet-multilevel speedup. Nevertheless, we have inserted these problems because they are typical of real-life applications in petroleum reservoir engineering. Also they provide significant benchmarks for elliptic solvers of various types.

This first problem to be considered was communicated to us by petroleum reservoir engineers. It is a model for a discrete shale barrier and involves solving (1.1)-(1.3) where A is visualized in Figure 4.1, where we have used different scales for L and H since L is of the order of 300 feet and H is of the order of 20 feet implying an aspect ratio of 15. Also the thickness of the barrier is of order one foot. The ratio of permeability coefficients is 10^2.

DISCRETE SHALE BARRIER PROBLEM



Geometry of the Discrete Shale Barrier Problem

Figure 4.1

The arrows in Figure 4.1 indicate the flow direction.

Concerning the numerical solution of the above problem we have been using a 40×40 finite element grid and a (2, 2) domain decomposition. For comparison purposes we have treated the cases with aspect ratios 1 and 15.

Using the domain decomposition algorithm discussed in [1] we need 33 iterations if $R=1$ and 48 if $R=15$. We can expect the number of iterations to be practically independent of R once our V

In the same vein the second problem is also a real life problem (1.1)-(1.3) where $A=k(x, y)/\mu(c)$ and

$$\mu(c)=c\mu_1^{-1/4}+(1-c)\mu_2^{-1/4},$$

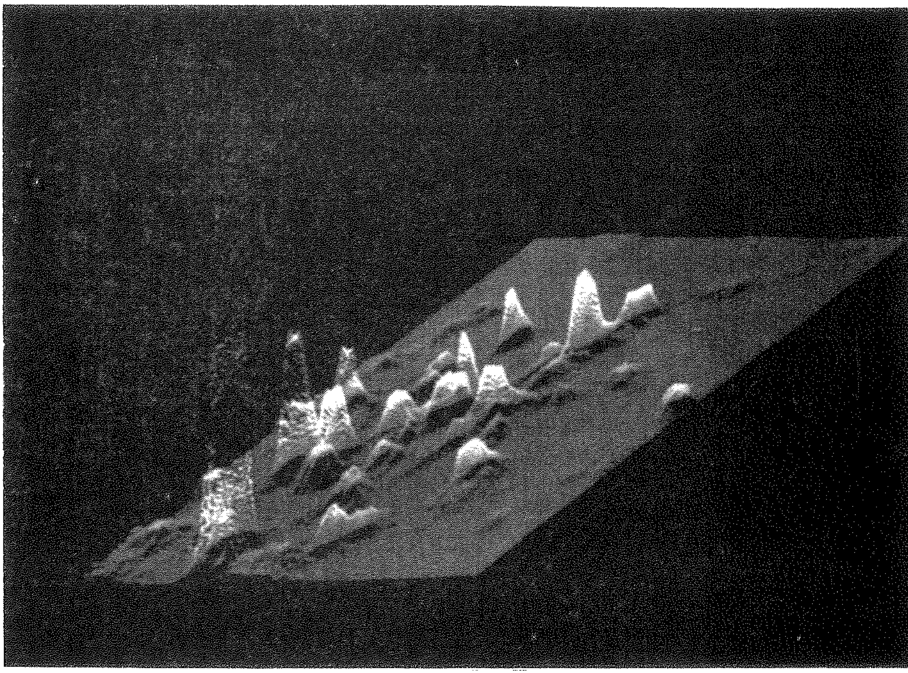
with $\mu_1,\mu_2>0$.

Applying the domain decomposition-mixed finite element methods of [1] to the above problem, with a 80×80 finite element grid and a (10, 10) domain decomposition, the solution was obtained in 9 conjugate gradient iterations. This represents a substantial improvement over a preconditioned conjugate gradient solution of the same discrete problem (without domain decomposition) since the convergence was requiring then about 150 iteration, (taking advantage of a good unital guess). Incidentally the lowest order Raviart-Thomas space ($r=0$ (in 4.1)) or cell-centered finite differences [11] do not work well on this type of problems due to the impossibility for these low order approximations to reproduce correctly flows which are not parallel to the coordinate axes; this drawback disappears if we chose $r=1$.

In Figure 4.2 we have visualized the permeability k(x, y), this data was measured by researchers at Atlantic Richfield Corporation and kindly communicated to us. Similarly the function $A=k/\mu$ is visualized in Figure 4.3.

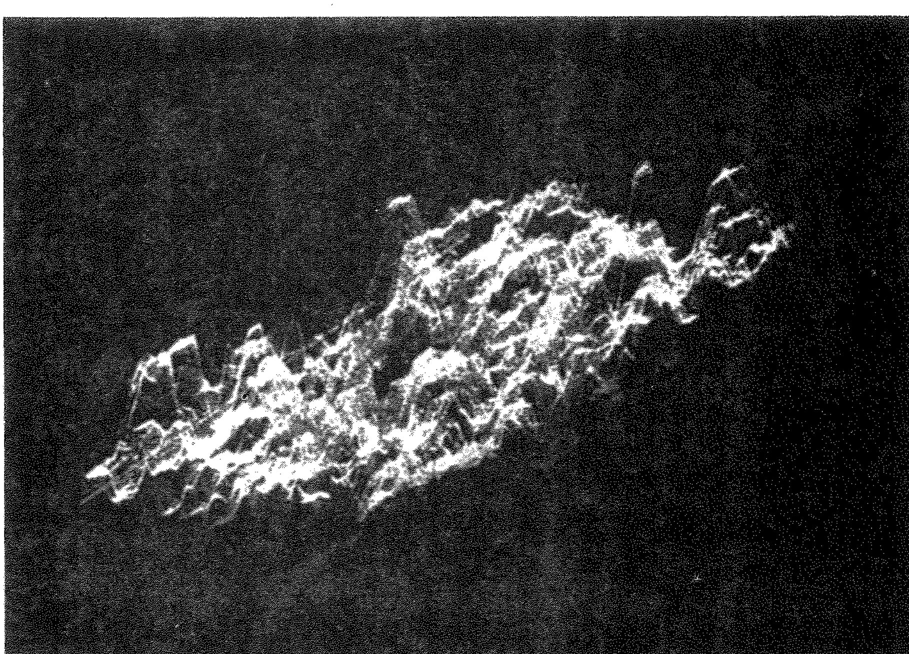
5. Mesh Refinements Via Domain Decomposition

Mesh refinements are necessary when strong gradients arise locally. In view of saving computer storage and avoiding complicated data structures it is interesting to incorporate local grid refinement over subdomains where the strong variations are arising and retain coarser grids elsewhere. The concept of domain decomposition provides an elegant and systematic way to implement the above ideas. In this section we would like to present a particular implementation of our scheme, new to our knowledge, relying again on a combination of Raviart-Thomas mixed finite element and domain decomposition methods.



Representation of k(x, y)

Figure 4.2



Representation of $A=k/\mu$.

Figure 4.3

5.1 Mesh Refinement Via a Modified Raneart-Thomas Mixed Finite Element Method

Consider the situation depicted in Figure 5.1 where a local refinement is necessary in a subregion Ω^* of Ω. The basic idea is to employ essentially mixed finite elements of Raviart-Thomas type inside and outside subregion Ω^*; the main issue here is clearly the matching between the "fine" and "coarse" approximations. To realize this matching we introduce the following finite dimensional spaces of mixed type.

Let $\Omega^*=(a^*, b^*)\times(c^*, d^*)$ and define Δ_x^* and Δ_y^* be partitions of $[a^*, b^*]$ and $[c^*, d^*]$, respectively. Generalizing the notation of Section 4.1, we denote by

$$W_{h^*}^{-1, r^*}(\Omega^*)=M_{-1}^{r^*}(\Delta x^*) \otimes M_{-1}^{r^*}(\Delta y^*), \tag{5.1}$$

$$V_{h^*}^{*-1, r^*}(\Omega^*)=(M_0^{r^*+1}(\Delta x^*) \otimes M_{-1}^{r^*}(\Delta y^*)) \times (M_{-1}^{r^*}(\Delta x^*) \otimes M_0^{r^*+1}(\Delta y^*)), \tag{5.2}$$

and

$$V_{h^*,0}^{*-1, r^*}(\Omega^*)=V_h^{*-1, r^*}(\Omega^*) \cap \{q : q\cdot\nu=0 \textit{ on } \partial\Omega^*\}. \tag{5.3}$$

Similarly we define the corresponding "coarse" spaces by

$$W_h^{-1, r}(\Omega-\Omega^*)=W_h^{-1, r}\big|_{\Omega-\Omega^*}, \tag{5.4}$$

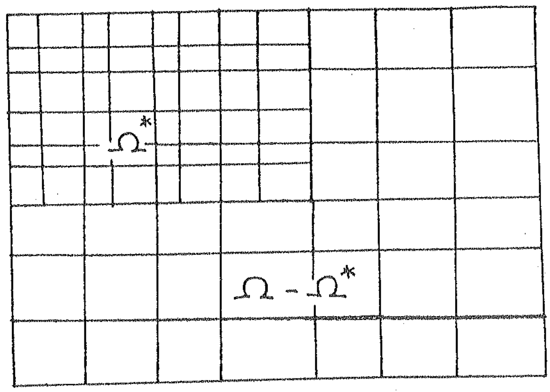


Figure 5.1

and

(5.5) $$V_h^{-1,r}(\Omega-\Omega^*)=V_h^{-1,r}\,|_{\Omega-\Omega^*},$$

with $W_h^{-1,r}$ and $V_h^{-1,r}$ as defined in Section 4.1. We set

(5.6) $$W_h^R=W_{h^*}^{-1,r^*}(\Omega^*)\cup W_h^{-1,r}(\Omega-\Omega^*),$$

(5.7) $$V_h^R=V_{h^*}^{-1,r^*}(\Omega^*)\cup V_h^{-1,r}(\Omega-\Omega^*),$$

(5.8) $$V_{h,0}^R=V_h^R\cap\{q:\ q\cdot\nu=0 \text{ on } \partial\Omega\}.$$

Strictly speaking W_h^R and V_h^R are not Raviart-Thomas spaces, however, they share the same approximating properties which include div $V_h^R\subset W_h^R$ and the order approximation is the same if $r^*=r$.

From a computational point of view this refinement technique is well suited for domain decomposition with Ω^* and Ω-Ω^* as subdomains.

The above approach is well suited for a multi-level solution of problem (2.1) in which we shall use different number of grid levels in the subdomains (usually more grid levels in the more refined regions). Domain decomposition allow a lot of flexibility by the fact that in one of the phases of their realization they decouple the computation to be done in each subdomain.

6. Conclusions

From the numerical results described in this paper the combination of mixed finite element, domain decomposition and multilevel methods discussed in Sections 2, 3 and 4 provides a robust, accurate and fast technique for solving elliptic problem with non-smooth coefficients like those arising in flow in porous media and other applications from Mechanics and Physics.

These methods are quite interesting from a parallel computing point of view since the ratio

$$\frac{\text{Work in Solving Subdomain Problems}}{\text{Communication Costs}}$$

is of order $0(h^{-1})$.

Here the communication involves the transfer of the boundary data at the subdomain interfaces.

We are presently cooperating with the computer scientists at the National Science Foundation Center for Research in Parallel Computation in the parallel implementation of the methods discussed in this paper.

Acknowledgement. We acknowledge the support of Cray Research, the National Science Foundation under Grants DMS8814841, DMS8822522 , and INT8612680 and the Air Force Office of Scientific Research under Grant AF0SR-89-0025 . Additional acknowledgements are given to Clint Dawson, Todd Dupont, David Moissis, and Steve Poole for help and suggestions and to Lasonya Jones for her diligence in processing this paper.

References

[1] R. GLOWINSKI and M. F. WHEELER, Domain Decomposition and Mixed Finite Element Methods for Elliptic Problems, in *Domain Decomposition Methods for Partial Differential Equations*, R. Glowinski, G. H. Golub, G. Meurant, J. Periaux eds., SIAM, Philadelphia, 1988, pp. 144-172.

[2] P. A. RAVIART and J. M. THOMAS, A Mixed Finite Element Method for Second Order Elliptic Problems, in *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 1977.

[3] J. E. ROBERTS and J. M. THOMAS, Mixed Finite Element Method, in *Numerical Analysis Handbook*, P. G. Ciarlet, J. L. Lions eds., North-Holland, Amsterdam (to appear).

[4] J. DOUGLAS JR. and J. E. ROBERTS, Global estimates for Mixed finite element methods for second order elliptic equations, *Math. Comp.* 44, (1985), pp. 39-52.

[5] W. HACKBUSH, *Multigrid Methods and Applications*, Springer-Verlag, Berlin, 1985.

[6] W. HACKBUSH and U. TROTTENBERG (eds.), *Multigrid Methods*, Lecture Notes in Mathematics, Vol. 960, Springer-Verlag, Berlin, 1982.

[7] S. F. McCORMICK (ed.), *Multigrid Methods*, SIAM, Philadelphia, 1987.

[8] J. H. BRAMBLE, J. E.

[9] D. MOISSIS, *Simulation of Viscous Fingering During Miscible Displacements in Nonuniform Porous Media*, Ph.D. Dissertation, Rice University, Houston, 1988.

[10] D. MOISSIS, C. A. MILLER and M. F. WHEELER, A parametric study of viscous fingering in miscible displacement by numerical simulation, in *Numerical Simulation in Oil Recovery*, M. F. Wheeler ed. Springer-Verlag, 1988, pp. 228-249.