Acceleration of Domain Decomposition Algorithms for Mixed Finite Elements by Multi-Level Methods

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Abstract. In this paper we consider the numerical solution of elliptic partial differential equations by a combination of domain decomposition algorithms, mixed finite element methods and multi-level procedures. The multi-level procedures are used to accelerate convergence of the algorithm which iteratively adjusts the matching conditions at the interfaces of the subdomains. Numerical results are included in this paper which exhibit improvements in convergence by applying this multi-level approach, compared to more traditional iterative methods.

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0. Introduction. In [1] Glowinski and Wheeler defined domain decomposition algorithms for solving mixed finite element approximations of elliptic problems with non-constant coefficients. A key result in [1] was the formulation of the matching conditions at the interfaces of the subdomains as variational problems defined over convenient trace space. These new problems were solved by conjugate gradient algorithms using simple preconditioners resulting in a \( O(h^{-5}) \) number of iterations to achieve convergence. In this paper we shall discuss a procedure for accelerating the convergence of the above algorithms which is essentially based on a multi-level technique acting on the trace space associated to the interfaces.

In Section 1, we shall give some examples of elliptic problems originating from flow in porous media. Compared to more traditional solution methods the algorithm described in this paper have been quite successful as we shall demonstrate in Section 4. In Section 2 which follows closely [1] we shall recall the mixed variational formulation of elliptic problems, the mixed finite element approximations and the associated domain decomposition methods. In Section 3 we shall discuss a multilevel method to speed up convergence of the domain decomposition algorithms discussed in Section 2. Results of numerical experiments will be discussed in Section 4. Finally some mesh refinement methods well suited for domain decomposition and mixed finite element methods will be discussed in Section 5.

1. Motivation for Robust Elliptic Solvers.

In our first example we consider the pressure equation which arises from miscible displacements in porous media. The equation has the form

\[
\begin{align*}
(1.1) & \quad u = -A \text{ grad } p \text{ in } \Omega, \\
(1.2) & \quad \nabla \cdot u = q \text{ in } \Omega, \\
(1.3) & \quad \nu \cdot u = 0 \text{ on } \partial \Omega,
\end{align*}
\]

where

\[ A = k(x, y)/\mu(c). \]
In this problem $\Omega$ is the flow region, $u$ is the Darcy velocity, $p$ is the pressure, $q$ is a source or sink term, $k$ is the permeability of the porous media, $\mu$ is the viscosity of the concentration $c$ of the fluid which is flowing through the porous media. In this example we use a permeability field and a form of the viscosity which has been previously obtained from laboratory experiments. In Figure 1.1, a visualization of $A$ is shown. In this case we have

$$\max A = 810 \times 10^{-2} \quad \text{and} \quad \min A = 282 \times 10^{-3},$$

implying that (1.1)-(1.3) is badly conditioned. However, as it will be seen with more detail in Section 4, we have been able to solve this problem, using domain decomposition, in less than 10 iterations.

![Variation of coefficient $A$](image)

Figure 1.1

2.1 The Model Problem.

We consider on $\Omega \subset \mathbb{R}^n$ the following Neumann problem

\[
\begin{cases}
-\nabla \cdot A \nabla p = f & \text{in } \Omega, \\
A \nabla p \cdot \nu = g & \text{on } \partial \Omega (= \Gamma),
\end{cases}
\]

(2.1)

where $\nu$ is the outward normal vector. We assume the compatibility condition

\[
\int_{\Omega} f \, dx + \int_{\Gamma} g \, d\Gamma = 0.
\]

(2.2)

Our formalism is motivated from flow in porous media where (2.1) is the pressure equation, but the method to be described applies to other branches of science and engineering. Also we have been considering the pure Neumann problem since it is the one occurring most frequently in applications. In fact, it is also the most difficult case.

2.2 A Mixed Variational Formulation of Problem (2.1)

Define $u$ by

\[
u = -A \nabla p.
\]

(2.3)

We then have

\[
\nabla \cdot u - f = 0,
\]

(2.4)

and

\[
\text{Multiply by } a
\]

(2.5)

\[
\text{and}
\]

(2.6)

\[
\text{where}
\]

(2.7)

\[
\text{Here}
\]

(2.8)

\[
\text{with } a
\]

(2.9)
\[ (2.5) \quad \nabla p = -A^{-1} u. \]

Multiplying (2.4) and (2.5) by \( q \) and \( v \) respectively, we obtain

\[ (2.6) \quad \int_{\Omega} (\nabla \cdot u - q) q dx = 0, \quad \forall q \in L^2(\Omega), \]

and

\[ (2.7) \quad \int_{\Omega} A^{-1} u \cdot v dx - \int_{\Omega} p \nabla \cdot v dx = 0, \quad \forall v \in V_o, \]

where

\[ (2.8) \quad V_o = \{ v \mid v \in H(\Omega, \text{div}), \quad v \cdot n = 0 \text{ on } \Gamma \}. \]

Here

\[ (2.9) \quad H(\Omega; \text{div}) = \{ v \in (L^2(\Omega))^p \text{ and } \text{div } v \in L^2(\Omega) \}. \]

Suppose \( f \in L^2(\Omega), \quad g \in H^{-1/2}(\Gamma) \) and \( A \) is symmetric such that \( A \in C^0(\Omega) \) and

\[ A(x) \xi \cdot \xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^p, \text{ a. e. on } \Omega, \]

with \( \alpha \) a positive constant.

If (2.2) holds then (2.1) has a unique solution on \( H^1(\Omega)/\mathbb{R} \) implying the uniqueness of \( u \). An alternative formulation of (2.1) is given by

\[ \text{Find } p \in L^2(\Omega), \quad u \in H(\Omega; \text{div}), \text{ such that} \]
Let $u \cdot \nu + g = 0$ on $\Gamma$,

$$\int_{\Omega} (\nabla \cdot u - f) \, q \, dx = 0, \quad \forall \, q \in L^2(\Omega),$$

(2.10)

$$\int_{\Omega} A^{-1} u \cdot \nu \, dx - \int_{\Omega} p \, \nabla \cdot v \, dx = 0, \quad \forall \, v \in V_0.$$

2.3 Finite Element Approximation of Problem (2.10).

We denote by $W^h$ and $V^h$ finite dimensional subspaces of $L^2(\Omega)$ and $H(\Omega; \text{div})$, respectively. In addition we set $V_0^h = V^h \cap V_0$. We shall assume that $\text{div} \, V^h \subset W^h$.

It is natural then to approximate problem (2.1), using its mixed equivalent formulation, by

Find $p_h \in W^h$, $u_h \in V^h$ satisfying

$$\int_{\Gamma} (u_h \cdot \nu + g) \nu \cdot \nu \, d\Gamma = 0, \quad \forall \, V^h,$$

(2.11)

$$\int_{\Omega} (\nabla \cdot u_h - f) q \, dx = 0, \quad \forall \, q \in W^h,$$

$$\int_{\Omega} A^{-1} u_h \cdot \nu \, dx - \int_{\Omega} p_h \nabla \cdot v \, dx = 0, \quad \forall \, v \in V_0^h.$$

Examples of particular finite element spaces for which (2.11) is well posed and for which $\lim_{h \to 0} u_h = u$ and $\lim_{h \to 0} p_h = p$ can be found in [2]. Additional convergence results including error estimates can be found in [3, 4].

2.4 Domain Decomposition Method for Problem (2.1), (2.11).

We follow here the notation and methodology developed in [1]. Considering first the continuous problem whose formula is much simpler we suppose that $\Omega$ has been decomposed in two subdomains $\Omega_1$ and $\Omega_2$.
subdomains $\Omega_1$ and $\Omega_2$. Figures 2.1a and 2.1b show such domain decompositions and corresponding notation.

If we denote by $\{p_i, u_i\}$ the restriction of $\{p, u\}$ to $\Omega_i$ there is equivalence between (2.10) and

\[
\begin{align*}
\int_{\Omega_1} (\nabla \cdot u_i - f) \, q_i \, dx &= 0, \quad \forall \, q_i \in L^2(\Omega_1), \\
\int_{\Omega_1} \left( A^{-1} u_i \cdot v_i - p_i \nabla \cdot v_i \right) \, dx &= 0, \quad \forall \, v_i \in V_{i0}, \; i = 1, 2,
\end{align*}
\]

and for which

\[
\begin{align*}
\sum_{i=1}^{2} u_i \cdot v_i &= 0 \quad \text{on} \; \gamma, \\
\sum_{i=1}^{2} p_i &= 0 \quad \text{on} \; \gamma,
\end{align*}
\]

with

\[
\begin{align*}
\int_{\Omega_1} \left( A^{-1} u_i \cdot v_i - p_i \nabla \cdot v_i \right) \, dx &= 0, \quad \forall \, v_i \in V_0,
\end{align*}
\]
\[ V_{10} = \{ v_i | v_i \in H(\Omega_i, \text{div}), v_i \cdot n_i = 0 \text{ on } \partial \Omega_i \}. \]  

Since \( V_{\alpha} = V_{10} \oplus V_{20} \oplus V_{\gamma_0} \) (where \( V_{\gamma_0} \) is a complementary subspace of \( V_{10} \oplus V_{20} \) in \( V_{\alpha} \)) it follows from (2.12) and (2.15) that (2.15) can be replaced by

\[ \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot v - p_i \nabla \cdot v) \, dx = 0, \quad \forall v \in V_{\gamma_0}. \]  

In addition to (2.12)-(2.14), \( \{ p_i, u_i \} \) must satisfy the compatibility condition

\[ \int_{\Omega_i} f \, dx + \int_{\partial \Omega_i \cap \Gamma} g \, d\Gamma + \int_{\gamma} u_i \cdot n_i \, d\gamma = 0. \]  

From (2.12)-(2.14), the local solutions satisfy at the interface \( \gamma \) the matching conditions (2.14) and (2.16). From this observation we can generate two classes (at least) of iterative methods for solving problem (2.11) by domain decomposition. In both approaches we assume that one of the matching conditions is satisfied by an appropriate choice of boundary conditions on \( \gamma \) and we try iteratively to satisfy the other matching condition. In this paper we shall concentrate on the case where the balance given by (2.14) is satisfied; we try therefore to verify (2.16).

This leads to the introduction of a variational problem involving functional spaces defined on \( \gamma \). Precisely such a functional space is \( V_{\gamma_0}^{\Omega} \) defined by

\[ V_{\gamma_0}^{\Omega} = \{ \mu | \mu \in V_{\gamma_0}, \int_{\gamma} \mu \cdot \nu \, d\gamma = 0 \}. \]  

We define a bilinear form \( a(\cdot, \cdot) \) over \( V_{\gamma_0}^{\Omega} \times V_{\gamma_0}^{\Omega} \) as follows:

\[ \text{Consider } \mu \in V_{\gamma_0}^{\Omega}; \text{ we associate to } \mu, u_i(\mu) \text{ and } p_i(\mu) \text{ by solving} \]
\[(2.19) \quad \int_{\Omega_1} \nabla \cdot u_1(\mu) v_1 \, dx = 0, \quad \forall v_1 \in L^2(\Omega_1),\]

\[(2.20) \quad \int_{\Omega_1} (A^{-1} u_1(\mu) \cdot v_1 - p_1(\mu) \nabla \cdot v_1) \, dx = 0, \quad \forall v_1 \in V_{10},\]

\[(2.21) \quad u_1(\mu) \cdot v_1 = 0 \quad \text{on} \quad \Gamma \cap \partial \Omega_1, \quad u_1(\mu) \cdot v_1 = \mu \cdot v_1 \quad \text{on} \quad \gamma.\]

Since \( \int_{\partial \Omega_1} u_1(\mu) \cdot v_1 \, d\Gamma = 0, \) the above problem is well posed in \( H(\Omega_1; \text{div}) \times L^2(\Omega_1)/R. \) To insure uniqueness of \( p_1(\mu) \) we enforce the conditions

\[(2.22) \quad \int_{\Omega_1} p_1(\mu) \, dx = 0, \quad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_1(\mu) \cdot v_1 - p_1(\mu) \nabla \cdot v_1) \, dx = 0\]

where \( \Pi(V_{\gamma 0} - V_{\gamma 0}^\circ). \) Finally we define \( a(\cdot, \cdot) \) by

\[(2.23) \quad a(\mu, \mu') = \sum_{i=1}^{2} \int_{\Omega_i} \left( A^{-1} u_1(\mu) \cdot \mu' - p_1(\mu) \nabla \cdot \mu' \right) \, dx, \quad \forall \mu, \mu' \in V_{\gamma 0}^\circ.\]

It has been shown in [1] that the bilinear form \( a(\cdot, \cdot) \) is symmetric and positive semi-definite over \( V_{\gamma 0}^\circ \times V_{\gamma 0}^\circ. \) Moreover, it is elliptic for the norm induced by \( H(\Omega; \text{div}) \) over the quotient space \( V_{\gamma 0}^\circ / \hat{R}, \)

where \( \hat{R} \) is the equivalence relation defined by \( \mu \hat{R} \mu' \rightarrow (\mu - \mu') \cdot \nu = 0 \) on \( \gamma. \)

From the above result we can interpret (2.12)-(2.17) as a linear variational problem in \( V_{\gamma 0}^\circ. \)

To formulate this latter problem consider \( \lambda_0 \in H(\Omega; \text{div}) \) such that

\[(2.24) \quad \lambda_0 \cdot \nu + g = 0 \quad \text{on} \quad \Gamma,\]

\[(2.25) \quad \int_{\Omega_1} f \, dx + \int_{\Gamma \cap \partial \Omega_i} g \, d\Gamma + \int_{\gamma} \lambda_0 \cdot v_1 \, d\gamma = 0, \quad \forall \ i = 1, 2;\]
solve then for $i=1, 2,$

\begin{align}
(2.26) & \int_{\Omega_i} (\nabla \cdot u_{oi} - \varphi_i) \, dx = 0, \quad \forall \varphi_i \in L^2(\Omega_i), \\
(2.27) & \int_{\Omega_i} (A^{-1} u_{oi} \cdot \nu_i - p_{oi} \cdot \nu_i) \, dx = 0, \quad \forall \nu_i \in V_i, \\
(2.28) & u_{oi} \cdot \nu_i + g = 0 \text{ on } \gamma \cap \partial \Omega_i, \\
(2.29) & u_{oi} \cdot \nu_i = \lambda_0 \cdot \nu_i \text{ on } \gamma.
\end{align}

The constants associated to the $p_{oi}$ are adjusted as follows:

\begin{align}
(2.30) & \int_{\Omega_i} p_{oi} \, dx = 0, \\
(2.31) & \sum_{i=1}^2 \int_{\Omega_i} (A^{-1} u_{oi} \cdot \nu_i - p_{oi} \cdot \nu_i) \, dx = 0.
\end{align}

Let us now denote by $u_0$ the element of $H(\Omega; \text{div})$ such that $u_0|_{\Omega_i} = u_{oi}$. If we define $\bar{u}$ by

\begin{align}
(2.32) & \bar{u} = u - u_0,
\end{align}

we clearly have $\bar{u} \in V_0$. Denoting $\bar{x} \in V_{\gamma_0}$ as the component of $\bar{u}$ in the decomposition

\begin{align}
V_0 = V_{10} \oplus V_{20} \oplus V_{\gamma_0}
\end{align}

we have from (2.17), (2.25), (2.28), (2.29) that

\begin{align}
(2.33) & \int_{\gamma} \bar{x} \cdot \nu_i \, d\gamma = 0, \text{ i.e. } \bar{x} \in V_{\gamma_0}^0.
\end{align}

We have then

\begin{align}
(2.34) & \\
(2.35) & \\
(2.36) & \\
(2.37) & \\
(2.38) & \text{ from the def.} \\
(2.39) & \\
(2.40) & \text{ from the def.}
\end{align}
define similarly $\bar{p}_i$ by $\bar{p}_i = p_i - p_{oi}$.

We have then

\begin{equation}
\int_{\Omega_i} \nabla \cdot q_i d\Omega = 0, \quad \forall q_i \in L^2(\Omega_i),
\end{equation}

\begin{equation}
\int_{\Omega_i} (A^{-1} \bar{u}_i \cdot \nabla \cdot \bar{p}_i) d\Omega = 0, \quad \forall \nu_i \in V_{\gamma_0},
\end{equation}

\begin{equation}
\bar{u}_i \cdot \nu_i = 0 \quad \text{on } \partial \Omega_i \cap \Gamma, \quad \bar{u}_i \cdot \nu_i = \bar{\nu}_i \cdot \nu_i \quad \text{on } \gamma,
\end{equation}

\begin{equation}
\int_{\Omega_i} \bar{p}_i d\Omega = 0, \quad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot \nabla \cdot \bar{p}_i) d\Omega = 0.
\end{equation}

It follows from (2.16) that

\begin{equation}
\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot \mu - p_i \nabla \cdot \mu) d\Omega = 0, \quad \forall \mu \in V_{\gamma_0}^0.
\end{equation}

From the definition of $\bar{u}_i$, $p_i$ and from (2.38) we obtain

\begin{equation}
\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot \mu - p_i \nabla \cdot \mu) d\Omega = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi} \nabla \cdot \mu) d\Omega, \quad \forall \mu \in V_{\gamma_0}^0.
\end{equation}

It follows from (2.23) and (2.33) that $\bar{\lambda}$ is the unique solution of the linear variational equation

\begin{equation}
\begin{aligned}
\text{Find } \bar{\lambda} \in V_{\gamma_0}^0 \text{ such that } \\
&= -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi} \nabla \cdot \mu) d\Omega, \quad \forall \mu \in V_{\gamma_0}^0.
\end{aligned}
\end{equation}
In [1], we showed that the variational problem (2.40) can be approximated by a finite dimensional problem of the same nature, obtained by combining the mixed approximation of Section 2.3 with the domain decomposition principle of Section 2.4. In addition, a conjugate gradient method for solving this finite dimensional problem approximating (2.40) was discussed in detail in the above reference.

In the following Section 3, we shall describe multilevel techniques for solving the finite dimensional problem approximating (2.40); it can be seen as a multigrid method operating on the interface \( \gamma \).

3. Multilevel Solution of Problem (2.40).

3.1. Domain Decomposition of the Discrete Problem.

Following Section 2.3, it is easily shown that the discrete mixed problem (2.11) is equivalent to finding \( \{u_{h,i}, p_{h,i}\} \), \( i = 1, 2 \), satisfying

\[
\int_{\Omega_i} (\nabla \cdot u_{h,i} - f) \, q_i \, dx = 0, \quad \forall q_i \in W_{h,i},
\]

\[
\int_{\Omega_i} (A^{-1} u_{h,i} \cdot v_i - p_{h,i} \nabla \cdot v_i) \, dx = 0, \quad \forall v_i \in V_{0,h,i}
\]

\[
\int_{\partial \Omega_i \cap \Gamma} (u_{h,i} \cdot v + g) \, v \, d\Gamma = 0, \quad \forall v \in V_{h,i},
\]

\[
\sum_{i=1}^2 u_{h,i} \cdot v_i = 0 \quad \text{on} \ \gamma,
\]

\[
\sum_{i=1}^2 \int_{\Omega} (A^{-1} u_{h,i} \cdot v - p_{h,i} \nabla \cdot v) \, dx = 0, \quad \forall v \in V_{h,i}
\]

where \( V_{h,i} \) (resp. \( W_{h,i} \)) is equal to \( V_{oh,i} \cap \Omega_i \) (resp. \( W_{h,i} \cap \Omega_i \)). As in the continuous case we associate to \( \gamma \) a complementary subspace \( V_{oh,\gamma} \) of \( V_{oh,1} \oplus V_{oh,2} \) in \( V_{oh} \); that is

\[
\text{It follows from}
\]

\[
\sum_{i=1}^2 \int_{\Omega} (A^{-1} u_{h,i} \cdot v - p_{h,i} \nabla \cdot v) \, dx = 0, \quad \forall v \in V_{h,i}
\]

In addition to

\[
\int_{\Omega} f \, q \, dx = 0, \quad \forall q \in W_{h,i}
\]

Finally

\[
V_h
\]

where

\[
V
\]

and

\[
V_{h,i}
\]

Following the
\[ V_{oh} = V_{oh,1} \oplus V_{oh,2} \oplus V_{oh,\gamma}. \]

It follows from (3.1) and (3.2) that (3.5) can be replaced by

\[ \sum_{i=1}^{2} \int_{\Omega_i} \left( (A^{-1}v_{h,i} \cdot v - p_{h,i} \nabla \cdot v) \ dx = 0, \ \forall v \in V_{oh,\gamma}. \right. \]

In addition to (3.5) and (3.6) \( \{v_{h,i}, p_{h,i}\} \) has to satisfy the compatibility conditions

\[ \int_{\Omega_i} \frac{f}{\partial t} + \int_{\partial \Omega_i \cap \Gamma} g d\gamma + \int_{\partial \Omega_i \cap \Gamma} u_{h,i} \cdot \nu d\gamma = 0, \ i=1, 2. \]

Finally we decompose \( V_{oh,\gamma} \) as the direct sum,

\[ V_{oh,\gamma} = V_{oh,\gamma}^{0} \oplus V_{oh,\gamma}^{\Pi}. \]

where

\[ V_{oh,\gamma}^{0} = \{ \gamma \in \Gamma_{oh,\gamma} \mid \int_{\gamma} z \cdot d\gamma = 0 \}, \]

and

\[ V_{oh,\gamma}^{\Pi} = \{ \gamma \in \Gamma_{oh,\gamma} \mid \int_{\gamma} \Pi \cdot d\gamma \neq 0 \}. \]

3.2. Discretization of the Boundary Problem (2.40).

Following the development in Section 2.4, we approximate (2.40) by the following variational problem
in \( V_{oh,\gamma}^0 \times V_{oh,\gamma}^\gamma \):

\[
\begin{cases}
\text{Find } \tilde{X}_h \in V_{oh,\gamma}^0 \text{ such that } \\
\alpha_h (\tilde{X}_h, \mu) = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oh,i} \cdot \mu - p_{oh,i} \nabla \cdot \mu) dx, \quad \forall \mu \in V_{oh,\gamma}^\gamma
\end{cases}
\]

(3.11)

where \( \tilde{X}_h, u_{oh,i} \) and \( p_{oh,i} \) are obtained as discrete analogues of \( X, u_{oi} \) and \( p_{oi} \) in Section 2.4 (see [1] for all the details).

3.3. Multilevel Algorithms for Solving Problem (3.11).

3.3.1. Synopsis

We first introduce a discretization parameters \( h_j \) to which we associate all the above discrete spaces. For simplicity we denote by \( Z^j \) the space \( V_{oh,\gamma}^0 \). We assume that the sequence \( \{Z^j\} \) satisfies the following inclusion property

(3.12)

\[
Z^0 \subset Z^1 \subset \ldots \subset Z^J.
\]

At level \( J \) (the finest level) we wish to solve problem (3.11) with \( h = h_J \).

Before defining a multilevel algorithm for solving problem (3.11), we describe in the following Section 3.3.2 the solution of general variational problems by multilevel methods. The application to the specific problem (3.11) will be discussed in Section 3.3.3.

3.3.2. A Multilevel Method for Linear Variational Problem in Hilbert Spaces.

Let \( V \) be a Hilbert space with \((\cdot, \cdot)\) as inner product and \( \|\cdot\|_V \) the corresponding norm. We consider the following problem

\[
\text{where }
\]

\[
\text{We consider the idea here is to }
\]

\[
\text{where a}_j \text{ and } L^j \text{ mixed finite eler V respectively).}
\]

\[
\text{The bas the form (3.14) multilevel metho}
\]

\[
\text{Step 0: Suppose}
\]

\[
\text{Step 1: Starting}
\]

\[
\text{Step 2: New for of sc}
\]
\[ \text{Find } u \in V \text{ such that } \]
\[ a(u, v) = L(v), \quad \forall v \in V, \]
where

1. \( a: V \times V \to \mathbb{R} \) is bilinear, continuous and \( V \)-elliptic,
2. \( L: V \to \mathbb{R} \) is linear and continuous.

We consider now a family of finite dimensional subspaces \( V^0 \subset V^1 \subset V^2 \subset \ldots \subset V^J \subset V \). The idea here is to approximate (3.13) by

\[ \text{Find } u^J \in V^J \text{ such that } \]
\[ a^J(u^J, v) = L_J(v), \quad \forall v \in V^J, \]  

where \( a^J \) and \( L_J \) are approximations to \( a(\cdot, \cdot) \) and \( L \) respectively (for those applications associated to mixed finite element approximations, \( a^J \) and \( L_J \) are never the restrictions of \( a(\cdot, \cdot) \) and \( L \) to \( V \times V \) and \( V \) respectively).

The basic principle of all multilevel methods is to solve (3.14) using solutions of problems of the form (3.14) defined on \( V^j \), \( j = 0, 1, \ldots J - 1 \). A classical way to handle this is to use a \( V \)-cycle multilevel method [5, 6, 7, 8]. For problem (3.14) the \( V \)-cycle with \( J \) levels takes the following form:

**Step 0:** Suppose that \( u^J \in V^J \) is known.

**Step 1:** Starting from \( u^J \), iterate \( V^j \) steps of some iterative method and call the result \( u_n^{aJ} \).

**Step 2:** Now for \( j = J - 1, \ldots, 1 \), assuming that \( u_n^{aJ+1} \) is known and starting from 0 perform \( V^j \) steps of some iterative procedure for solving the following variational residual equation.
\[ a_0^j (u_n, v) = L_0^j(v) - \sum_{l=1}^{j+1} a_l (u_{n+1}^j, v), \forall v \in V^j. \]
\[ u_n^0 = v^0. \]

Call \( u_n^j \) the result of this smoothing.

Step 3: For \( j = 0 \) solve exactly the residual equation (3.15). Set \( u_n^{p0} = u_n^0 \).

Step 4: For \( j = 1, 2, \ldots, J \), assuming \( u_n^{p_j-1} \) is known, take \( u_n^{p_j-1} + u_n^{s_j} \) as an initial condition. Perform \( \mu_j \) steps of some iterative procedure for solving (3.15). Call the result \( u_n^{p_j} \).

Step 5: Take \( u_n^{p_{j+1}} = u_n^{p_j} \).

3.3.3 Application of the V-cycle Method to the Solution of Problem (3.11).

Problem (3.11) is a particular case of problem (3.14). Thus, it can be solved by the multilevel method described in Section 3.3.2. Once the basic iterative methods involved in the V-cycle have been specified, thus applying the above multilevel method is canonical.

The numerical results discussed in Section 4 have been obtained using conjugate gradient as a smoother in Steps 1 and 2, taking \( \nu_j = 2 \). For \( j = 0 \) we also used conjugate gradient to obtain \( u_n^0 \). In Step 4 we employed one iteration of steepest descent.

The conjugate gradient algorithm for solving problem (3.11) is described in Section 4 of [1].

4. Numerical Results

In this section we shall present the results of numerical experiments where the mixed element/multi-level domain decomposition methods described in Section 2.3 have been applied to the solution of test problems. The examples considered here include both some standard cases as well as physical problems arising in flow in porous media, such as (1.1)-(1.3) of Section 1. In all our examples, the discrete problem (2.11) approximating the elliptic problem (2.1) has been obtained using for \( W^h \) and \( V^h \) the Raviart-Thomas mixed finite element spaces. A full description of these elements can be found in [1] and [2]; however for completeness we shall describe these spaces in the following Section 4.1.
4.1 Mixed Finite Element Approximations of Problem (2.1).
Let $\Omega$ be the rectangular domain $(0, x_L) \times (0, y_L)$ and let $\Delta_x$: $0 = x_0 < x_1 < \ldots < x_{N_x} = x_L$ and $\Delta_y$: $0 = y_0 < y_1 < \ldots < y_{N_y} = y_L$ define partitions of $[0, x_L]$ and $[0, y_L]$, respectively. For $\Delta$ a partition, define the piecewise polynomial space

$$M^s_0(\Delta) = \{ v \in C^0([0, 1]) : v \text{ is a polynomial of degree } \leq r \text{ on each subinterval of } \Delta \},$$

where $s = -1$ refers to the discontinuous functions. We define now the following approximations of $L^2(\Omega)$, $H(\Omega; \text{ div})$ and $V_0$ respectively

$$W^{r,r}_h = M^r_0(\Delta_x) \otimes M^r_0(\Delta_y),$$

$$V^{r,r}_h = \left[ M^{r+1}_0(\Delta_x) \otimes M^r_0(\Delta_y) \right] \times \left[ M^r_0(\Delta_x) \otimes M^{r+1}_0(\Delta_y) \right],$$

$$V^{r,r}_{h,0} = V^{r,r}_h \cap \{ v : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \},$$

where $h = \max_{ij} \{ (x_{i+1} - x_i), (y_{j+1} - y_j) \}$. We remark that these spaces satisfy

$$\nabla \cdot v \in W^{r,r}_h, \forall v \in V^{r,r}_h \text{ (i.e. } \nabla \cdot V^{r,r}_h \subset W^{r,r}_h).$$

In our numerical experiments we set $r = 1$.

4.2 Solution of Standard Test Problems

Motivated by applications in reservoir engineering we are considering now the following class of test problems:

$$\begin{cases}
- \nabla \cdot (A \nabla p) = \delta(1, 0)^- \delta(0, 1)^+ \quad & (4.1) \\
A \nabla p \cdot n = 0 \text{ on } \partial \Omega,
\end{cases}$$

where $\Omega = (0, 1)^2$ and where $A$ is defined by either
(i) \( A = A_1 = I \),

or

(ii) \( A = A_2 = \frac{1}{1 + 100(x^2 + y^2)} I \),

or

(iii) \( A = A_3 = \alpha I \), where \( \alpha = 100 \) if \( 0 \leq x \leq 0.5 \) and \( \alpha = 1 \) if \( 0.5 < x \leq 1 \).

The partitionings of \( \Omega \) used to implement the domain decomposition are those shown in Section 8 of [1]. In particular, a \((N_x, N_y)\) decomposition involves a partitioning into \( N_x N_y \) rectangular subdomains whose edges are parallel to the coordinate axis.

Table 4.1 depicts the number of multi-level V cycles versus mesh and subdomain partitions:

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>( h^{-1} )</th>
<th>(#Subdomains, #V cycles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>20</td>
<td>(4, 6)</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>(4, 6); (16, 7)</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>(4, 9); (16, 8); (64, 7)</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>20</td>
<td>(4, 6)</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>(4, 8); (16, 7)</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>(4, 10); (16, 8); (64, 7)</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>20</td>
<td>(4, 7)</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>(4, 6); (16, 7)</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>(4, 10); (16, 8); (64, 7)</td>
</tr>
</tbody>
</table>

Number of Cycles versus Mesh Size and Subdomain Partition for the 3-Level V-Cycle.

Table 4.1
Interestingly the above table applies for the three cases (i)–(iii). We also observe that the number of grid points by subdomain is the same for the three decompositions considered and that the number of \( V \) cycles is practically independent of \( h \) despite the fact that the dimension of the interface problem is growing like \( h^{-1} \).

To further illustrate the efficiency of the above methods we are providing in Table 4.2 below the dimensions of the various finite element and boundary spaces involved in our combined domain decomposition/mixed finite elements methodology (below, \( \gamma \) is defined by an \( N \times M \) decomposition).

<table>
<thead>
<tr>
<th>( h^{-1} )</th>
<th>( \text{Dim } W^h )</th>
<th>( \text{Dim } V^h )</th>
<th>( \text{Dim } V^o_{\text{oh,}\gamma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1600</td>
<td>3120</td>
<td>40 ( (N + M) - 79 ) – NM</td>
</tr>
<tr>
<td>40</td>
<td>6400</td>
<td>12640</td>
<td>80 ( (N + M) - 159 ) – NM</td>
</tr>
<tr>
<td>80</td>
<td>25600</td>
<td>50800</td>
<td>160 ( (N + M) - 319 ) – NM</td>
</tr>
</tbody>
</table>

Dimension of the Discrete Spaces  
Table 4.2
This insensitivity to the smooth or fast variation of coefficient $A$ over $\Omega$ is a remarkable property which shows that this methodology has attractive potential for the solution of badly conditioned practical problems arising in porous media ([9, 10].)

The above results represent a substantial improvement in terms of robustness and speedup compared to the results obtained in [1] for the same test problems with the same grids and decompositions.

Another interesting property of the above methodology (already observed in [1]) is that the subdomain problems need not be solved exactly. We also observed, concerning the multilevel solution of the matching problem, that one to two $V$ cycles are sufficient in practice to achieve the solution within truncation error; in particular, with $\nu_j=\mu_j=2$ in the algorithm of Section 3.3.2, the initial residual is reduced by six orders of magnitude in six to seven iterations, the largest reduction taking place in the first $V$-cycle.

4.3. Solution of Real-Life Test Problems.

To be honest the test cases discussed here are more relevant to [1] since the domain decomposition methodology is exactly the one described in the above reference, i.e. without-yet-multilevel speedup. Nevertheless, we have inserted these problems because they are typical of real-life applications in petroleum reservoir engineering. Also they provide significant benchmarks for elliptic solvers of various types.

This first problem to be considered was communicated to us by petroleum reservoir engineers. It is a model for a discrete shale barrier and involves solving (1.1)-(1.3) where $A$ is visualized in Figure 4.1, where we have used different scales for $L$ and $H$ since $L$ is of the order of 300 feet and $H$ is of the order of 20 feet implying an aspect ratio of 15. Also the thickness of the barrier is of order one foot. The ratio of permeability coefficients is $10^2$. The arrows in the element grid $i$ with aspect ratio $15$.

Using $R=15$. 48
DISCRETE SHALE BARRIER PROBLEM

Geometry of the Discrete Shale Barrier Problem
Figure 4.1

The arrows in Figure 4.1 indicate the flow direction.

Concerning the numerical solution of the above problem we have been using a 40×40 finite element grid and a (2, 2) domain decomposition. For comparison purposes we have treated the cases with aspect ratios 1 and 15.

Using the domain decomposition algorithm discussed in [1] we need 33 iterations if R=1 and 48 if R=15. We can expect the number of iterations to be practically independent of R once our V
In the same vein the second problem is also a real life problem (1.1)-(1.3) where

\[ A = k(x, y)/\mu(c) \]

and

\[ \mu(c) = c\mu_1^{-1/4} + (1-c)\mu_2^{-1/4}, \]

with \( \mu_1, \mu_2 > 0 \).

Applying the domain decomposition-mixed finite element methods of [1] to the above problem, with a 80 \times 80 finite element grid and a (10, 10) domain decomposition, the solution was obtained in 9 conjugate gradient iterations. This represents a substantial improvement over a preconditioned conjugate gradient solution of the same discrete problem (without domain decomposition) since the convergence was requiring then about 150 iteration, (taking advantage of a good initial guess). Incidentally the lowest order Raviart-Thomas space (r = 0 in 4.1) or cell-centered finite differences [11] do not work well on this type of problems due to the impossibility for these low order approximations to reproduce correctly flows which are not parallel to the coordinate axes; this drawback disappears if we chose \( r = 1 \).

In Figure 4.2 we have visualized the permeability \( k(x, y) \), this data was measured by researchers at Atlantic Richfield Corporation and kindly communicated to us. Similarly the function \( A = k/\mu \) is visualized in Figure 4.3.

5. Mesh Refinements Via Domain Decomposition

Mesh refinements are necessary when strong gradients arise locally. In view of saving computer storage and avoiding complicated data structures it is interesting to incorporate local grid refinement over subdomains where the strong variations are arising and retain coarser grids elsewhere. The concept of domain decomposition provides an elegant and systematic way to implement the above ideas. In this section we would like to present a particular implementation of our scheme, new to our knowledge, relying again on a combination of Raviart-Thomas mixed finite element and domain decomposition methods.
above problem, was obtained in preconditioned iteration since the unital guess. Differences [11] approximations \( k \) disappears if

\[
A = k/\mu
\]

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Representation of \( k(x, y) \)
Figure 4.2

Representation of \( A = k/\mu \).
Figure 4.3
5.1 Mesh Refinement Via a Modified Rannard-Thomas Mixed Finite Element Method

Consider the situation depicted in Figure 5.1 where a local refinement is necessary in a subregion $\Omega^*$ of $\Omega$. The basic idea is to employ essentially mixed finite elements of Raviart-Thomas type inside and outside subregion $\Omega^*$; the main issue here is clearly the matching between the "fine" and "coarse" approximations. To realize this matching we introduce the following finite dimensional spaces of mixed type.

Let $\Omega^*=(a^*, b^*) \times (c^*, d^*)$ and define $\Delta^*_x$ and $\Delta^*_y$ be partitions of $[a^*, b^*]$ and $[c^*, d^*]$, respectively. Generalizing the notation of Section 4.1, we denote by

\begin{equation}
W_h^{r^*}(\Omega^*) = M_1^{r^*}(\Delta x^*) \otimes M_1^{r^*}(\Delta y^*),
\end{equation}

\begin{equation}
V_h^{r^*}(\Omega^*) = (M_0^{r^*+1}(\Delta x^*) \otimes M_1^{r^*}(\Delta y^*)) \times (M_1^{r^*}(\Delta x^*) \otimes M_0^{r^*+1}(\Delta y^*)),
\end{equation}

and

\begin{equation}
V_h^{r^*}(\Omega^*) = V_h^{r^*}(\Omega^*) \cap \{ q : q \cdot \nu = 0 \text{ on } \partial \Omega^* \}.
\end{equation}

Similarly we define the corresponding "coarse" spaces by

\begin{equation}
W_h^{r}(\Omega - \Omega^*) = W_h^{r}(\Omega - \Omega^*),
\end{equation}

with $W_h^{-1, r} \equiv W_h^{r}$

\begin{equation}
V_h \equiv V_h
\end{equation}

\begin{equation}
W_h \equiv W_h
\end{equation}

\begin{equation}
V_h \equiv V_h
\end{equation}

\begin{equation}
\text{Strict approximations}
\end{equation}

\begin{equation}
r^* = r.
\end{equation}

From decomposition
necessary in a
riart-Thomas
en the "fine"
-dimensional
and $[c^*, d^*],

\begin{align}
\text{Figure 5.1}
\end{align}

and

\begin{align}
V_h^{r-1} \cap (\Omega^* - \Omega^*) &= V_h^{r-1} \cap \Omega^* \cap (\Omega - \Omega^*)
\end{align}

with $W_h^{r-1}$ and $V_h^{r-1}$ as defined in Section 4.1. We set

\begin{align}
W_h^R &= W_h^{r*} \cap V_h^{r*} (\Omega^*) \cup W_h^{r*} (\Omega - \Omega^*), \\
V_h^R &= V_h^{r*} \cap V_h^{r*} (\Omega - \Omega^*), \\
V_h^R_{h, 0} &= V_h^R \cap \{q: q \cdot n = 0 \text{ on } \partial\Omega\}.
\end{align}

Strictly speaking $W_h^R$ and $V_h^R$ are not Raviart-Thomas spaces, however, they share the same approximating properties which include $\text{div } V_h^R \subset W_h^R$ and the order approximation is the same if $r^* = r$.

From a computational point of view this refinement technique is well suited for domain decomposition with $\Omega^*$ and $\Omega - \Omega^*$ as subdomains.
The above approach is well suited for a multi-level solution of problem (2.1) in which we shall use different number of grid levels in the subdomains (usually more grid levels in the more refined regions). Domain decomposition allow a lot of flexibility by the fact that in one of the phases of their realization they decouple the computation to be done in each subdomain.

6. Conclusions

From the numerical results described in this paper the combination of mixed finite element, domain decomposition and multilevel methods discussed in Sections 2, 3 and 4 provides a robust, accurate and fast technique for solving elliptic problem with non-smooth coefficients like those arising in flow in porous media and other applications from Mechanics and Physics.

These methods are quite interesting from a parallel computing point of view since the ratio

\[
\frac{Work \ in \ Solving \ Subdomain \ Problems}{Communication \ Costs}
\]

is of order \(0(h^{-1})\).

Here the communication involves the transfer of the boundary data at the subdomain interfaces.

We are presently cooperating with the computer scientists at the National Science Foundation Center for Research in Parallel Computation in the parallel implementation of the methods discussed in this paper.

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References

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the ratio


