CHAPTER 13

An Additive Schwarz Algorithm for Nonselfadjoint Elliptic Equations*

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Abstract

In this paper, we consider the solution of linear systems of algebraic equations that arise from elliptic finite element problems. We study the additive Schwarz method for general, not necessarily selfadjoint, linear, second order, elliptic partial differential equations in R^2 and R^3. We use the GMRES method to solve the resulting linear system of equations. We show that the algorithm is optimal in the sense that the rate of convergence does not depend on the mesh size, nor on the number of substructures.

1 Introduction

Domain decomposition techniques are very powerful iterative methods for solving linear systems of equations that arise from finite element problems. In each step of an iteration, a number of smaller linear systems, which correspond to the restriction of the original problem to subregions, are solved instead of the large original system. The number of subproblems can be potentially large and these methods therefore are promising for parallel computation. They are divide and conquer methods and the central mathematical question is to obtain a bound on the rate of convergence of the iteration. Borrowing a term from structural engineering computations, the subregions are often called substructures. We thus have two partitions of the region into substructures, which sometimes defines a coarse, global model, and the elements of the finite element model.

The iterative methods, most commonly used, are the conjugate gradient method for the symmetric, positive definite case and generalized conjugate residual methods for the general, nonsymmetric case. If the symmetric part of the operator is positive definite, with respect to a suitable inner product, convergence can be guaranteed. In this paper, the rate of convergence of a domain decomposition algorithm is estimated.

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We show that the additive Schwarz algorithm is optimal for elliptic problems in both R^2 and R^3 in the sense that the rate of convergence is independent of both the coarse mesh size, defined by the substructures, and the fine mesh size. All theorems and lemmas are stated without proof. For details, see [5]. For related works, see [3] [4] [2] [6] [12] [15] [13].

This paper is organized as follows. We first briefly introduce the GMRES method, which is our main iterative method for solving our linear system of equations. Then, we develop an abstract theory for the additive Schwarz method, which unifies a number of separate results developed for some different domain decomposition algorithms in recent years. Then, we apply our abstract theory to study the additive Schwarz method (ASM) for the stationary convection-diffusion problems in R^2 and R^3. Finally, we present some results of our numerical experiments.

2 The GMRES method

In Eisenstat, Elman and Schultz [9], the conjugate gradient method was generalized to solve nonsymmetric linear system of equations. In particular they considered the so called generalized minimum residual method, GMRES, which has been shown in practice to be very powerful for a large class of problems. In their paper, the GMRES method and the corresponding theory in $L^2(\Omega)$ space were given, but in fact this algorithm and the theory can be extended easily to any Hilbert spaces. We shall describe the algorithm and state the theory without proof.

Let P be an linear operator on the finite dimensional space R^n with the inner product $[\cdot,\cdot]$ and norm $\|\cdot\| = \sqrt{[\cdot,\cdot]}$, which are chosen to take advantage of some special properties of P. In our applications, P is not symmetric but positive definite with respect to $[\cdot,\cdot]$.

We are interested in using GMRES method to solve the following linear system of equations on R^n:

$$Px = b,$$

where b is given in R^n.

The method begins with an initial approximate solution $x_0 \in R^n$ and an initial residual $r_0 = b - Px_0$. At the m^{th} iteration, a correction vector z_m is computed in the Krylov subspace

$$\mathcal{K}_m(r_0) = \text{span}\{r_0, Pr_0, \cdots, P^{m-1}r_0\},$$

which minimizes the residual $\min_{z\in\mathcal{K}_m(r_0)} \| b - P(x_0+z) \|$. The m^{th} iterate is then $x_m = x_0 + z_m$.

It can be shown that if we perform exact arithmetic operations then the solution would be reached in no more than n iterations.

According to the theory of Eisenstat, Elman and Schultz, the rate of convergence of the GMRES method can be characterized by the ratio of the minimal eigenvalue of the symmetric part of the operator and the norm of the operator. Those two quantities are defined as follows:

$$c_p = \inf_{x\neq 0} \frac{[x,Px]}{[x,x]} \quad \text{and} \quad C_p = \sup_{x\neq 0} \frac{\| Px \|}{\| x \|}.$$

We have the following theorem for the rate of convergence. In the case that $[\cdot,\cdot] = (\cdot,\cdot)$, the proof is given in [9].

Theorem 1 *If $c_p > 0$, then the GMRES method converges and at the m^{th} iteration, the residual is bounded as*

$$\| r_m \| \leq (1 - \frac{c_p^2}{C_p^2})^{m/2} \| r_0 \| .$$

3 The abstract theory for the additive Schwarz method

Let V be a Hilbert space consisting of real functions defined on $\Omega \subset R^d$, where d is the dimension, with an inner product $(u, v)_V$ and a corresponding norm $\|u\|_V$. Let ω be a subdomain of Ω; we define $(u, v)_{V(\omega)}$ to be the inner product for functions obtained by restricting on ω.

Suppose that $\mathcal{B}(\cdot, \cdot)$ is a bilinear form on $V \times V$ and $\mathcal{F}(\cdot)$ a linear functional on V such that

(i) $\mathcal{B}(\cdot, \cdot)$ is continuous, i.e. there exists a constant $C > 0$ such that

$$| \mathcal{B}(u, v) | \leq C \|u\|_{V(\omega)} \|v\|_{V(\omega)}, \quad \forall u, v \in V,$$

where $\omega = \{\text{supp } u\} \cap \{\text{supp } v\}$.

(ii) $\mathcal{B}(\cdot, \cdot)$ is V-elliptic, i.e. there exists a constant $c > 0$ such that

$$\mathcal{B}(u, u) \geq c \|u\|_V^2, \quad \forall u \in V.$$

(iii) $\mathcal{F}(\cdot)$ is continuous, i.e. there exists a constant $C > 0$ such that

$$| \mathcal{F}(u) | \leq C \|u\|_V, \quad \forall u \in V.$$

We define

$$\mathcal{A}(u, v) = 1/2(\mathcal{B}(u, v) + \mathcal{B}(v, u)),$$

which will be called the symmetric part of $\mathcal{B}(\cdot, \cdot)$, and

$$\mathcal{N}(u, v) = 1/2(\mathcal{B}(u, v) - \mathcal{B}(v, u)),$$

which will be called the skewsymmetric part of $\mathcal{B}(\cdot, \cdot)$.

Of course $\mathcal{A}(u, v)$ satisfies (i) and (ii), which implies that the norm corresponding to $\mathcal{A}(\cdot, \cdot)$ is equivalent to the V-norm. In the following, we shall use $(\cdot, \cdot)_{\mathcal{A}}$ instead of $(\cdot, \cdot)_V$.

Our abstract variational equation reads as following: Find $\mathbf{u} \in V$, such that

$$\mathcal{B}(\mathbf{u}, v) = \mathcal{F}(v), \quad \forall v \in V. \tag{1}$$

In order to define the additive Schwarz method, we assume that there exists a decomposition of V. Let V_i, $i = 0, \cdots, N$, be subspaces of V, such that

$$V = \sum_{i=0}^{N} V_i,$$

i.e., for any $v \in V$, there exist $v_i \in V_i$, $i = 0, \cdots, N$ such that

$$v = \sum_{i=0}^{N} v_i. \tag{2}$$

Moreover, we assume that there exists a constant C_0^2 such that

$$\sum_{i=0}^{N} \|v_i\|^2_{V(\omega_i)} \leq C_0^2 \|v\|^2_V, \quad \forall v \in V, \tag{3}$$

where ω_i is the support of V_i defined as

$$\omega_i = \{x | \in \Omega, \exists u \in V_i \text{ such that } u(x) \neq 0\}.$$

Note that the constant C_0^2 may depend on the number of subregions N and also some other parameters of V, which may be introduced in specific applications.

We assume that there exists a constant $C_\omega > 0$, such that

$$\sum_{i=0}^{N} \|v\|^2_{V(\omega_i)} \leq C_\omega \|v\|^2_V, \quad \forall v \in V. \tag{4}$$

In fact this constant C_ω is the maximum number of ω_i's to which a point $x \in \Omega$ can belong.

For each subspace V_i, $0 \leq i \leq N$, we define a projection

$$P_i^{\mathcal{B}} = P_{V_i}^{\mathcal{B}} : V \longrightarrow V_i$$

with respect to the bilinear form $\mathcal{B}(\cdot, \cdot)$: For any $u \in V$, $P_i^{\mathcal{B}} u \in V_i$ is the solution of the problem

$$\mathcal{B}(P_i^{\mathcal{B}} u, v) = \mathcal{B}(u, v), \quad \forall v \in V_i.$$

We introduce a mapping $P^{\mathcal{B}} : V \longrightarrow V$ as

$$P^{\mathcal{B}} = P_0^{\mathcal{B}} + \cdots + P_N^{\mathcal{B}}.$$

Let $\mathbf{u}$ be the solution of (1), and denote

$$\mathbf{b}^{\mathcal{B}} = \sum_{i=0}^{N} P_i^{\mathcal{B}} \mathbf{u},$$

where $P_i^{\mathcal{B}} \mathbf{u}$ can be obtained by solving

$$\mathcal{B}(P_i^{\mathcal{B}} \mathbf{u}, v) = \mathcal{F}(v), \quad \forall v \in V_i, \tag{5}$$

for each i. $\mathbf{b}^{\mathcal{B}}$ is thus the sum of the solutions, obtained in (5).

We now introduce a linear equation: Find $\mathbf{u} \in V$, such that

$$P^{\mathcal{B}} \mathbf{u} = \mathbf{b}^{\mathcal{B}}. \tag{6}$$

We shall call this equation the projected equation with respect to the bilinear form $\mathcal{B}(\cdot, \cdot)$ and the decomposition $\{\omega_i\}$.

The following theorem can be established trivially if $P^{\mathcal{B}}$ is invertible.

Theorem 2 *The equations (1) and (6) have the same solution.*

The additive Schwarz algorithm can be stated as

Additive Schwarz Algorithm ($A\S M$): Find the the solution **u** of equation (1) by solving equation (6).

In practice, the operator $P^{\mathcal{B}}$ corresponds to the product of a large, sparse matrix and the inverse of some other sparse matrices. The explicit matrix for $P^{\mathcal{B}}$ is not known except in some special cases, however, the matrix vector multiply $P^{\mathcal{B}}u$ can be computed by solving one linear system for each subregion. Therefore iterative methods are natural candidates for problem (6). The rate of convergence of any iterative methods depends on the conditioning of the operator $P^{\mathcal{B}}$. The main results can be formulated as follows.

Theorem 3 *(1) There exists a constant $C > 0$, such that*

$$\|P^{\mathcal{B}}u\|_{\mathcal{A}} \leq CC_{\omega}\|u\|_{\mathcal{A}}, \quad \forall u \in V.$$

(2) If there exists $0 < \delta < 1$, such that $| \mathcal{N}(u, P^{\mathcal{B}}u) | \leq \delta \mathcal{B}(u, P^{\mathcal{B}}u)$, $\forall u \in V$, then

$$(u, P^{\mathcal{B}}u)_{\mathcal{A}} \geq c_2(1-\delta)C_0^{-2}(u,u)_{\mathcal{A}}, \quad \forall u \in V,$$

where $c_2 > 0$ is independent of C_0 and δ.

4 Application to the convection-diffusion problems

In this section, we apply the abstract theory for the $A\S M$ given in the previous section to convection-diffusion problems.

4.1 A stationary convection-diffusion problem

Let Ω be an open bounded polygon in R^2 or R^3, with boundary $\partial\Omega$. d denotes the dimension of the space; $d = 2$ or $d = 3$. Consider the homogenous Dirichlet boundary value problem:

$$\begin{cases} Lu &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where L is a strongly elliptic operator of the following form

$$Lu(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x),$$

where $a_{ij}(x) = a_{ji}(x)$ for all i, j and $x \in \Omega$.

Let L^* denote the adjoint operator of L which satisfies

$$(Lu, v) = (u, L^*v),$$

for all sufficiently smooth elements of $H_0^1(\Omega)$.

We assume that $f \in L^2(\Omega)$. The existence and uniqueness of the solution of equation (7), as well as its adjoint equation, are well understood. It is well known that if the domain boundary is not smooth enough, for example if Ω is a nonconvex polygon, we cannot expect, in general, the solution to be in $H^2(\Omega)$. According to the elliptic regularity theory on Lipschitz region, see [11], we make our regularity assumption as follows: There exists a constant $\gamma \in (0, 1/2)$, which depends on the geometry of Ω, such that the adjoint equation has a unique solution $u \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, and furthermore there exists a constant C, such that

$$\|u\|_{H^{1+\gamma}(\Omega)} \leq C\|Lu\|_{L^2(\Omega)}. \tag{8}$$

The weak form of equation (7) reads as: Find u such that

$$B(u, v) = F(v), \quad \forall v \in H_0^1(\Omega), \tag{9}$$

where the bilinear form $B(u, v)$ is defined as

$$B(u,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v dx + \int_\Omega cuv dx$$

and the linear functional $F(v) = (f, v)$. By $A(\cdot, \cdot)$ and $N(\cdot, \cdot)$, we denote the symmetric part and skewsymmetric part of $B(\cdot, \cdot)$. With suitable assumptions on the coefficients of L, $B(\cdot, \cdot)$ and $F(\cdot)$ satisfy conditions (i), (ii) and (iii).

In addition, we assume that the bilinear form $N(\cdot, \cdot)$ is bounded, i.e. there exists a constant C, such that

$$| N(u, v) | \leq C \| u \|_{H_0^1(\Omega)} \| v \|_{L^2(\Omega)}, \quad \forall u, v \in H_0^1(\Omega).$$

This bound can be established if the coefficients $b_i(x)$ and $div(b)$ are bounded. We note that the bounds for $A(\cdot, \cdot)$ and $N(\cdot, \cdot)$ are different. If we look at the integral formulas for $A(\cdot, \cdot)$ and $N(\cdot, \cdot)$, it is obvious that the terms in $N(\cdot, \cdot)$ are one order lower than the terms in $A(\cdot, \cdot)$. This is a key factor that makes our analysis possible.

4.2 The Galerkin finite element method

We solve equation (9) by the Galerkin conformal finite element method. For simplicity, we use piecewise linear triangular elements in R^2 and the corresponding tetrahedral elements in R^3. In the following, we first introduce a two level triangulation of $\Omega \in R^2$ and the corresponding finite element spaces. We then give the Galerkin approximation.

1. Two level triangulation

For a given polygonal region $\Omega \in R^d$, in the first step, we define $\{\Omega_i\}$ to be a regular finite element triangulation of Ω where $\{\Omega_i\}$ is a set of non-overlapping d-dimensional simplices, i.e. triangles if $d = 2$ and tetrahedron if $d = 3$, such that no vertex of one simplex lies on the edge or face of another simplex and

$$\Omega \subset \bigcup_{i=1}^{N} \overline{\Omega_i}.$$

Let H_i be the diameter of Ω_i and $\tilde{H}_i$ the diameter of the largest inscribed ball in Ω_i. We assume that the ratio $H_i/\tilde{H}_i$ is uniformly bounded from above, i.e. that the triangulation is shape regular.

We introduce the mesh parameter

$$H = \max\{H_1, \cdots, H_N\}.$$

We call Ω_i a substructure and $\{\Omega_i\}$ the coarse mesh or H-level subdivision of Ω.

In our second step, we further divide each substructure Ω_i into smaller simplices, denoted as τ_i^j, $j = 1, \cdots$. We assume that $\cup_{i,j}\tau_i^j$ form a shape regular finite element subdivision in the same sense as above. Let h_i^j be the diameter of τ_i^j and introduce the fine mesh parameter by

$$h = \max_{i,j}(h_i^j)$$

We call $\{\tau_i^j\}$ the fine mesh or h-level subdivision of Ω .

We define the piecewise linear finite element function spaces over both the H-level and the h-level subdivision of Ω.

$$V^H = \{v^H \mid \text{ continuous on } \Omega, \text{ and } v^H|_{\Omega_i} \text{ is linear on } \Omega_i,\ v^H = 0 \text{ on } \partial\Omega\}$$

$$V^h = \{v^h \mid \text{ continuous on } \Omega, \text{ and } v^h|_{\tau_i^j} \text{ is linear on } \tau_i^j,\ v^h = 0 \text{ on } \partial\Omega\}$$

It is obvious that $V^H \subset V^h$.

2. Galerkin finite element approximation

The Galerkin approximation of equation (9) reads as follows: Find $\mathbf{u}^h \in V^h$, such that

$$B(\mathbf{u}^h, v^h) = F(v^h), \quad \forall v^h \in V^h. \tag{10}$$

The existence and uniqueness of $\mathbf{u}^h$ has been extensively studied in the literature, see [1]. By using the nodal base functions, equation (10) can be transformed into a linear system of equations, which is large, sparse and relatively ill-conditioned.

4.3 An algorithm and the main results

The additive variant of the Schwarz alternating method was originally proposed by M. Dryja and O. Widlund [8] and S. Nepomnyaschikh [16] for the selfadjoint elliptic problems. In this section, we generalize this method to the nonselfadjoint elliptic cases.

We first form a basic decomposition of the domain Ω and we then define the projections which define our additive Schwarz algorithm.

In the previous section, we introduced the H-level subdivision $\{\Omega_i\}$ of Ω. Since the Schwarz type domain decomposition methods use overlapping subregions, we extend each subregion Ω_i to a larger region Ω_i' such that $\Omega_i \subset \Omega_i'$. Moreover, we assume that there exists a constant $\alpha > 0$ such that

$$\text{distance}(\partial\Omega_i', \partial\Omega_i) \geq \alpha H_i, \quad \forall i.$$

To simplify the notation, we denote $\Omega_0' = \Omega$.

We suppose that $\partial\Omega_i'$ does not cut through any h-level elements. We make the same constructions for the subregions that meet the boundary except that we cut off the parts that are outside Ω.

The assumption on the existence of such a constant α is important. We shall see that the larger the α is the fewer the total number of iterations will be needed. However if we increase the overlap, the size of the subproblems will also be increased, therefore, the cost for solving the subproblems in each iteration will be increased. To balance the total number of iterations and the cost of solving subproblems is an important practical issue.

For each Ω_i', there is a regular finite element subdivision which is naturally induced from the h-level subdivision of Ω. The corresponding finite element function space, denoted as V_i^h, is defined as

$$V_i^h = V^h \cap H_0^1(\Omega_i'),$$

which can be regarded as a subspace of V^h if we extend each function by zero to the complement of Ω_i'. It is known that this extension is continuous. We also use the subspace

$$V_0^h = V^H.$$

It can be seen easily that our finite element function space V^h can be represented as the sum of the $N+1$ subspaces, i.e.

$$V^h = V_0^h + V_1^h + \cdots + V_N^h.$$

Let $P_{V_i^h}^B = P_i^B$ denote the projection from the finite element space V^h to the subspace V_i^h with respect to the bilinear form $B(\cdot,\cdot)$, for $i = 0, \cdots, N$, and denote $P^B = P_0^B + P_1^B + \cdots + P_N^B$. Then, the projected equation with respect to $B(\cdot,\cdot)$ and the decomposition $\{\Omega_i'\}$ is

$$P^B \mathbf{u}^h = \mathbf{b}^B, \tag{11}$$

By Theorem 2, the equation (11) is equivalent to the Galerkin equation (10). However, the conditioning is improved as shown in the following theorem.

We make use of the following convention. All the constants, denoted by C, c, c_p, C_P etc., are positive and independent of the mesh parameters H and h.

Theorem 4 *The operator P^B is uniformly well conditioned. i.e.,*

(1) There exists a constant C_p, such that

$$\| P^B u^h \|_A \leq C_p \| u^h \|_A, \quad \forall u^h \in V^h.$$

(2) There exists a constant $H_0 > 0$, independent of H and h, and a constant $c_p(H_0)$, such that for $H \leq H_0$,

$$(u^h, P^B u^h)_A \geq c_p (u^h, u^h)_A, \quad \forall u^h \in V^h.$$

Remarks: (a) The operator P_0^B is very important. It provides for global information transportation. All the other P_i^Bs are local mappings. Without using P_0^B, in each iteration the information can travel from one substructure only to its next neighbors. Therefore, it takes at least $O(1/H)$ iterations for the information to move across the region.

(b) In the case that $B(\cdot,\cdot)$ is symmetric, this additive Schwarz algorithm is identical to the one proposed by M. Dryja and O. Widlund.

(c) Theorem 4 part (2) shows that if the coarse mesh size is fine enough the symmetric part is positive definite, which guarantees that the GMRES method converges. Since both the constants c_p and C_p are independent of the mesh parameters H and h, according to Theorem 1, the convergence rate of the GMRES method does not dependent on the size of the discrete problems, nor the number of substructures.

(d) The constant H_0 is important, since it determines the size of coarse mesh problem. H_0 depends on the problem. In general, H_0 decreases if we increase the coefficients of the nonselfadjoint terms, while it increases if we use larger overlap. It also depends on the shape of the domain Ω. We do not have an explicit formula for H_0. However, we will give some idea about how to determine this constant numerically later.

(e) If the skewsymmetric part of the elliptic operator vanishes, then $H_0 = +\infty$, and we have no restrictions on the coarse mesh size H.

4.4 The condition number estimates

In order to prove Theorem 4, we need only to show that all the assumptions for Theorem 3 hold. Since $\{\Omega_i'\}$ is a finite covering of Ω, the maximum number of Ω_i's to which any point in Ω can possibly belong is finite, and denoted as C_ω. It is easy to verify that

$$\sum_{i=0}^{N} \| u^h \|^2_{A(\Omega_i')} \leq C_\omega \| u^h \|^2_A, \quad \forall u^h \in V^h, \tag{12}$$

i.e. assumption (4) holds.

Lemma 1 *For any $u^h \in V^h$, there exist $u_i^h \in V_i^h(\Omega_i')$, such that*

$$u^h = \sum_{i=0}^{N} u_i^h.$$

Moreover, there exists a constant C_0, such that

$$\sum_{i=0}^{N} \| u_i^h \|^2_{A(\Omega_i')} \leq C_0^2 \| u^h \|^2_A.$$

In order to prove part (2), we need the following estimates for the skewsymmetric part.

Lemma 2 *There exists a constant C, such that*

$$| N(u^h, P^B u^h) | \leq C H^\gamma B(u^h, P^B u^h), \quad \forall u^h \in V^h.$$

5 Numerical results

1. The model problems

We consider the following problem defined on $\Omega = [0,1] \times [0,1] \subset R^2$.

$$Lu = -\frac{\partial}{\partial x}(\xi \frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\eta \frac{\partial u}{\partial y}) + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + \gamma u = f,$$

with the homogenous Dirichlet boundary condition. f is so defined that the solution has the form $u = xe^{xy} sin(\pi x) sin(\pi y)$. The coefficients are specified as follows.

Example 1. $\xi = 1 + x^2 + y^2$, $\eta = e^{xy}$, $\alpha = 5(x + y)$, $\beta = 1/(1 + x + y)$ and $\gamma = 0$.

Example 2. $\xi = \sigma$, $\eta = \sigma$, $\alpha = 1$, $\beta = 1$ and $\gamma = 1$. σ will be specified later.

Let us introduce some notations. **ite** denotes the total number of iterations required for the GMRES algorithm. **max err** denotes the maximum error between the numerical solution of the Galerkin's equation and the exact solution of the continuous equation. **ovlp** denotes the size of overlap. In our Fortran program, all the subproblems are solved exactly by the band solver from LINPACK. The stopping criterion for the GMRES method is $\| r_i \|_A / \| r_0 \|_A \leq 10^{-4}$. The program is run in single precision on CONVEX C-1 machine at NYU.

2. Test of the additive Schwarz method

Based on the 2-level subdivision of Ω, we introduce our basic substructure Ω_i' obtained by extending each triangle Ω_i to a larger triangle such that each side is parallel to its corresponding side, and does not cut through any h-level triangles. See the following picture. We assume the extension in each side of the Ω_i is uniform in the sense that they all extended the same number of h-level triangles, which will be called the size of overlap and denote by **ovlp**. Of course, we cut off anything which is outside Ω.

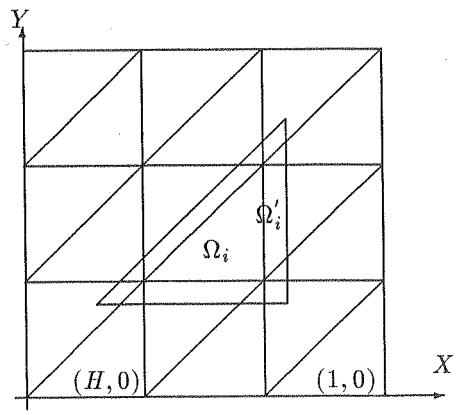

Figure 1. extended subregions

Example 1. In this example, we test the effectiveness of the algorithm and how it depends on the mesh sizes h and H with a fixed overlapping factor α. But, since the overlap must set to be an integer multiple of the fine mesh size h, it turns out that we can not always use the same overlapping factor for all h and H. In the following table,

if $H = 1/3$, we set the overlap to be approximately $0.4H$, and if $H = 1/5$, to be $0.33H$.

h	H	ite	H	ite	max err
1/15	1/3	14	1/5	12	5.0×10^{-3}
1/30	1/3	14	1/5	14	1.3×10^{-3}
1/45	1/3	15	1/5	15	5.8×10^{-4}
1/60	1/3	15	1/5	15	3.0×10^{-4}

With fixed H, the optimality in h can be seen clearly, i.e. **ite** changes very slowly if we use finer h. However if we change H, the number of iteration required to achieve the same accuracy does change . Note that the parameter δ in Theorem 3 is increased if the coarse mesh is further refined subject to the constraint $H \leq H_0$. This can be seen more clearly if we solve problems with higher Reynold's number as in the following examples.

In the theory for the $A_S M$, we assume that the distance between the interior boundary of Ω_i and the boundary of Ω_i' is bounded from below by a constant α times the coarse mesh size H. Now, in the following table we show how this constant affect the number of iterations. We use the same model problem as above and set $H = 1/5$ and $h = 1/60$. We run the same program for different size of overlap, which range from one fine grid size to six grid size. The results are shown in the following table.

ovlp	h	$2h$	$3h$	$4h$	$5h$	$6h$
ite	18	18	16	15	14	13

Indeed, the increase of overlap can reduce the number of iterations. But we have to note that this also increases the size of the subproblems and the cost per iteration is increased also. Further studies on how to determine the optimal overlap so that the parallel CPU time or the serial CPU time can be minimized are needed. Some discussions about this issue for symmetric problems can be found in [10].

Example 2. (1) $\sigma = \sqrt{2}/30$.

We test the algorithm with a higher Reynold's number. We take $h = 1/45$ and vary the parameter H.

H	1/3	1/5	1/9	1/15
ite	+50	37	20	11
ovlp	$5h$	$3h$	$2h$	$1h$

We believe that H_0 is approximately $1/9$ in this case.

Shown in Theorem 4, the observed parameter H_0 is independent of h. This is demonstrated in the following table.

H=1/10	h=1/80,ovlp=$2h$	h=1/100,ovlp=$3h$	h=1/120,ovlp=$4h$
ite	19	20	20

(2) $\sigma = \sqrt{2}/100$

In this example, we increase the Reynold's number to 100. It can be seen that by choosing a suitable $H \leq H_0$, we can still control the total number of iterations taken to achieve a specified accuracy. By running the same example on a coarser fine mesh space the approximate H_0 can be obtained, which has been shown to be independent of the fine mesh size. We then refine the mesh so that the required accuracy can be achieved. According to Theorem 4 this further refinement of the fine mesh should not increase the number of iterations. A further refinement of the coarse mesh reduces the total number of iterations.

H=1/15	h=1/45,**ovlp**=1h	h=1/60,**ovlp**=1h	h=1/75,**ovlp**=2h
ite	23	23	23

H=1/20	h=1/80,**ovlp**=1h	h=1/100,**ovlp**=2h	h=1/120,**ovlp**=2h
ite	15	21	21

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