CHAPTER 12

Augmented Lagrangian Interpretation of the Nonoverlapping Schwarz Alternating Method*

R. Glowinski†
P. Le Tallec**

Abstract. We present below an interpretation of the nonoverlapping Schwarz alternating method proposed by P. L. Lions. In an augmented Lagrangian framework, we can interpret such an algorithm either as a classical saddle-point algorithm or as a time integration scheme of Peaceman-Rachford type. It is hoped that such a point of view can give insight on the choice of the algorithm parameters and on its extension to nonlinear situations.

1. INTRODUCTION OF A SADDLE - POINT FORMULATION AND ALGORITHM

We first formally introduce this formulation on the following model problem

$$\begin{cases} -\Delta u = f & in\ \ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

the domain Ω being decomposed in two subdomains as indicated in the figure below, the interface being denoted by $S = \gamma_{12} = \gamma_{21}$.

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†Department of Mathematics, University of Houston, Houston, TX 77204-3476 and INRIA, France.

**Ceremade, Université Parius Dauphine, Palace de Lattre de Tassigny, 75775 Paris 16, France.

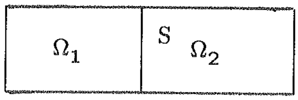

Figure 1

Obviously such a problem can be rewritten as

$$\underset{v \in H_0^1(\Omega)}{\text{Min}} \left[\int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 - fv\right)dx \right],$$

that is

$$\underset{\substack{v_i \in H^1(\Omega_i) \\ v_i = 0 \text{ on } \partial\Omega,\ v_1 = v_2 \text{ on } S}}{\text{Min}} \left[\int_{\Omega_1} \left(\tfrac{1}{2}|\nabla v_1|^2 - fv_1\right)dx + \int_{\Omega_2} \left(\tfrac{1}{2}|\nabla v_2|^2 - fv_2\right)dx \right]$$

Adding the extra variable $q = v_1 = v_2$ on S, we finally get the following constrained minimisation problem

$$\begin{cases} \underset{\substack{v_i \in H^1(\Omega_i), \\ v_i = 0 \text{ on } \partial\Omega}}{\text{Minimize}} \left[\sum_{i=1}^{2} \left\{ \int_{\Omega_i} \left(\tfrac{1}{2}|\nabla v_i|^2 - fv_i\right)dx + \tfrac{r}{2}\int_S |v_i - q|^2 ds \right\} \right] \\ \textit{under the linear constraints } v_1|_S = q \text{ and } v_2|_S = q. \end{cases}$$

Introducing the Lagrange multipliers λ_i of these linear constraints and the corresponding Lagrangian

$$\mathcal{L}(v_i, q; \mu_i) = \sum_{i=1}^{2} \left\{ \int_{\Omega_i} \left(\tfrac{1}{2}|\nabla v_i|^2 - fv_i\right)dx + \tfrac{r}{2}\int_S |v_i - q|^2 ds + \int_S \mu_i (v_i - q) ds \right\}$$

our original problem takes the final form:

<u>FIND A SADDLE-POINT $((u_i, p); \lambda_i)$ OF $\mathcal{L}$ OVER A WELL CHOSEN PRODUCT SPACE.</u>

Such a saddle point formulation is particularly interesting because of the many algorithms that are available for its solution. For example one can use the following algorithm (denoted by ALG3 in references [2] and [3]):

λ_i^0 and q^{-1} given. Then for $n \geq 0$, λ_i^n and p^{n-1} being given, solve successively

$$\frac{\partial \mathcal{L}}{\partial v_i}(u_i^n, p^{n-1}; \lambda_i^n)=0,$$

$$\lambda_i^{n+\frac{1}{2}} = \lambda_i^n + r\frac{\partial \mathcal{L}}{\partial \mu_i}(u_i^n, p^{n-1}; \lambda_i^n),$$

$$\frac{\partial \mathcal{L}}{\partial q}(u_i^n, p^n, \lambda_i^{n+\frac{1}{2}}) = 0,$$

$$\lambda_i^{n+1} = \lambda_i^{n+\frac{1}{2}} + r\frac{\partial \mathcal{L}}{\partial \mu_i}(u_i^n, p^n; \lambda_i^{n+\frac{1}{2}}).$$

From the definition of $\mathcal{L}$, this algorithm takes the form

$$\begin{cases} -\Delta u_i^n = f \ \textit{in} \ \Omega_i, \\ u_i^n = 0 \ \textit{on} \ \partial\Omega, \ \dfrac{\partial u_i^n}{\partial n_i} + r u_i^n = r p^{n-1} - \lambda_i^n \ \textit{on} \ S, \end{cases}$$

$$\lambda_i^{n+\frac{1}{2}} = \lambda_i^n + r(u_i^n - p^{n-1}),$$

$$2rp^n = r(u_1^n + u_2^n) + \lambda_1^{n+\frac{1}{2}} + \lambda_2^{n+\frac{1}{2}},$$

$$\lambda_i^{n+1} = \lambda_i^{n+\frac{1}{2}} + r(u_i^n - p^n).$$

Therefore we have

$$\begin{aligned} \frac{\partial u_i^{n+1}}{\partial n_i} + r u_i^{n+1} &= r p^n - \lambda_i^{n+1} \\ &= 2rp^n - \lambda_i^{n+\frac{1}{2}} - r\, u_i^n \\ &= r\,(u_1^n + u_2^n) + \lambda_1^{n+\frac{1}{2}} + \lambda_i^{n+\frac{1}{2}} - r u_i^n \\ &= r\, u_j^n + \lambda_j^{n+\frac{1}{2}} \\ &= 2r u_j^n + \lambda_j^n - r p^{n-1} \\ &= r u_j^n - \frac{\partial u_j^n}{\partial n_j}. \end{aligned}$$

In other words, after elimination of λ_i and q, our algorithm writes

$$\begin{cases} -\Delta u_i^n = f & in\ \Omega_i, \\ \dfrac{\partial u_i^n}{\partial n_i} + r u_i^n = -\dfrac{\partial u_j^{n-1}}{\partial n_j} + r u_j^{n-1} & on\ S. \end{cases}$$

This is precisely the nonoverlapping Schwarz alternating method proposed by P. L. Lions [1], that we have recovered by a mathematical programming approach.

2. ABSTRACT FRAMEWORK.

2.1 The original problem.

We take the notations of P. L. Lions [1]. Thus Ω is a bounded, smooth open set of $\mathbb{R}^n$, decomposed into

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_m \cup \Sigma,$$

$$\Sigma = \bigcup_{1 \leq i \neq j \leq m} \gamma_{ij},$$

$$\gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j.$$

On Ω, we want to solve the elliptic variational problem below

$$\sum_{i=1}^{m} \left\{a_i(u, v) - L_i(v)\right\} = 0,\ \forall v \in H_0^1(\Omega; \mathbb{R}^p),\ u \in H_0^1(\Omega; \mathbb{R}^p),$$

under the notations

$$a_i(u, v) = \int_{\Omega_i} A\nabla u : \nabla v dx,$$

$$L_i(v) = \int_{\Omega_i} f \cdot v dx.$$

Above A is a symmetric definite tensor, possibly depending on x, and f belongs to $L^2(\Omega; \mathbb{R}^p)$. The problem to solve therefore corresponds to the partial differential equation

$$\begin{cases} -\text{div}\,(A\nabla u) = f & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases}$$

2.2 Notations

Let us introduce

$$V = \prod_{i=1}^{m} H^1(\Omega_i),$$

$$H = \prod_{1 \le i \ne j \le m} H^{-\frac{1}{2}}(\gamma_{ij}),$$

$(\cdot\,,\cdot)_{ij}$=scalar product on $H^{-\frac{1}{2}}(\gamma_{ij})$,

$\eta_{ij} = \eta_{ji}$ = Riesz map from $H^{\frac{1}{2}}(\gamma_{ij})$ into $H^{-\frac{1}{2}}(\gamma_{ij})$ associated to the scalar product $(\cdot,\cdot)_{ij}$,

$$B \quad : \quad V \to H$$

$$(u_i)_i \longrightarrow \left(\eta_{ij}\ \mathrm{tr}\ u_{i|\gamma_{ij}}\right)_{ij},$$

$$E \quad : \quad \left\{q \in H,\ q_{ij}=q_{ji}\right\} = B(H^1(\Omega)),$$

$$F \quad = \quad \mathrm{Ind}_E = \text{indicator functional of E in H},$$

$$G \quad : \quad V \to \mathbb{R}$$

$$(u_i)_i \to \sum_{i=1}^{m} \int_{\Omega_i} \left(\frac{1}{2} A \nabla u_i : \nabla u_i - f \cdot u_i\right) dx.$$

An alternative choice of notation would be to introduce positive numbers η_{ij}, and to set

$$(Bu)_{ij} = \eta_{ij}\ \mathrm{tr} u_{i|\gamma_{ij}},$$

$$(\cdot\,,\cdot)_{ij} = \frac{1}{\eta_{ij}} (\cdot\,,\cdot)_{H^{-\frac{1}{2}}(\gamma_{ij})}.$$

This last choice is the strict equivalent to what is done in P. L. Lions [1] (with $\underline{\lambda}_{ij} = r\eta_{ij}$), and leads to a much easier numerical implementation. But then, BB^t is not an isomorphism on H, which will translate in more fragile convergence properties of the algorithm to come.

2.3 Lagrangian Formulation.

In the above notations, our elliptic problem takes the abstract form (see §1)

$$(P) \qquad \operatorname*{Min}_{v \in V} \left\{ F(Bv) + G(v) \right\}.$$

Such problems have been extensively studied in nonlinear programming (see Fortin-Glowinski [2] Glowinski-LeTallec [3]). After introduction of the augmented Lagrangian

$$\mathcal{L}_r(v, q; \mu) = F(q) + G(v) + \tfrac{r}{2}(Bv-q, Bv-q) + (\mu, Bv-q)$$

it reduces to the saddle-point problem

$$\begin{cases} \mathcal{L}_r(u, p; \mu) \leq \mathcal{L}_r\,(u, p; \lambda) \leq \mathcal{L}_r(v, q; \lambda), \\ \forall\ (v, q; \mu) \in V\times H\times H,\ (u, p; \lambda) \in V\times H\times H, \end{cases}$$

which can be solved by the algorithm ALG3 of §1

λ^o *and* p^{-1} *known*

then for $n \geq 0$, λ^n *and* p^{n-1} *known, solve successively*

$$\frac{\partial \mathcal{L}}{\partial v}\,(u^n, p^{n-1}; \lambda^n) = 0,$$

$$\lambda^{n+\frac{1}{2}} = \lambda^n + r\,\frac{\partial \mathcal{L}}{\partial \mu}\,(u^n, p^{n-1}; \lambda^n),$$

$$\frac{\partial \mathcal{L}}{\partial q}\left(u^n, p^n; \lambda^{n+\frac{1}{2}}\right) = 0,$$

$$\lambda^{n+1} = \lambda^{n+\frac{1}{2}} + r\,\frac{\partial \mathcal{L}}{\partial \mu}\left(u^n, p^n; \lambda^{n+\frac{1}{2}}\right).$$

As in §1, this algorithm is the nonoverlapping Schwarz alternating method proposed in [1].

2.4 Dual evolution problem.

As seen in [3], the analysis of the above augmented Lagrangian algorithm is best done by introducing the equivalent dual formulation

$$0 \in \partial F^{-1}(\lambda) - B\partial G^{-1}(-B^t\lambda).$$

In our case, we have

$$\partial F(q) = E^{\perp} \text{ if } q \in E, \quad = \emptyset \text{ if not,}$$

thus

$$\partial F^{-1}(\lambda) = E \text{ if } \lambda \in E^{\perp} \quad = \emptyset \text{ if not.}$$

Similarly, a direct computation characterizes $\partial G^{-1}(-B^t\lambda)$ as the solutions (u_i) of the problems

$$(P_i)\begin{cases} -\mathrm{div}(A\nabla u_i) = 0 & \text{in } \Omega_i \\ A\nabla u_i \cdot n_i = -\lambda_{ij} & \text{on } \gamma_{ij}. \end{cases}$$

Thus $-B\partial G^{-1}(-B^t\lambda)$ is the generalized Steklov-Poincaré operator which transforms the normal derivative of an harmonic function into its trace.

Then, the dual problem has the following form

$$\begin{cases} Bu \in E & (\textit{continuity of the function at the interfaces}), \\ \lambda \in E^{\perp} & (\textit{continuity of the normal derivatives at the interfaces}). \end{cases}$$

To this dual problem, we associate the evolution equation

$$\frac{d\lambda}{dt} + \partial F^{-1}(\lambda) - B\partial G^{-1}(-B^t\lambda) = 0,$$

which we solve by the Peaceman-Rachford algorithm

$$\frac{\lambda^{n+\frac{1}{2}} - \lambda^n}{\Delta t/2} + \partial F^{-1}(\lambda^n) - B\partial G^{-1}(-B^t\lambda^{n+\frac{1}{2}}) = 0,$$

$$\frac{\lambda^{n+1} - \lambda^{n+\frac{1}{2}}}{\Delta t/2} + \partial F^{-1}(\lambda^{n+1}) - B\partial G^{-1}(-B^t\lambda^{n+\frac{1}{2}}) = 0.$$

As proved in [3], this algorithm is identical to the algorithm ALG3 of §2.3 and therefore to the method of [1].

Moreover, under this last form, we can prove linear convergence of this algorithm (independently of any discretisation step h) as soon as BB^t is an isomorphism on H (Lions-Mercier [4]).

3. CONCLUSIONS

The alternating method of [1] has been rewritten first as a saddle point algorithm, second as a Peaceman-Rachford time integration scheme. Such interpretations guarantee linear convergence (for a proper choice of $\underline{\lambda}_{ij} = r\eta_{ij}$), and simplify its numerical implementation and its extension to nonlinear situations. But as it is the case for most augmented Lagrangian algorithms, this emphasizes the key importance of a proper choice of $r\eta_{ij}$ on the algorithm's convergence properties. An automatic efficient strategy for choosing $r\eta_{ij}$ is still to find.

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