Convergence Estimates for Some Multigrid Algorithms*

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Abstract

This work is devoted to the theoretical analysis of the multigrid algorithm applied to three different problems in finite elements. First of all, a piecewise linear nonconforming element is studied. By choosing the conforming elements as coarser spaces, we propose a multigrid algorithm whose convergence properties are similar to conforming elements. Secondly, for the 2nd order elliptic boundary problems with strongly discontinuous coefficients, we prove that the convergence rate of the multigrid method (with the natural weighted norms) is independent of the jumps of the discontinuous coefficients. Finally, we design a multigrid algorithm for arbitrarily refined meshes in any dimension and then prove that the algorithm has the uniform contraction property. Similar results are also obtained for interface problems with refined meshes. The theory is currently established under the condition that the number of levels is fixed.

1 Introduction

In this paper, we shall first propose and study a multigrid algorithm for some nonconforming element. In the family of nonconforming finite element spaces, the Crouzeix–Raviart (c.f. [6]) piecewise linear nonconforming element perhaps is the simplest one, but its multigrid analysis was still not well–developed. The difficulties lie in the nonnested multilevel spaces and the unnatural prolongation operators. Attentions has largely been paid to the construction of appropriate prolongations and some convergence results have been established under the condition that the number of smoothings is sufficiently large. For the work in this direction, we refer to Brenner [5] and Braess and Verfürth [1]. In this paper, the approach we will take for this kind of nonconforming element is different from that in [5] or

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[1]. Our focus here is on the choice of coarse level spaces. In the context of the Crouzeix–Raviart element, there seems to be no reason why we still have to use the nonconforming $\mathcal{P}_1$ element on the coarse levels where instead we use the conforming $\mathcal{P}_1$ element. This is the main point in our approach. It turns out that the resulting sequence of spaces are nested and the behavior between any two coarse levels is exactly same as the conforming element and hence there is nothing new in its analysis. The only problem is the transition from the finest space to the next coarser space which is chosen to be the conforming $\mathcal{P}_1$ on the *same* triangulation on which the finest space is defined. We are able to show that the uniform contraction property is still valid between these two grids. Therefore a theory similar to the conforming element can be established, in particular the resulted algorithm is uniformly convergent with only one smoothing step.

The second problem we are interested in is the multigrid analysis for interface problems. What we mean by interface problem is that the coefficients appearing in the paritial differential equations may possess discontinuous jumps and specifically these jumps may be extremely large. Since the usual multigrid estimates for this problem depend on such jumps, the multigrid convergence rate could be deteriorated as the jumps get large and therefore the algorithm may no longer be efficient. In practical computations, one can use some properly weighted discrete L^2 products for improvement. It is observed numerically the convergence rate of the corresponding multigrid algorithm is independent of the jumps. This phenomenon will be justified in this paper for two level or fixed number of level schemes.

Multigrid algorithm for nonquasiuniform meshes is another problem to be studied in this paper. There are many situations where local mesh refinements are important in the finite element approximation. A typical example is the case where the solution of the partial differential equation possesses singularities near the corner of a non–convex domain, singularity also occurs in the interface problem mentioned above. Near a singularity, the mesh should be refined in order to maintain the accuracy. In this way, nonquasiuniform triangulations arise. It is quite natural to try the multigrid method for the refined meshes. As a matter of fact, much attention has been paid to this problem in the literature. Some numerical examples actually demonstrate the efficiency of the algorithm. However, the theoretical aspect of the algorithm gets much more complicated and it seems that very little was done in this direction. In [12] and [13], Yserentant presented some results for some systematically refined meshes, but it is not clear how to get a sequence of nested meshes that still satisfy the required conditions. In this paper, we shall design an algorithm for some quite general nonquasiuniform grids and show an optimal convergence result under the condition that the number of levels is fixed. An important point in our analysis is to make the right choice of the discrete L^2 product.

In the usual multigrid analysis, one of the main ingredient is the so-called elliptic regularity and the resulting approximation estimate (cf. [14]). The difficult in the analysis of the problems mentioned above is that it is very hard to get the right approximation estimates by using the elliptic regularity. Our strategy is somehow to skirt the elliptic regularity. Although we have not reached the optimal result for all the problems considered, at least our theory provides a rigorous theoretical justification for some special cases and further research for more general cases is under development.

As is done by the author in [14], we are going to use the following notation:

$$x \lesssim y, \quad f \gtrsim g \quad \text{and} \quad u \asymp v$$

which mean that

$$x \leq Cy \quad , \quad f \geq cg \quad \text{and} \quad cv \leq u \leq Cv$$

where C and c, are positive constants independent of the variables appearing in the inequalities and any other parameters related to meshes, spaces etc.

The remainder of this paper is organized as follows. In Section 2, we will describe an abstract multigrid algorithm with applications to a second order elliptic boundary value problem with finite element discretizations. Section 3 contains a major technical lemma that is used throughout this paper. Section 4 is devoted to the study of the Crouzeix and Raviart nonconforming element. In Section 5, interface problems, mesh refinement and interface problem with mesh refinements are discussed. Section 6 is a special remark on a forthcoming new result related to the current paper.

2 Multigrid Algorithm and Model Problem

In this section, we shall give a brief description for a multigrid algorithm and its applications to a model elliptic boundary value problem with finite element discretizations. The multigrid algorithm will be presented in an abstract fashion in the first subsection. In the second subsection, we will state the model problem with finite element discretizations and indicate how the multigrid algorithm may be applied.

2.1 A Multigrid Algorithm

Assume we are given a hierarchy of real finite dimensional spaces as follows:

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_j.$$

In addition, let $A(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ be symmetric positive definite bilinear forms on $\mathcal{M}_k$. We shall develop multigrid algorithms for the solution of the following problem: Given $f \in \mathcal{M}_j$, find $u \in \mathcal{M}_j$ satisfying

$$A(u,\phi) = (f,\phi)_j \quad \forall \phi \in \mathcal{M}_j.$$

To define the multigrid algorithms, we need to define some auxiliary operators. For $k = 1,\ldots,j$, the operator $A_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ is defined by

$$(A_k w, \phi)_k = A(w,\phi) \quad \forall w, \phi \in \mathcal{M}_k.$$

Clearly the operator A_k is symmetric positive definite (in both the $A(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ inner products). Operators $\mathcal{I}_k^t : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ and $P_{k-1} : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ are defined by

$$(\mathcal{I}_k^t w, \phi)_{k-1} = (w,\phi)_k \quad \forall w \in \mathcal{M}_k, \phi \in \mathcal{M}_{k-1}, \tag{2.1}$$

and

$$A(P_{k-1}w,\phi) = A(w,\phi) \quad \forall w \in \mathcal{M}_k, \phi \in \mathcal{M}_{k-1}. \tag{2.2}$$

In other words, $\mathcal{I}_k^t$ and P_{k-1} are the adjoints of the inclusion operators with the inner products $(\cdot,\cdot)_k$ and $A(\cdot,\cdot)$ respectively. $\mathcal{I}_k^t$ is often called *restriction* operator. P_{k-1} is just the ordinary Galerkin projection.

With the framework and notation given above, we are now in a position to define our multigrid algorithm and it will be characterized in terms of a sequence of recursively defined operators $B_k : \mathcal{M}_k \mapsto \mathcal{M}_k$. In the following, p, m_k are given positive integers and λ_k is either equal to $\rho(A_k)$ or an upper bound of $\rho(A_k)$ such that $\lambda_k \lesssim \rho(A_k)$.

Algorithm S *First of all, $B_1 = A_1^{-1}$. Now assume B_{k-1} is defined. Then B_k is defined, for $g \in \mathcal{M}_k(A_k w = g)$, by $B_k g = w^{2m_k+1}$ with w^{2m_k+1} being defined as follows:*

1. *Pre-smoothing on $\mathcal{M}_k$:*

$$\begin{aligned} &w^0 = 0 \\ &w^l = w^{l-1} + \lambda_k^{-1}(g - A_k w^{l-1}) \\ &l = 1, 2, \cdots, m_k. \end{aligned}$$

2. *Correction on $\mathcal{M}_{k-1}$: $w^{m_k+1} = w^{m_k} + q^p$ where $q^p \in \mathcal{M}_{k-1}$ is defined as follows*

$$\begin{aligned} &q^0 = 0 \\ &q^l = (I - B_{k-1}A_{k-1})q^{l-1} + B_{k-1}\mathcal{I}_k^t(g - A_k w^{m_k}) \\ &l = 1, 2, \cdots, p. \end{aligned}$$

3. *Post-smoothing on $\mathcal{M}_k$:*

$$\begin{aligned} &w^l = w^{l-1} + \lambda_k^{-1}(g - A_k w^{l-1}) \\ &l = m_k + 2, \cdots, 2m_k + 1. \end{aligned}$$

Denoting $E_k = I - B_k A_k$, by a direct calculation we can show that

$$E_k = K_k^{m_k}(I - P_{k-1} + E_{k-1}^p P_{k-1})K_k^{m_k} \tag{2.3}$$

and

$$A(E_k v, v) = A((I - P_{k-1})v, v) + A(E_{k-1}^p P_{k-1}v, P_{k-1}v) \tag{2.4}$$

where $v = K_k^{m_k} v$.

By induction, it is easy to see that E_k is symmetric positive definite under the inner product $A(\cdot,\cdot)$. To show the convergence of the algorithm, it suffices to establish the estimate of the following type:

$$|||E_k|||_A \le \delta_k < 1 \tag{2.5}$$

where

$$|||E_k|||_A = \max_{v \in \mathcal{M}_k} \frac{A(E_k v, v)}{A(v, v)}.$$

Remark 2.1 If $m_k (= m)$ are independent of k, the above algorithm is often called $V--$cycle if $p = 1$ or $W--$cycle if $p = 2$. Otherwise (for $p = 1$), we may choose, for example, $m_k = 2^{j-k} m$, the algorithm is then called variable V-cycle. Unless otherwise specified, all the results in this paper will be for these three algorithms and moreover m is any fixed positive integer.

Remark 2.2 For simplicity, we have chosen Richardson method as the smoother, but all results in this paper hold for some other smoothers as well, e.g. Gauss–Seidel iteration, see [14].

2.2 Finite Element Equations

We consider the the following model second order elliptic boundary value problem:

$$\begin{aligned} -\nabla\cdot(a\nabla)U &= F \quad \mathit{in}\ \Omega, \\ U &= 0 \quad \mathit{on}\ \partial\Omega \end{aligned} \tag{2.6}$$

where a is a positive function [1] with a positive lower bound on $\bar{\Omega}$.

Correspondingly, we have the following bilinear form:

$$A(v,w)=\int_\Omega a\nabla v\cdot\nabla w dx \tag{2.7}$$

This form is defined for all v and w in the Sobolev space $H^1(\Omega)$ that consists of square integrable functions on Ω with square integrable first derivatives. Clearly, $U\in H_0^1(\Omega)$ (functions in $H^1(\Omega)$ that have zero trace on $\partial\Omega$) is the solution of

$$A(U,\chi)=(F,\chi), \quad \forall\chi\in H_0^1(\Omega), \tag{2.8}$$

where $(F,\chi)=\int_\Omega F\chi dx$.

The above problem often possess so–called elliptic regularity, namely there is a constant $\alpha\in(0,1]$ such that

$$\|u\|_{H^{1+\alpha}(\Omega)}\le C\|f\|_{H^{\alpha-1}(\Omega)}. \tag{2.9}$$

The above regularity estimate often plays a crucial role in the usual multigrid analysis. The analysis in this paper however does not directly depend on this regularity.

We will use the finite element method to discretize the above problem. To do this we first need to discretize the underlying domain Ω, namely, to construct a triangulation of Ω. We assume this triangulation is constructed by a successive refinement. More precisely, $\mathcal{T}=\mathcal{T}_j$ (for some integer j) and $\mathcal{T}_k$ (for $k\le j$) are a sequence of triangulations $\mathcal{T}_k=\{\tau_k^i\}$ of simplices of size h_k for $k=1,\dots,j$ such that $\Omega=\cup_i\tau_k^i$ and the minimal interior angles of all elements $\{\tau_k^i\}$ are uniformly bounded below. These triangulations should be nested in the sense that any simplex τ_{k-1}^l can be written as a union of simplices of $\{\tau_k^i\}$. The triangulations are said to be quasiuniform if the ratios of the maximum mesh size to the minimum ones on each level are uniformly bounded above. As an example for quasiuniform triangulations in two dimensional case, the finer grid is obtained by connecting the mid-points of the edges of the triangles of the coarser grid, starting with a given quasiuniform triangulation $\mathcal{T}_1$.

Hence we can assume a triangulation $\mathcal{T}_j$ is given on Ω. On this triangulation, finite element spaces can be constructed. First, we consider the conforming elements, namely the continuous piecewise linear polynomial space $\mathcal{M}_j\subset H_0^1(\Omega)$. Using this space, we can formulate the finite element approximation of the problem (2.8) by

[1] Except for the interface problem, a is also assumed to be sufficiently smooth.

Find $u_j \in \mathcal{M}_j$ such that

$$A(u_j, v) = (f, v), \quad \forall v \in \mathcal{M}_j. \tag{2.10}$$

Another type of finite element space we will study is a nonconforming element [6], denoted by $\bar{\mathcal{M}}_j$ which is the space of piecewise linear polynomials that assume the same value at the midpoint of each edge of any element and vanish at the midpoints of the edges on $\partial\Omega$. With this space, the finite element approximation of (2.8) is given by

Find $u_j \in \bar{\mathcal{M}}_j$ such that

$$A(u_j, v) = (f, v), \quad \forall v \in \bar{\mathcal{M}}_j \tag{2.11}$$

where

$$A(v, w) = \sum_{\tau \in \mathcal{T}_j} \int_\tau a \nabla v \cdot \nabla w dx. \tag{2.12}$$

Our primary purpose is to develop multigrid algorithms to solve the equation (2.10) and (2.11). More specifically, the **Algorithm S** described in Section 2 will be used. To define the **Algorithm S** , we need to choose the ingredients such as the multilevel spaces $\mathcal{M}_k$ and the bilinear forms $(\cdot,\cdot)_k$ for $k = 1, \cdots, j$. In all the following applications, all the quadratic forms $A(\cdot,\cdot)$ will all be the same as given by (2.7) or (2.12). Other components of the **Algorithm S** will be described later in each concrete case.

3 Some Lemmas

As we mentioned above, the technique that makes use of ellptic regularity in the usual multigrid analysis does not always work. Therefore it is desirable to develop a technique in which the regularity does not play a crucial role. What is to be described next is just for such purpose. Similar technique has been used by some other authors before, c.f. Mandel [9], Hackbusch [7],Brandt [4], Kočava and Mandel [8], Ruge and Stüben [11] etc.

Assume we are given two finite dimensional spaces $\mathcal{M}_{k-1} \subset \mathcal{M}_k$ and a symmetric positive definite bilinear form $A(\cdot,\cdot)$ defined on $\mathcal{M}_k$. Each $\mathcal{M}_l$ is equipped with an inner product $(\cdot,\cdot)_l$ with $l = k-1, k$. $P_{k-1} : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ is the standard Galerkin projection satisfying

$$A(P_{k-1}u, v) = A(u, v), \quad \forall u \in \mathcal{M}_k, v \in \mathcal{M}_{k-1}.$$

For $l = k-1, k$, the operator $A_l : \mathcal{M}_l \mapsto \mathcal{M}_l$ is defined by

$$(A_l u, v)_l = A(u, v), \quad \forall u, v \in \mathcal{M}_l.$$

We make the following principal assumption:

$$\inf_{\chi \in \mathcal{M}_{k-1}} \|v - \chi\|_k{}^2 \le C_1 \lambda_k{}^{-1} A(v, v) \quad \forall v \in \mathcal{M}_k, \tag{3.1}$$

where $\lambda_k = \rho(A_k)$ and C_1 is a positive constant.

Lemma 3.1 *Assume $K_k = I - \lambda_k^{-1} A_k$. Then, under the assumption* (3.1), *we have*

$$\begin{aligned} A((I-P_{k-1})K_k^m u, K_k^m u) &\le (1-\tfrac{1}{C_1})A(K_k^{2m-1}u,u) \\ &\le (1-\tfrac{1}{C_1})A(u,u), \quad \forall u \in \mathcal{M}_k. \end{aligned}$$

Proof. By the definition of P_{k-1}, for any $v \in \mathcal{M}_k$ and $\chi \in \mathcal{M}_{k-1}$, we have

$$\begin{aligned} A((I-P_{k-1})v,v) &= A((I-P_{k-1})v,(I-P_{k-1})v-\chi) \\ &= (A_k(I-P_{k-1})v,(I-P_{k-1})v-\chi)_k \\ &\le \|A_k(I-P_{k-1})v\|_k\|(I-P_{k-1})v-\chi\|_k. \end{aligned}$$

Applying 3.1 yields

$$A((I-P_{k-1})v,v) \le C_1\lambda_k{}^{-1}\|A_k(I-P_{k-1})v\|_k{}^2.$$

Using the hypothesis for K_k, we get that

$$\begin{aligned} & A(K_k(I-P_{k-1})v,(I-P_{k-1})v) \\ \le\ & A((I-P_{k-1})v,v) - \lambda_k{}^{-1}\|A_k(I-P_{k-1})v\|_k{}^2 \\ \le\ & (1-\frac{1}{C_1})A((I-P_{k-1})v,v). \end{aligned}$$

Using Schwarz inequality and the above estimate with $v = K_k^m u$, we deduce that

$$\begin{aligned} & A((I-P_{k-1})K_k^m u, K_k^m u)^2 \\ \le\ & A(K_k(I-P_{k-1})K_k^m u, K_k^m u)A(K_k^{2m-1}u,u) \\ \le\ & (1-\frac{1}{C_1})A((I-P_{k-1})K_k^m u, K_k^m u)A(K_k^{2m-1}u,u). \end{aligned}$$

The desired result then follows.

As a consequence of Lemma 3.1, we have

Lemma 3.2 *Assume $E_l : \mathcal{M}_l \mapsto \mathcal{M}_l$, for $l = k-1, k$ are nonnegative self adjoint operators that are related by*

$$E_k = K_k^m(I - P_{k-1} + E_{k-1}^p P_{k-1})K_k^m,$$

where $p \ge 1$ is an integer. Then, under the assumptions of Lemma 3.1

$$|||E_k|||_A \le (1-\eta)|||E_{k-1}|||_A^p + \eta,$$

where $\eta = 1 - \frac{1}{C_1}$. Hence $|||E_{k-1}|||_A < 1$ implies that $|||E_k|||_A < 1$.

Proof. Denote $\delta_{k-1} = \|E_{k-1}\|_A$. It is routine to see that

$$\begin{aligned}
A(E_k u, u) &\leq (1-\delta_{k-1}^p)A((I-P_{k-1})K_k^m u, K_k^m u) + \delta_{k-1}^p A(K_k^m u, K_k^m u) \\
&\leq ((1-\delta_{k-1}^p)\eta + \delta_{k-1}^p)A(K_k^{2m-1}u, u) \\
&\leq ((1-\eta)\delta_{k-1}^p + \eta)A(K_k^{2m-1}u, u) \\
&\leq ((1-\eta)\delta_{k-1}^p + \eta)A(u, u),
\end{aligned}$$

as desired.

4 A Nonconforming Element

Assume we are given a nested sequence of quasiuniform triangulations $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_j$. On the finest triangulation $\mathcal{T}_j$, the Crouzeix-Raviart space $\bar{\mathcal{M}}_j$ is defined, which is a space of piecewise linear functions on $\mathcal{T}_j$ that assume the same value at two adjacent elements at the midpoint of their common edge and vanish at the midpoint of each edge on $\partial\Omega$.

To define the coarse level space, we temporiarily employ the notation $\tilde{\mathcal{M}}_k$ to denote the continuous piecewise linear functions on $\mathcal{T}_k$ that belongs to $H_0^1(\Omega)$. The multilevel spaces defining the **Algorithm S** will be given by

$$\mathcal{M}_k = \begin{cases} \bar{\mathcal{M}}_j, & \text{if } k = j; \\ \tilde{\mathcal{M}}_{k+1}, & \text{if } k < j, \end{cases}$$

Observing that $\tilde{\mathcal{M}}_j \subset \bar{\mathcal{M}}_j$, we then get a sequence of nested spaces

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_{j-1} \subset \mathcal{M}_j.$$

On $\mathcal{M}_j$, the bilinear form $A(\cdot,\cdot)$ is now defined by (2.12) and the bilinear form $(\cdot,\cdot)_j$ is defined by

$$(u, v)_j = h_j^2 \sum_{x \in \bar{\mathcal{N}}_j} u(x)v(x),$$

where $\bar{\mathcal{N}}_j$ = set of all midpoints of the edges in $\mathcal{T}_j$. On $\mathcal{M}_k$, for $k < j$, the bilinear forms $A(\cdot,\cdot)$ are all the same as given by (2.7) and the bilinear form $(\cdot,\cdot)_k$ is defined by

$$(u, v)_k = h_k^2 \sum_{x \in \mathcal{N}_{k+1}} u(x)v(x)$$

where $\mathcal{N}_k$ is the set of all nodes of $\mathcal{T}_k$.

For the verification of (3.1), we define $\pi_j : \bar{\mathcal{M}}_j \mapsto \mathcal{M}_j$ by

$$(\pi_j u)(x) = \begin{cases} 0, & \text{if } x \in \partial\Omega \cap \mathcal{N}_j; \\ \frac{1}{|\bar{\mathcal{N}}_x|}\sum_{y \in \bar{\mathcal{N}}_x} u(y), & \text{if } x \in \mathcal{N}_j \setminus \partial\Omega, \end{cases}$$

where $\bar{\mathcal{N}}_x$ is the set of midpoints of the edges with x as one of their endpoints.

The following result is essential for the analysis in this section.

Lemma 4.1

$$\|(I - \pi_j)u\|_j^2 \lesssim \lambda_j^{-1} h_j^2 A(u, u), \quad \forall u \in \bar{\mathcal{M}}_j.$$

Proof. For a given $x \in \bar{\mathcal{N}}_j \setminus \partial\Omega$, in the following, x_1 and x_2 will denote the endpoints of the edge where x is located. We have

$$
\begin{aligned}
\|(I-\pi_j)u\|_j^2 &\lesssim h_j^2 \sum_{x\in\bar{\mathcal{N}}_j} |(I-\pi_j)u(x)|^2 \\
&= h_j^2 \sum_{x\in\bar{\mathcal{N}}_j} |u(x) - \frac{1}{2}[(\pi_j u)(x_1) + (\pi_j u)(x_2)]|^2 \\
&\lesssim h_j^2 \sum_{x\in\bar{\mathcal{N}}_j} \sum_{i=1}^{2} |u(x) - (\pi_j u)(x_i)|^2 \\
&\lesssim h_j^2 \sum_{x\in\bar{\mathcal{N}}_j} \sum_{i=1}^{2} \sum_{y\in\bar{\mathcal{N}}_{x_i}} |u(x) - u(y)|^2.
\end{aligned}
$$

It is straightforward to check that

$$
\sum_{y\in\bar{\mathcal{N}}_{x_i}} |u(x)-u(y)|^2 \le \sum_{\tau\in\mathcal{T}_j, x\in\tau} \sum_{x',x''\in\bar{\mathcal{N}}_j\cap\tau} |u(x')-u(x'')|^2.
$$

Consequently

$$
\begin{aligned}
\|(I-\pi_j)u\|^2 &\lesssim h_j^2 \sum_{\tau\in\mathcal{T}_j} \sum_{x',x''\in\bar{\mathcal{N}}_j\cap\tau} |u(x')-u(x'')|^2 \\
&\lesssim h_j^2 A(u,u).
\end{aligned}
$$

Since $\lambda_j \lesssim h_j^{-2}$ as usual, the proof is then complete.

It follows from Lemma 4.1 and Lemma 3.2 that

Theorem 4.1 *Under the assumptions described above, the* **Algorithm S** *for solving* (2.11) *satisfies*

$$
\|\!|E_j|\!\|_A \le \delta < 1, \tag{4.1}
$$

where, for variable V–cycle or W–cycle, δ is independent of j and for V–cycle, $\delta = 1 - O(j^{-\gamma})$

Proof. By the multigrid theory for conforming elements (cf. [2, 10, 14]), we have

$$
\|\!|E_{j-1}|\!\|_A \le \delta_j < 1 \tag{4.2}
$$

where δ_j is independent of j for variable V–cycle and W–cycle, and $\delta_j = 1 - O(j^{-\gamma})$ for V–cycle.

The desired result then follows from Lemma 4.1 and Lemma 3.2.

Remark 4.1 We observe that Theorem 4.1 only depends on Lemma 4.1 and the estimate (4.2) and it has nothing to do with other multilevel triangulations. Consequently any multigrid algorithm for conforming elements would correspond to an algorithm that has the same convergence rate. For example, as is done in [3, 14], we can establish a similar nonnested multigrid theory for nonconforming elements.

5 Interface Problems and Mesh Refinements

This section is devoted to the analysis of multigrid algorithms for interface problems and mesh refinements.

5.1 Interface Problems

We call problem (2.6) to be an interface problem if the coefficient has discontinuous jumps. More precisely, Ω admits the following decomposition

$$\bar{\Omega} = \bigcup_{i=1}^{J} \bar{\Omega}_i \tag{5.1}$$

where Ω_i are mutually disjoint open polygon or tetrahedral and

$$a(x) = \omega_i, \qquad \forall x \in \Omega_i, i = 1, \cdots, j \tag{5.2}$$

for some positive constants ω_i. We will call the set $\cup_{i=1}^{J} \partial\Omega_i \setminus \partial\Omega$ to be the interface of the decomposition of (5.1).

The point here is that the ratio $\max \omega_i / \min \omega_i$ may be extremely large. Our purpose is to design a multigrid algorithm and show that the above mentioned ratio does not affect the convergence rate.

To begin our analysis, we introduce, with respect to the positive constants $\{\omega_i\}_{i=1}^{J}$ above, the following weighted inner products:

$$(u, v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i (u, v)_{L^2(\Omega_i)}, \tag{5.3}$$

and

$$(u, v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i (\nabla u, \nabla v)_{L^2(\Omega_i)}, \tag{5.4}$$

with the induced norms denoted by $\|\cdot\|_{L^2_\omega(\Omega)}$ and $|\cdot|_{H^1_\omega(\Omega)}$, respectively. Notice that

$$A(\cdot, \cdot) = |\cdot|^2_{H^1_\omega(\Omega)}.$$

As is done in Section 2, we assume that Ω is triangulated by a nested sequence of quasiuniform meshes $\{\mathcal{T}_k : k = 1, \ldots, j\}$. An additional assumption we need here is that these triangulations are lined up with the subdomains Ω_i's. Namely the restriction of each $\mathcal{T}_k$ on each Ω_i is also a triangulation of Ω_i itself. Corresponding to these triangulations, as in Section 4, we have the multilevel spaces as follows:

$$\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_j.$$

Namely $\mathcal{M}_k \subset H^1_0(\Omega)$ is a space of piecewise linear polynomials.

For the definition of **Algorithm S** , we choose to define the discrete inner product by

$$(u, v)_k = h_k^d \sum_{i=1}^{J} \omega_i \sum_{x \in \mathcal{N}_k \cap \bar{\Omega}_i} u(x) v(x). \tag{5.5}$$

It is easy to see that the norm induced by the above inner product is equivalent to $\|\cdot\|_{L^2_\omega(\Omega)}$ defined from (5.3). Defining operators $A_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ similarly as before, by the well-known inverse inequality, we have for the largest eigenvalue of A_k, $\lambda_k \lesssim h_k^{-2}$.

Again, we will use the technique described in Section 3 to study this problem. To begin with, we need

Lemma 5.1 *Let $I_{k-1} : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ is the standard nodal value interpolation, then*

$$\|(I - I_{k-1})v\|_k^2 \lesssim \lambda_k^{-1} A(v,v), \quad \forall v \in \mathcal{M}_k.$$

The idea of the proof of this lemma is quite similar to that of the forthcoming Lemma 5.2, the detail is omitted here.

Hence we can apply Lemma 3.2 to conclude that

Theorem 5.1 *Under assumptions described above, the* **Algorithm S** *in Section 2 satisfies:*

$$|||E_k|||_A \leq \delta_k = 1 - (1-\eta)^{k-1}$$

where $\eta \in (0,1)$ is a constant independent of k.

The above theorem shows that the multigrid algorithm converges uniformly with respect to the jumps of the coefficients, provided that the number of levels is fixed.

5.2 Nonquasiuniform Meshes

Assuming we are given a nested sequence of triangulations $\{\mathcal{T}_k, k \in \mathcal{I}\}$, which are not necessarily quasiuniform. As usual we have the corresponding finite element spaces $\{\mathcal{M}_k, k = 1, \ldots, j\}$.

We will follow the notation and assumptions in Section 2. The first step is to make a proper choice of the inner products $(\cdot,\cdot)_k$, which can be defined by

$$(u,v)_k = h_k^2 \sum_{i=1}^{J} \sum_{\tau \in \mathcal{T}_k \cap \Omega_i} h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} u(x)v(x) \tag{5.6}$$

where $h_\tau = \mathrm{diam}(\tau)$.

For any $x \in \Omega$, we define a local mesh size

$$h_{k,x} = \frac{1}{|A_{k,x}|} \sum_{\tau \in A_{k,x}} h_\tau$$

where $A_{k,x} = \{\tau \in \mathcal{T}_k : x \in \tau\}$ and $|A_{k,x}|$ is the number of elements in $A_{k,x}$.

We need to assume that any consecutive meshes are comparatively close in the sense that

$$h_{k-1,x} \lesssim h_{k,x} \leq h_{k-1,x}, \quad \forall k, x.$$

Roughly speaking, the number of elements of $\mathcal{T}_k$ contained in any element of $\mathcal{T}_{k-1}$ is uniformly bounded.

The discrete inner product $(\cdot,\cdot)_k$ will be still defined as in (5.6). It is trivial to see that the induced norm $\|\cdot\|_k$ satisfies:

$$\|u\|_k{}^2 \asymp h_k^2 \sum_{\tau\in\mathcal{T}_k} h_\tau^{d-2} \sum_{x\in\mathcal{N}_k\cap\tau} |u(x)|^2. \tag{5.7}$$

Lemma 5.2 *For any $v \in \mathcal{M}_k$,*

$$\|(I - I_{k-1})v\|_k^2 \lesssim \lambda_k^{-1} A(v,v).$$

Proof. For any $\tau \in \mathcal{T}_{k-1}$, it is routine to show that

$$\|(I - I_{k-1})v\|_{L^2(\tau)}^2 \lesssim h_\tau^2 |v|_{H^1(\tau)}^2, \quad \forall v \in \mathcal{M}_k.$$

But

$$h_\tau^d \sum_{x\in\mathcal{N}_k\cap\tau} |(I - I_{k-1})v(x)|^2 \lesssim \|(I - I_{k-1})v\|_{L^2(\tau)}^2,$$

and hence

$$h_\tau^{d-2} \sum_{x\in\mathcal{N}_k\cap\tau} |(I - I_{k-1})v(x)|^2 \lesssim |v|_{H^1(\tau)}^2.$$

Summing over all $\tau \in \mathcal{T}_{k-1}$, we then get

$$\sum_{\tau\in\mathcal{T}_{k-1}} h_\tau^{d-2} \sum_{x\in\mathcal{N}_k\cap\tau} |(I - I_{k-1})v(x)|^2 \lesssim A(v,v).$$

It is easy to see that $\lambda_k \lesssim h_k^{-2}$, hence the desired result follows because of (5.7).

From Lemma 5.2 and Lemma 3.2, we conclude that

Theorem 5.2 (Non–quasiuniform meshes) *Under assumptions described above, the* **Algorithm S** *satisfies:*

$$\|E_k\|_A \le \delta_k = 1 - (1-\eta)^{k-1}$$

where $\eta \in (0,1)$ is a constant independent of k.

5.3 Interface Problems with Refined Meshes

In this subsection, we shall combine the results in the preceding two sections to study the interface problems with refined meshes. According to the theory of partial differential equations, we know that the solution of the interface problems usually possess singularities. Hence mesh refinement is important in this case. In this way, we are confronted with two difficulties at the same time, namely the large jumps and the nonuniform grids. But fortunately our argument given above is completely local and hence still applies to this case.

The nested sequence of triangulations are given as in the preceding subsection with a constraint that the interfaces of $\partial\Omega_i$'s are lined up with each restriction $\mathcal{T}_k$. The discrete inner products on the corresponding spaces $\mathcal{M}_k$ are now defined by

$$(u, v)_{k,\omega} = h_k^2 \sum_{i=1}^{j} \omega_i \sum_{\tau \in \mathcal{T}_k \cap \bar{\Omega}_i} h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} u(x)v(x).$$

With a similar local argument, we have the analog of Lemma 5.2 as follows:

Lemma 5.3 *For any $v \in \mathcal{M}_k$,*

$$\|(I - I_{k-1})v\|_{k,\omega}^2 \lesssim h_k^2 A(v, v).$$

Therefore Lemma 3.2 is satisfied. Similar to the preceding section, we have

Theorem 5.3 (Interface problems with refined meshes) *Under assumptions described above, the* **Algorithm S** *satisfies:*

$$|||E_k|||_A \leq \delta_k = 1 - (1 - \eta)^{k-1}$$

where $\eta \in (0, 1)$ is a constant independent of k.

6 A Remark

After this work was finished, in a joint work of this author with Bramble, Pasciak and Wang, we have developed a new technique to study the problems in Section 5. The main idea is again to avoid using the elliptic regularity. In most cases for the problems in Section 5, we can show that

$$0 \leq A((I - B_j A_j)v, v) \leq (1 - \frac{1}{j^\gamma})A(v, v), \quad \forall v \in \mathcal{M}_j$$

where $\gamma = 1$ or 2. These results will be reported elsewhere.

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