CHAPTER 6

Domain Decomposition for a Boundary-Value Problem with a Shock Layer*

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1. Introduction. In this paper we remove some of the geometric constraints from our domain decomposition method [6], and we extend the method to a nonlinear problem. We consider a quasilinear elliptic equation

$$\epsilon \Delta u = a(x,y,u)\partial_x u + b(x,y,u)\partial_y u + c(x,y,u) \tag{1.1}$$

on a domain Ω in the plane. We assume that (1.1) is singularly perturbed in the sense that the coefficient ϵ of the Laplacian is a small, positive number. It is easy to solve the reduced hyperbolic equation obtained by setting $\epsilon = 0$ in (1.1), and it is known [5] that the solutions of the two problems are close to each other over most of the domain Ω. We are interested, however, in numerical methods which resolve the differences.

Our algorithms are based on results of asymptotic analysis as presented, for example, in the book of Kevorkian and Cole [7]. Our work is therefore in the spirit of Chin and Krasny [3] and Brown *et al.* [1]. Chin and Krasny considered 1-dimensional problems without a convection term $a\partial_x u$. Brown *et al.* [solved singularly perturbed problems in which the singular behavior was only an outflow boundary layer. We [6] and Rodrigue and Reiter [9] considered linear problems with parabolic boundary layers. In a related method for a parabolic equation Scroggs and Sorensen [10] treat a nonlinear shock layer and an outflow boundary.

In our previous paper [6] we required that the solution of (1.1) have a single line of turning points (a linear approximation to a shock) which is parallel to one of the coordinate axes. We now permit the shock to have arbitrary position in Ω, and more importantly, we no longer require the shock to be straight.

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The paper is organized as follows. In Sections 2 and 3 we examine the analytic behavior of singularly-perturbed 2-point boundary-value problems with shocks, and we use this information to construct a numerical method. Section 4 summarizes the asymptotic analysis of solutions of (1.1), and Section 5 presents our algorithm based on this information. We discuss parallelism in Section 6.

2. A 1-dimensional problem. In this section we discuss a version of the convection-reaction-diffusion problem, restricted to an interval. In particular, we consider the equation

$$\epsilon u'' = a(x,u)u' + b(x,u) \tag{2.1}$$

on the interval $0 < x < 1$ in the case when its solution has an interior shock layer. We require that a and b be continuous functions. We also require that $\partial_u a$ and $\partial_u b$ exist and be continuous. The boundary conditions for (2.1) are

$$u(0) = A, \qquad u(1) = B. \tag{2.2}$$

We assume that ϵ is a small, positive number. In this section we give a description of the analytic properties of solutions of (2.1–2). For more details the reader is referred to the books of Kevorkian and Cole [7] and Chang and Howes [2].

Let us discuss conditions under which the solution to (2.1–2) has a single internal shock layer. Because we are interested in the case of small ϵ, it is natural to examine solutions of the reduced equation obtained by setting $\epsilon = 0$ in (2.1),

$$a(x,U)U' + b(x,U) = 0. \tag{2.3}$$

In order to discuss solutions of (2.3) with shocks, it is convenient to introduce the conservation-law form. Consequently, we introduce a function F such that $\partial_U F(x,U) = a(x,U)$. Then (2.3) takes the form

$$\frac{dF(x,U)}{dx} + c(x,U) = 0 \tag{2.4}$$

with $c = b - \partial_x F$. It is useful to think of a shock in a solution of (2.4) as a stationary shock in a solution of the related hyperbolic partial differential equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(x,U)}{\partial x} + c(x,U) = 0. \tag{2.5}$$

The Rankine-Hugoniot condition for a solution U of (2.4) to have a shock at a point x_0 is therefore that there exist solutions U_L and U_R

$$U(x) = \begin{cases} U_L(x), & \text{for } 0 < x < x_0, \\ U_R(x), & \text{for } x_0 < x < 1 \end{cases} \tag{2.6}$$

with

$$F\big(x_0, U_L(x_0)\big) = F\big(x_0, U_R(x_0)\big) \tag{2.7}$$

while $U_L(x_0) \neq U_R(x_0)$.

The functions U_L and U_R are obtained as follows. It is to be expected that solutions of (2.3) do not fulfill both boundary conditions (2.2). We may, however, define the two solutions determined by these conditions separately. That is, we define $U_L(x)$ to be the solution of (2.3) which satisfies the condition that $U_L(0) = A$ and $U_R(x)$ to be the solution such that $U_R(1) = B$.

It so happens that not all functions U in (2.6) defined in this way correspond to shock-layer solutions of (2.1–2)—we need to add what amounts to an entropy condition. A consideration of the characteristic curves of (2.5) shows that this additional requirement may be taken in the form

$$a(x_0, U_L(x_0)) > 0, \qquad a(x_0, U_R(x_0)) < 0. \tag{2.8}$$

In order to ensure that the solution of (2.1–2) have no boundary layers and only one shock layer, we strengthen condition (2.8) to

$$a(x, U_L(x)) > 0, \qquad 0 < x \leq x_0, \tag{2.9}$$

$$a(x, U_R(x)) < 0, \qquad x_0 \leq x < 1. \tag{2.10}$$

For the time-dependent equation (2.5) conditions (2.9–10) imply that both boundary points $x = 0$ and $x = 1$ are inflow boundaries and that the direction of the characteristic curves is toward the shock at $x = x_0$.

By using the Nagumo-Westphal bounding principle, it is easy to show that under conditions (2.9–10) the solution $u(x)$ of (2.1–2) differs from the solution $U(x)$, as given by (2.6), of the reduced equation (2.3) by

$$|u(x) - U(x)| \leq C\epsilon$$

for x outside an interval $\mathcal{N}$ containing the point x_0. (Here, the constant C depends on the neighborhood $calN$.) See the book by Chang and Howes [2]. In particular, we see that under these conditions $u(x)$ has no boundary layers and only one shock layer.

Heuristic information about the behavior of the solution u within the shock layer may be obtained by making the change of variable $x = x_0 + \epsilon\xi$ in (2.1),

$$\frac{d^2u}{d\xi^2} = a(x_0 + \epsilon\xi, u)\frac{du}{d\xi} + \epsilon b.$$

Thus, in terms of the original variable x, we expect solutions of

$$\epsilon V'' = a(x_0, V)V' \tag{2.11}$$

to reflect the behavior of the solution of (2.1–2) in the vicinity of $x = x_0$.

For our computational example we choose Lagerström's special case of (2.1),

$$\epsilon u'' = -uu' + u. \tag{2.12}$$

For this equation the reduced equation (2.3) takes the form

$$UU' = U, \tag{2.13}$$

and its solutions are given by

$$U = 0 \qquad \text{or} \qquad U = x + \text{const.} \tag{2.14}$$

Thus, we see that if $A \neq 0$ and $B \neq 0$, we have

$$U_L(x) = x + A, \qquad U_R(x) = x + B - 1. \tag{2.15}$$

The strengthened entropy condition (2.9–10) is equivalent to the requirements that

$$x_0 + A < 0 \qquad \text{and} \qquad x_0 + B - 1 > 0. \tag{2.16}$$

Let us now determine the form taken by the Rankine-Hugoniot condition for (2.12). The conservation-law form of (2.12) is

$$\epsilon u'' = -\frac{1}{2}(u^2)' + u.$$

The Rankine-Hugoniot condition (2.7) for a stationary shock is therefore that

$$U_L^2(x_0) = U_R^2(x_0).$$

Upon combining this equality with the entropy condition (2.16), we find that the location x_0 of the shock is determined by the equation

$$U_L(x_0) = -U_R(x_0).$$

Hence, by (2.15) we obtain

$$x_0 = \frac{1}{2}(1 - A - B). \tag{2.17}$$

In order for x_0 as given by (2.17) to be an interior point on the interval $0 < x < 1$, we require that the boundary data A and B satisfy the condition

$$-1 < A + B < 1. \tag{2.18}$$

Let us also note that for (2.14) the shock-layer equation (2.11) takes the form

$$\epsilon V'' = -VV'. \tag{2.19}$$

This equation is easily integrated, and we find that its solutions are given by

$$V(x) = \beta \tanh\left\{\frac{\beta(x - x_0)}{2\epsilon}\right\} \tag{2.20}$$

for arbitrary constants β and x_0. It is clear that (2.20) represents a smooth transition from $V(x) \approx -\beta$ for $x \ll x_0 - \epsilon$ to $V(x) \approx \beta$ for $x \gg x_0 + \epsilon$.

3. An algorithm for the 1-dimensional problem. In this section we present a computational algorithm for the problem (2.1–2). The method is in the spirit of matched asymptotic expansions as described, for example in Kevorkian and Cole [7]. The method uses a technique introduced by Brown *et al.* [1] to obtain the solution in the outer region away from the shock layer. Within the shock layer we use a method specifically designed to resolve the shock. We iterate between the two regions in order to improve the accuracy. These iterations are of the function-iteration kind that is standard in numerical analysis, and they converge. This is in contrast to the iterations used in matched asymptotics to obtain successive terms of a formal series expansion which may well be divergent.

Step 1, initialization. The first step of the algorithm is the determination of an initial approximation u_0. This is done by finding the solution (2.6) of the reduced equation (2.3). We do this analytically if that is possible, otherwise we solve (2.3) numerically for U_L with initial data $U_L(0) = A$ using a nonstiff integrator, and we save the values of $U_L(x)$. Similarly, we solve (2.3) for U_R with initial data $U_R(1) = B$. We then locate the point

x_0 at which the Rankine-Hugoniot condition (2.7) is satisfied. The iteration count n is set to 1.

We assume that there exists such a point x_0 in the interior of the interval $0 < x < 1$. Otherwise, we have a different type of problem, and other methods would be used. Note also that the point x_0 need not be located precisely—we need only a good initial approximation.

Step 2, domain decomposition. We select a positive number δ and define the *shock-layer region* to be the interval $x_0 - \delta < x < x_0 + \delta$. We require that

$$0 < x_0 - \delta < x_0 + \delta < 1.$$

The *outer region* is made up of the two intervals $0 < x < x_0 - \delta$ and $x_0 + \delta < x < 1$.

Based on the heuristic arguments leading to (2.11), we expect the thickness of the shock-layer region to be of the order of ϵ, and $\delta \approx 5\epsilon$ is a reasonable value. It should also be stated that for our algorithm the precise value taken for δ is not particularly important, provided that the shock layer is contained in the interval $x_0 - \delta < x < x_0 + \delta$. If δ is chosen to be too small, then a portion of the shock layer will extend into the outer region, and the shock may not be well resolved. On the other hand, the only drawback to taking δ large is that doing so entails additional computational effort. In fact, in numerical tests we have had no difficulty with treating the entire interval $0 < x < 1$ as the shock-layer region.

Step 3, the outer region. This step of the algorithm is the same as that used by Brown *et al.* [1] in the outer region for a 2-point boundary-value problem with a boundary layer. We discuss the subinterval $0 < x < x_0 - \delta$, and we assume that we are given a grid $\{\xi_j\}$ $(j = 0, 1, \ldots, J)$ on this interval with $\xi_J = x_0 - \delta$. For use in a discretization of the term $\epsilon u''$ we define the operator $D(u_n, u_{n-1})$ by

$$D(u_n, u_{n-1})(\xi_j) = \frac{2}{\xi_{j+1} - \xi_{j-1}} \left(\frac{u_{n-1}(\xi_{j+1}) - u_n(\xi_j)}{\xi_{j+1} - \xi_j} - \frac{u_n(\xi_j) - u_n(\xi_{j-1})}{\xi_j - \xi_{j-1}} \right) \tag{3.1}$$

when $j = 1, \ldots, J-1$. For $j = J$ we take

$$D(u_n, u_{n-1})(\xi_J) = \frac{2}{\xi_J - \xi_{J-1}} \left(u'_{n-1}(\xi_J^+) - \frac{u_n(\xi_J) - u_n(\xi_{J-1})}{\xi_J - \xi_{J-1}} \right). \tag{3.2}$$

On the grid $\{\xi_j\}$ we use a nonstiff 1-leg method as in Dahlquist [4] to solve the initial-value problem

$$a(x, u_{n-1})u'_n + b(x, u_n) = \epsilon D(u_n, u_{n-1}) \tag{3.3}$$

for u_n with $u_n(0) = A$. On the interval $x_0 + \delta < x < 1$ the initial value is $u_n(1) = B$, and the direction of integration is reversed. The reversal of direction entails obvious modifications in the definition of D.

Note that we could replace (3.3) by

$$a(x, u_n)u'_n + b(x, u_n) = \epsilon D(u_n, u_{n-1})$$

if it is desirable to do so. (We have found that these iterations converge so quickly that it makes little difference.) In the solution of (3.3) we use a 1-leg method in order to make it easy to incorporate the operator D into the scheme. In our implementations we use a Runge-Kutta method. Note that the definition (3.1) means that the term $\epsilon u''$ in (2.1) is treated in a Gauss-Jacobi fashion. This detail differentiates our method from a numerical implementation of matched asymptotic expansions, and it greatly improves the convergence. (For a discussion of the restrictions required when D depends on y_{n-1} alone, see Hedstrom and Howes [6].) Note also that information from the shock-layer region is fed into the outer

region via the term ϵD and that it propagates at the speed of one grid point per iteration. That is one reason why the shock layer should be completely contained in the interval $x_0 - \delta < x < x_0 + \delta$.

Step 4, grid the shock region. Suppose that we are given a partition $\{\eta_k\}$ of the range of u_{n-1}. (We use a uniform grid, $\eta_k - \eta_{k-1} = \Delta u$ for some positive number Δu.) The values of $u_{n-1}^{-1}(\eta_k)$ are used to define a partition $\{\zeta_m\}$ of the shock-layer region $x_0 - \delta < x < x_0 + \delta$. In particular, determine $\{\zeta_m\}$ $(m = 0, 1, \ldots, M)$ with

$$x_0 - \delta = \zeta_0 < \zeta_1 < \cdots < \zeta_M = x_0 + \delta$$

so that for each m $(1 \le m \le M)$ there exists a k for which

$$\eta_{k-1} \le u_{n-1}(x) \le \eta_k \tag{3.4}$$

when $\zeta_{m-1} < x < \zeta_m$. This inverse mapping of a grid was also used by Chin and Krasny [3] in their numerical method for singularly perturbed reaction-diffusion equations. It should be noted that it is not necessary that inequalities (3.4) be satisfied exactly. It is sufficient to find approximate roots. It should also be noted that we expect u'_{n-1} to change sign at most twice in the shock-layer region, so that it isn't difficult to determine the points ζ_k. In addition, we remark that M, the number of subintervals, may vary with the iteration count n. For this reason we wrote the code in C, and we represent the data in a linked list.

Step 5, set up the linear problem in the shock-layer region. We next define step functions α_n, β_n, and γ_n for the equation

$$\epsilon u_n'' = \alpha_n(x)u_n' + \beta_n(x)u_n + \gamma_n(x). \tag{3.5}$$

Here, we are replacing the nonlinear equation (2.1) by a sequence of linear equations, so that we have an iterative version of the Preuss method [8]. Consider integers k and m connected by the relation (3.4). Define the midpoints

$$\overline{\zeta}_m = \frac{1}{2}(\zeta_{m-1} + \zeta_m), \qquad \overline{\eta}_k = \frac{1}{2}(\eta_{k-1} + \eta_k).$$

Define the coefficient $\alpha_n(x)$ to be the constant

$$\alpha_n(x) = a(\overline{\zeta}_m, \overline{\eta}_k)$$

on the interval $\zeta_{m-1} < x < \zeta_m$. Based on the linearization

$$b(x,u) \approx b(x,\eta) + \partial_u b(x,\eta)(u - \eta),$$

define β_n and γ_n by

$$\beta_n(x) = \partial_u b(\overline{\zeta}_m, \overline{\eta}_k),$$
$$\gamma_n(x) = b(\overline{\zeta}_m, \overline{\eta}_k) - \partial_u b(\overline{\zeta}_m, \overline{\eta}_k)\eta_k$$

on the interval $\zeta_{m-1} < x < \zeta_m$.

We remark that for the Lagerström equation (2.12) in place of the iterative Preuss method (3.5), we have also tried using a finite-element method using basis functions composed of constants and approximate shocks (2.20). We found that in order to get smooth transitions between the shock-layer and outer regions, we had to add a solution of the reduced equation (2.13) to the basis in the shock-layer region. In addition, we found that although the first iterate gives a good approximation when ϵ is small, subsequent iterates

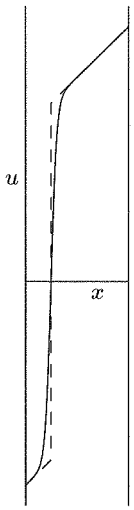

Fig. 1. A shock layer.

converge quite slowly. (With $A = -2$, $B = 2.5$, and $\epsilon = 0.05$ the first iterate gives approximate answers with 3 correct digits, but we don't get 4 correct digits until the tenth iterate.) Besides that, the iterative Preuss method (3.5) is applicable to a much wider class of equations because it requires only the solution of linear differential equations with constant coefficients.

Step 6, solution of the shock-layer problem. We solve the linear differential equation (3.5) for u_n on the interval $x_0 - \delta < x < x_0 + \delta$ subject to the boundary conditions that the values of u_n at the end points agree with those obtained in Step 3.

Step 7, increment n. If the iterations have not converged, increment n and return to Step 3.

As a numerical example we used our method to solve the Lagerström equation (2.12) with $\epsilon = 0.05$ and with boundary data $A = -2$ and $B = 2.5$. The result is shown in Fig. 1, along with the solution (2.15) of the reduced equation (2.13). The value of ϵ is sufficiently large so this problem would cause no difficulty for a standard numerical method for 2-point boundary-value problems. We chose to present this case in order to show that our method works for moderate values of ϵ, even though it is designed for ϵ very small.

4. A 2-dimensional problem. The 2-dimensional version of the convection-reaction-diffusion equation (2.1) is

$$\epsilon \Delta u = a(x, y, u)\partial_x u + b(x, y, u)\partial_y u + c(x, y, u), \tag{4.1}$$

where Δ denotes the Laplacian $\partial_x^2 + \partial_y^2$. We suppose that (4.1) is prescribed on a domain Ω in the plane and that Dirichlet boundary conditions are imposed,

$$u = f(x, y) \qquad \text{for } (x, y) \in \partial\Omega. \tag{4.2}$$

The asymptotic behavior of solutions of (4.1–2) as $\epsilon \downarrow 0$ is discussed in the book by Eckhaus [5], and we summarize the parts of the theory that we require.

As in the 1-dimensional case, it is instructive to introduce the reduced equation obtained by setting $\epsilon = 0$,

$$a(x,y,U)\partial_x U + b(x,y,U)\partial_y U + c(x,y,U) = 0. \tag{4.3}$$

Once we know what boundary conditions to apply, this equation may be solved by the method of characteristics

$$\begin{aligned} \partial_s x &= a(x,y,U), \\ \partial_s y &= b(x,y,U), \\ \partial_s U &= -c(x,y,U). \end{aligned} \tag{4.4}$$

Boundary conditions for (4.4) require use to define the concept of *inflow boundary*. A point (x,y) on the boundary $\partial\Omega$ is said to be an inflow boundary point for a function U if at (x,y,U) the dot product of the vector (a,b) with the exterior normal to $\partial\Omega$ is negative.

The first step in the determination of the solution of (4.3) is the solution of the system (4.4) with the boundary data (4.2) prescribed at inflow boundary points. It may happen that the characteristic curves (4.4) so constructed cross inside Ω, apparently causing U to have multiple values on a subdomain. This difficulty is resolved by the introduction of shocks as follows. We introduce functions F, and G such that

$$\partial_U F(x,y,U) = a(x,y,U), \qquad \partial_U G(x,y,U) = b(x,y,U). \tag{4.5}$$

Because (4.3) is a time-independent problem, its shocks are stationary. That is, at every point on a shock the vector n normal to the shock satisfies the equation

$$n \cdot ([F],[G]) = 0,$$

where $[F]$ denotes the jump in $[F]$ across the shock. A shock may therefore be determined by solving the differential equations

$$\frac{dx}{ds} = [F], \qquad \frac{dy}{ds} = [G]. \tag{4.6}$$

Note that in (4.6) we may wish to replace $([F],[G])$ by its negative in order to reverse the direction of travel along the shock. In the vicinity of a shock it is useful to transform to coordinates (s,t) parallel and normal to the shock, because to a first approximation only normal derivatives $\partial_t u$ and $\partial_t^2 u$ are important there.

Fig. 2 illustrates a particular example of the theory in which (4.1) takes the form

$$\epsilon\Delta u = u\partial_x u - \partial_y u + u \tag{4.7}$$

and Ω is the square $0 < x < 1$ and $0 < y < 1$. The boundary conditions are that

$$u = x \quad \text{for} \quad y = 1, \tag{4.8}$$

$$u = -2 - y \quad \text{for} \quad x = 1. \tag{4.9}$$

$$u = 0 \quad \text{for} \quad x = 0, \tag{4.10}$$

$$u = 1 \quad \text{for} \quad y = 0. \tag{4.11}$$

The figure shows characterstic curves, inflow and outflow boundaries, and a shock for the reduced equation

$$U\partial_x U - \partial_y U + U = 0. \tag{4.12}$$

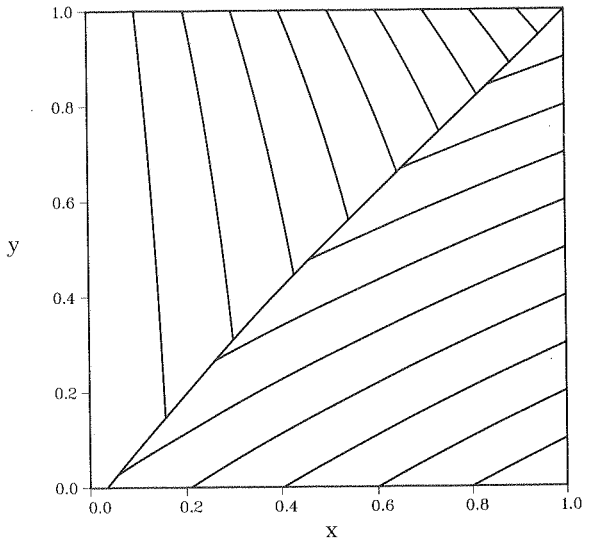

Fig. 2. A 2-dimensional shock.

Because $b = -1$ in (4.12), the top of the square $y = 1$ is necessarily an inflow boundary, while the bottom $y = 0$ is an outflow. The boundary condition (4.11) is therefore apt not to be fulfilled by the solution to the reduced equation (4.12). Consequently, we expect to find an ordinary boundary layer near $y = 0$.

Because $a = U$ and $U < 0$ on $x = 1$, the right-hand boundary $x = 1$ is an inflow boundary. The boundary condition (4.10) makes $x = 0$ a tangential boundary (neither an inflow nor an outflow). It so happens that the solution of (4.12) with boundary data (4.8) also satisfies the condition (4.10), so that there is no boundary layer near $x = 0$. If negative values of u were prescribed on $x = 0$, though, the solution of (4.7) would have a parabolic boundary layer there. Numerical methods for such problems were discussed by Hedstrom and Howes [6] and Rodrigue and Reiter [9]. If positive values of u were prescribed at $x = 0$, this boundary would also be an inflow boundary, and there would be another shock.

5. A numerical method for the 2-dimensional problem. Our numerical method for the problem (4.1–2) is a natural extension of the algorithm described in Section 3 for the problem (2.1–2). There are some additional geometrical complications, however. We give details for the problem illustrated in Fig. 2, in which the domain Ω is the square $0 < x < 1$, $0 < y < 1$ and there is a single shock starting at the boundary of Ω.

Step 1, initialization. We solve the reduced equation (4.3) with conditions (4.2) prescribed on the inflow portion of the boundary $\partial\Omega$. We do this by solving the systems (4.4) for the characteristic curves. We save the locations and directions of the characteristic curves (first partial derivatives), because these curves are used to generate a coordinate system in the outer region (away from the shock layer). We use methods of differential geometry (the Hodge $*$-operator) [11] to represent the Laplacian in the curved coordinate system.

In order to obtain an approximate location of the shock, we interpolate the solutions U of (4.4) onto a rectangular grid, and we solve (4.6) using these interpolated values. Note that the solution of (4.6) involves yet another interpolation because U is known only at grid

points. (It would be better to use just one interpolation between the two overlapping sets of characteristic curves. Our only reason for introducing the supplementary rectangular grid was that at the time this section of code was written, we had not yet implemented a direct interpolation between curvilinear grids.) We again save the location of the shock and information about its derivatives for later use in a local coordinate transformation.

Step 2, domain decomposition. Around the shock we introduce a strip $\mathcal{S}$ of width, say, 10ϵ. If the curvature of the shock is not too large, this may be done by generating an orthogonal coordinate system based on the shock. At the outflow portion of the boundary we introduce a strip $\mathcal{B}$ of width, say 5ϵ. Because this boundary is a straight line, the original coordinates may be used here. We cover the intersections of the shock with the boundary $\partial\Omega$ by rectangles oriented parallel to the original axes. Let us denote the rectangle at the entry of the shock by $\mathcal{E}_0$ and the one at the exit by $\mathcal{E}_1$. Note that some of the regions $\mathcal{S}$, $\mathcal{B}$, $\mathcal{E}_0$, and $\mathcal{E}_1$ intersect each other. Finally, we denote the outer region by $\mathcal{O}$, so that

$$\mathcal{O} = \Omega \setminus (\mathcal{S} \cup \mathcal{B} \cup \mathcal{E}_0 \cup \mathcal{E}_1).$$

Step 3, the outer region. Given an approximate solution u_{n-1}, we solve on $\mathcal{O}$ the equation

$$a(x, y, u_n)\partial_x u_n + b(x, y, u_n)\partial_y u_n + c(x, y, u_n) = \epsilon D(u_n, u_{n-1}). \tag{5.1}$$

As mentioned above, we use here a coordinate system generated by the characteristic curves for $u_0 = U$. The operator D is a discrete Laplacian using values of u_n at the center and upwind on the particular characteristic curve and using u_{n-1} on adjacent characteristic curves.

Step 4, shock entry region. On $\mathcal{E}_0$ we solve the full problem (4.1). On the top and right-hand edge of $\mathcal{E}_0$ use the prescribed boundary data (4.2). On the left-hand edge of $\mathcal{E}_0$ we use the value of u_n computed in Step 3. On the bottom edge of $\mathcal{E}_0$ we set $\partial_y u_n = \partial_y u_{n-1}$. We use a traditional multigrid method with a mesh so fine that the solution is smooth on the finest grid. This step is not yet implemented.

Step 5, shock region. In the region $\mathcal{S}$ we use coordinates s and t, parallel and normal to the shock. In $\mathcal{S}$ our discretization of (4.1) uses $\partial_t u_n$ and $\partial_t^2 u_n$ for the normal derivatives and $\partial_s u_{n-1}$ and $\partial_s^2 u_{n-1}$ for the tangential derivatives. Thus, we have a set of 2-point boundary-value problems of the sort considered in Sections 2 and 3. As boundary data for the portion of $\mathcal{S}$ interior to $\mathcal{E}_0$ we use the values of u_n obtained from Step 3 for the portion of $\partial\mathcal{S}$ which is in $\partial\mathcal{O}$. Otherwise get the boundary data from u_n in Step 4.

This transfer of boundary data requires an interpolation between two curvilinear coordinate systems. We represent the coordinate lines by polygonal lines, and we use a binomial search algorithm to locate the intersection of a pair of polygonal lines.

Step 6, outflow boundary layer. In the region $\mathcal{B}$ the variation of u in the y-direction is much more rapid than the variation in the x-direction. We therefore approximate (4.1) in $\mathcal{B}$ by

$$\epsilon(\partial_x^2 u_{n-1} + \partial_y^2 u_n) = a(x, y, u_{n-1})\partial_x u_{n-1} + b(x, y, u_n)\partial_y u_n + c(x, y, u_n). \tag{5.2}$$

As boundary data for (5.2) we use the condition (4.2) on the bottom ($y = 0$), and on the top we use the values of u_n in $\mathcal{O}$ obtained in Step 3. Thus, for (5.2) we again have a set of 2-point boundary-value problems, and the method of Section 3 may be applied. (This step is also not yet implemented.) We remark that Brown *et al.* [1] use a Galerkin method based on exponential splines for such problems.

Step 7, the shock exit region. On the rectangle $\mathcal{E}_1$ we solve (4.1) for u_n with boundary data given by (4.2) if $y = 0$. We use values of u_n obtained from Steps 3, 5, and 6,

depending on whether that portion of the boundary of $\mathcal{E}_1$ is also part of the boundary of $\mathcal{O}$, $\mathcal{S}$, or $\mathcal{B}$. As in Step 4, on $\mathcal{E}_1$ we would use a fine mesh and a traditional multigrid method.

Step 8, increment n. If the iterates have not converged, we increment the counter n and return to Step 3.

6. Parallelism. The method described in Section 3 for 2-point boundary-value problems is essentially just a serial method. In fact, this problem is too small to warrant the use of a parallel computer, and we used a PC for to generate the data for Fig. 1. The algorithm given in Section 5, however, has a high degree of parallelism. Step 3, the problem in the outer region $\mathcal{O}$, is a collection of independent systems of ordinary differential equations, one system for each characteristic curve. The problems in the shock region $\mathcal{S}$ (Step 5) and in the outflow boundary layer $\mathcal{B}$ (Step 6) consist of independent 2-point boundary-value problems.

The only subdomains in which we have to obtain parallelism by methods used by most of the other speakers at this conference are the regions $\mathcal{E}_0$ and $\mathcal{E}_1$ (Steps 4 and 7), where the shock interacts with the boundary.

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