

Hierarchical Preconditioning and Partial Orthogonalization for the p -Version Finite Element Method¹

Jan Mandel²

Abstract. We present a new preconditioning for the p -version finite element method, based on domain decomposition principles with each element treated as a subdomain. We give a simple theory which makes it possible to predict the condition number from the solution of eigenvalue problems formulated on one element at a time, and we use such computations to select a practically attractive preconditioning for hierarchical serendipity elements of order p in three-dimensional elasticity. The preconditioner consists of parallelizable element-by-element computations and the solution of an auxiliary problem for lower order elements.

Key words. Domain Decomposition, Elliptic Partial Differential Equations, Linear Elasticity, Iterative Methods, Conjugate Gradients

1. Introduction. In this paper, we study a new preconditioning method for large symmetric positive definite linear system arising from the p -version finite element method for three-dimensional elasticity.

The p -version finite element method achieves an increase of precision by increasing the degree of elements rather than decreasing their size as the h -version [1, 5, 6, 35]. The p -version also provides for easier modeling of complicated geometries and facilitates sophisticated post-processing [35]. Moreover, decreasing the element size as well as increasing the degree p gives exponential convergence as long as the singularities of the problem lie on interelement boundaries, which is the case in practice. Direct methods are fully satisfactory for the solution of the resulting system of linear equations in two dimensions; in three dimensions, however, they suffer from much larger fill-in and iterative methods are the only choice for problems with a large number of elements.

¹ This research was sponsored by the National Science Foundation under grant DMS-8704169. The finite element code PROBE and the crankshaft test problem were provided by Noetic Technologies Corporation.

² Computational Mathematics Group, University of Colorado at Denver, 1200 Larimer Street, Denver, CO 80204.

Because the elements of the p -version are fairly large, we treat them as subdomains and use a domain decomposition technique related to and extending the approach of Bramble, Pasciak, and Schatz [9]. For other related domain decomposition methods, see Dryja and Widlund [18, 38]. The method of this paper is closely related to a class of preconditioning for the p -version which was introduced and studied in previous papers [29, 28], and it is a direct extension of the preconditioner for the two-dimensional case developed in [2]. For further developments in the two-dimensional case see [3, 4, 30].

The present method is in fact a hybrid direct-iterative method. One iteration of the preconditioned conjugate gradient method [20, 24] requires the solution of a preconditioning problem, which in our case decomposes into a number of independent local subproblems associated with edges and faces of the finite element structure, and of a global auxiliary problem which is identical to the system obtained by elements of a lower order for the same structure. Because the p -version finite element we use is hierarchical, the matrix of this auxiliary global problem is immediately available as a submatrix of the global stiffness matrix. It also turns out that certain transformations of the stiffness matrix are needed to obtain low condition numbers. These transformations and the decomposition of the matrices of the systems mentioned above constitute a pre-processing phase. The solution of the auxiliary systems is then done by back substitutions in every iteration. The global auxiliary system has much less variables than the original problems, but its structure is same as that of the original problem, so one can use an existing direct solver for its solution.

Thanks to the existence of the global auxiliary problem, the condition number can be proved to be independent of the number of elements. We give a bound for the condition number which can be computed numerically from the data of one element at a time, and show how such a bound leads to the development of an efficient method. An implementation of the method is in progress.

The use a global auxiliary problem with fewer variables makes the present method related to multigrid methods [10, 22, 25, 32]. The present method also belongs to the class of methods based on a splitting the solution space and constructing an iterative method by solving independent problems on the subspaces. Some such methods are the additive Schwarz method [7, 16, 26], the robust multigrid method [23], iterative refinement methods [17, 31], and the domain reduction method [13, 14, 15]. For a general treatment of this class of methods, see [8]. For another method for the p -version finite elements using hierarchical type iterations, see [11, 19]. Conditioning of mass matrices, rather than stiffness matrices as here, was investigated by Wathen [36, 37].

The p -version is closely related to spectral methods and spectral element methods, which also use high order polynomial approximations. Domain decomposition techniques for systems arising from spectral methods were studied by Gottlieb and Hirsh [21] and Quarteroni and Sacci-Landriani [33]. Multigrid for spectral element methods was developed and analyzed by Rønquist and Patera [34] and Maday and Muñoz [27].

TABLE 2.1
 Number of degrees of freedom of elements K'_p and K_p for three-dimensional elasticity

Element	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8
K'_p	24	60	96	150	222	315	432	576
K_p	24	81	192	375	648	1029	1536	2187

In Section 2, we describe the finite element basis functions used. In Section 3, we give a naive hierarchical preconditioning and estimate the resulting condition numbers using an elementary theory and numerical computations. The basic idea of our theoretical analysis is to show that condition numbers can be estimated using quantities related to one element at a time, and then evaluate those quantities numerically. Sections 4 gives a more sophisticated theory, which leads to an improved preconditioner in Section 5. In Section 6, we estimate the operations counts for the preconditioner. In Section 7, we report on further extensions of the method and on the results of computations for several test problems.

2. Preliminaries. We consider the conforming finite element method with the serendipity element of order p [39] with hierarchical basis functions similar to those in the program PROBE [35]. We denote this element by K'_p and briefly describe its basis functions for the reference element $K = (-1, +1)^3$. First, there are nodal basis functions, which are the standard trilinear functions of the form

$$(x \pm 1)(y \pm 1)(z \pm 1).$$

If $p \geq 2$, then there are $p - 1$ basis functions of the form

$$(x - 1)(y - 1)L_n(z), \quad n = 2, \dots, p,$$

for every edge (with obvious changes of -1 to $+1$ and permutation of variables), where L_n is the integral of the Legendre polynomial P_{n-1} of order $n - 1$,

$$L_n(z) = \int_{-1}^z P_{n-1}(\zeta) d\zeta.$$

If $p \geq 4$, we have basis functions of the form

$$(x - 1)(1 - y^2)(1 - z^2)P_m(y)P_n(z), \quad m, n \geq 0, \quad m + n \leq p - 4,$$

for every face (again with obvious changes of -1 to $+1$ and permutation of variables). Finally, if $p \geq 6$, we have the interior functions

$$(1 - x^2)(1 - y^2)(1 - z^2)P_l(x)P_m(y)P_n(z), \quad l, m, n \geq 0, \quad l + m + n \leq p - 6.$$

Because we are interested in three-dimensional elasticity, there are three basis functions for each function described above, one for every displacement component in Cartesian coordinates.

We will also work with the tensor product element K_p , which is obtained similarly as above except that we have the basis functions of the form

$$(x - 1)(1 - y^2)(1 - z^2)P_m(y)P_n(z), \quad m, n \geq 0, \quad m, n \leq p - 4,$$

for the faces, and

$$(1-x^2)(1-y^2)(1-z^2)P_l(x)P_m(y)P_n(z), \quad l, m, n \geq 0, \quad l, m, n \leq p-6,$$

for the interior.

Note that only the trilinear functions are associated with nodes, namely the vertices of K . The remaining degrees of freedom are nodeless. We did not need to specify scaling of the basis functions, because the algorithms in the present paper will be independent of it. The number of degrees of freedom in the elements K_p and K'_p are in Table 2.1.

Functions from the (local or global) finite element space are denoted by u, v , and the corresponding vectors of degrees of freedom (in a given basis) are \tilde{u}, \tilde{v} , and so on.

For two matrices A and B , $A \leq B$ means that A and B are symmetric and $B - A$ is positive semidefinite.

3. A Simple Hierarchical Preconditioning. We are interested in the numerical solution of the problem

$$(3.1) \quad Ax = b,$$

where A is a symmetric positive definite matrix obtained by the usual finite element assembly of local stiffness matrices A_K ,

$$A = \sum_K N_K A_K N_K^T.$$

The matrices $N_K = (n_{K,ij})$ represent the mappings of local to global degrees of freedom: $n_{K,ij} = 1$ if j -th local degree of freedom in element K corresponds to i -th global degree of freedom, $n_{K,ij} = 0$ otherwise. The general idea of our preconditioner is to replace the local stiffness matrices A_K by matrices C_K such that

$$(3.2) \quad m_1 C_K \leq A_K \leq m_2 C_K, \quad 0 < m_1 \leq m_2 < \infty.$$

In every step of the preconditioned conjugate gradient method, we need to solve the linear system

$$(3.3) \quad Cx = r, \quad C = \sum_K N_K C_K N_K^T,$$

which should be much less expensive than solving the system (3.1). We have the following simple bound on the resulting condition number.

THEOREM 3.1. *If (3.2) holds with the same constants m_1 and m_2 for all elements K , then*

$$(3.4) \quad m_1 C \leq A \leq m_2 C.$$

In particular, the condition number of the problem (3.1) preconditioned by C can be bounded by m_2/m_1 independently of the number of elements.

Proof. Add (3.2) over all K using the mappings of local degrees of freedom given by N_K . \square

The bounds m_1 and m_2 in each element can be computed numerically as the smallest and the largest eigenvalue, respectively, of the generalized eigenvalue problem

$$(3.5) \quad A_K x = \lambda C_K x.$$

The problem (3.5) is degenerate; however, (3.2) holds if and only if

$$\ker A_K = \ker C_K,$$

and the common nullspace can be factored out from (3.5). Calculating the extreme eigenvalues numerically for representative elements makes it possible to obtain rigorous and practical a-priori bounds on the condition number of the preconditioned problem.

To choose the matrices C_K , we consider a decomposition of the local finite element space V_K on the element K into a direct sum of $n_K + 1$ subspaces,

$$(3.6) \quad V_K = V_{K,0} \oplus \cdots \oplus V_{K,n_K}.$$

Every function $u \in V_K$ is then decomposed uniquely as

$$(3.7) \quad u = u_0 + \cdots + u_{n_K}, \quad u_i \in V_{K,i}.$$

We now define C_K by

$$(3.8) \quad \tilde{u}^T C_K \tilde{v} = \tilde{u}_0^T A_K \tilde{v}_0 + \cdots + \tilde{u}_{n_K}^T A_K \tilde{v}_{n_K}.$$

For efficiency, we will choose the subspaces $V_{K,i}$ as the subspaces spanned by certain groups of basis functions on the element K . These groups will be induced by a decomposition of the set of global basis functions into disjoint subsets. In a suitable numbering of the degrees of freedom, the local preconditioning matrices C_K and the global assembled preconditioning matrix C are the corresponding block diagonal parts of A_K and A , respectively.

The choices of the decomposition (3.6) which give a condition number independent on the number of elements are limited.

THEOREM 3.2. ([28]) *Let C_K be constructed as in (3.6) to (3.8). Then the inequality (3.2) holds with some $m_1 > 0$ and $m_2 < +\infty$ if and only if $\ker A_K$ is contained in one of the subspaces $V_{K,i}$.*

For linear elasticity, the nullspace $\ker A_K$ is the space of small rigid body motions, or a subspace of it in the presence of boundary conditions on the element K , and it is contained in the space of all linear functions on K . We thus choose the space $V_{K,0}$ to contain all linear functions on K . Because V_K is the space of the element K'_p , it is natural to choose $V_{K,0}$ as the space of the element K'_q , $q < p$. We further choose the spaces $V_{K,i}$, $i \geq 2$, as the spans of all remaining basis functions corresponding to the edges, faces, and the interior of K , respectively. Then the preconditioning matrix C is the block diagonal part of A (after a renumbering of global degrees of freedom), determined as follows: Because of the hierarchical character of the elements, the first

TABLE 3.1

Condition numbers for element K'_p preconditioned by K'_q . Three-dimensional elasticity, Poisson ratio $\sigma = 0.3$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	71.9	83.9	327.2	385.1	1228.5	1394.0	3218.3
q=2		12.8	124.3	197.6	1017.6	1291.5	2989.3
q=3			116.1	147.8	1025.9	1120.9	3071.4
q=4				97.8	410.2	840.6	2258.6
q=5					410.6	479.7	2425.7
q=6						383.0	568.8
q=7							570.2

diagonal block is the stiffness matrix of the same problem discretized by the elements K'_q . Then there is one diagonal block for every edge in the finite element structure, one block for every face in the structure, and one for the interior of every element. Solution of the linear system associated with those blocks can be done locally on the element level, with some communication between neighboring elements.

The condition numbers m_2/m_1 for the element K'_p preconditioned by K'_q as described above were computed numerically and the results are summarized in Table 3.1. We can see that the condition numbers grow fast with increasing p . To obtain low condition numbers, one has to increase q as well, which results in a large auxiliary system to be solved in each iteration. But for $6 \leq p \leq 8$, one does not get condition numbers under 380 for any q .

4. Boundedness of the Local Decomposition in Energy. We need a further theoretical insight in order to obtain a better version of the method. Denote by $a_K(u, v)$ the local energy inner product on element K ; that is, if vectors of degrees of freedom \tilde{u}, \tilde{v} correspond to functions $u, v \in V_K$, respectively, then

$$(4.1) \quad a_K(u, v) = \tilde{u}^T A_K \tilde{v}.$$

Denote the local energy of $u \in V_K$ by

$$|u|_K^2 = a_K(u, u).$$

The poor conditioning we have observed in Section 3 is caused by the lack of orthogonality in the local energy inner product between the subspaces $V_{K,i}$. The following simple theorem shows that to bound the condition number, it is enough to bound the decomposition (3.6) in local energy. Various versions of this theorem were given in [2, 29, 28], and it is also closely related to a lemma by Lions [26, Lemma I.1], see also [8, 38].

THEOREM 4.1. *For any $u \in V_K$, let u_i denote its component $u_i \in V_{K,i}$ as in (3.7). Let b be the least possible number such that*

$$(4.2) \quad \sum_{i=0}^{n_K} |u|_K^2 \leq b |u|_K^2, \quad \forall u \in V_K,$$

TABLE 4.1

Condition numbers for element K'_p preconditioned by $K_q \cap K'_p$. Three-dimensional elasticity, Poisson ratio $\sigma = 0.3$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	71.9	83.9	327.2	385.1	1228.5	1394.0	3218.3
q=2		12.8	21.0	162.1	255.1	825.6	1198.3
q=3			9.9	13.8	169.7	230.8	1109.0
q=4				9.2	11.8	132.5	210.0
q=5					9.2	11.0	129.7
q=6						9.1	10.4
q=7							9.1

and m_1 be the largest possible and m_2 the least possible such that (3.2) holds. Then

$$(4.3) \quad b \leq \frac{m_2}{m_1} \leq (n_K + 1)b.$$

Proof. Let $u \in V_K$ and \tilde{u} be the corresponding vector of degrees of freedom. From the definition of C_K and (4.1),

$$(4.4) \quad \tilde{u}^T C_K \tilde{u} = \sum_{i=0}^{n_K} |u_i|_K^2, \quad \tilde{u}^T A_K \tilde{u} = |u|_K^2.$$

From the Cauchy-Schwarz inequality and the inequality $xy \leq \frac{1}{2}(x^2 + y^2)$,

$$\begin{aligned} \tilde{u}^2 A_K \tilde{u} &= |u|_K^2 = \sum_{i=0}^{n_K} \sum_{j=0}^{n_K} a_K(u_i, u_j) \leq \sum_{i=0}^{n_K} \sum_{j=0}^{n_K} |u_i|_K |u_j|_K \\ &\leq \frac{1}{2} \sum_{i=0}^{n_K} \sum_{j=0}^{n_K} (|u_i|_K^2 + |u_j|_K^2) = (n_K + 1) \sum_{i=0}^{n_K} |u_i|_K^2 \\ &= (n_K + 1) \tilde{u}^2 C_K \tilde{u}, \end{aligned}$$

so $m_2 \leq n_K + 1$. On the other hand, from (3.2), (4.2), and (4.4), we have using the optimality of m_1 and b that

$$m_1 = 1/b.$$

This proves the right-hand-side inequality in (4.3). For the other inequality, it suffices to note that $m_2 \geq 1$ from the definition of C_K . \square

Note that the number n_K is bounded by the geometry of the element. For the brick element, $n_K \leq 23$.

Theorem 4.1 provides a guidance for the choice of the subspaces V_K . We now remove one source of bad conditioning related to the choice of the subspace $V_{K,0}$. Consider a function u with minimum energy from all functions in V_K which have a fixed nonzero value at a vertex and are zero on all sides opposite to the vertex. Such function u decays fast with the distance from the vertex, and, in three dimensions,

TABLE 4.2
Number of degrees of freedom per element in $K_q \cap K'_p$ for three-dimensional elasticity.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	24	24	24	24	24	24	24
q=2		60	78	78	81	81	81
q=3			114	150	171	180	189
q=4				186	243	288	324
q=5					279	360	432
q=6						396	504
q=7							540

functions with less decay have much larger energy. (In the scalar case, such a function can be thought of as an approximation of the unbounded potential of a point source.) So, if $V_{K,0}$ does not contain sufficiently fast decaying functions, the component u_0 will have a large energy and by Theorem 4.1, the condition number will be also large. This gives us a hint how to decrease the condition number at a moderate expense: we add to the basis of $V_{K,0}$ few selected basis functions so that it contains faster decaying functions around vertices. Then the energy of the component u_0 can be expected to be much smaller than before. Because K'_q is the serendipity element, it does not contain all polynomial functions of the form

$$(4.5) \quad x^l y^m z^n, \quad 0 \leq l, m, n \leq q,$$

which form the tensor product element K_q . Recalling the definition of K'_q , we can see that it is exactly the missing basis functions which make a fast decay of a polynomial function possible along the diagonals and diagonal planes of the brick element, and all functions (4.5) are contained in K'_p for $p \geq 3q$. We thus add the functions from K_q which are also in V_K to the basis of the space $V_{K,0}$. The preconditioner is then constructed exactly as in Section 3. In Table 4.1, we give the resulting condition numbers. We can see that the condition numbers have indeed decreased; however, for q in a practical range, say, $q = 2$ or 3 , the condition numbers are still too large.

Table 4.2 contains the number ν of degrees of freedom per element in the first group, i.e., the dimension of $V_{K,0}$. Note that the time for the LL^T decomposition of the global auxiliary system grows about quadratically with ν (for the same finite element structure), because both the bandwidth and the total number of degrees of freedom of the global auxiliary system grow about linearly with ν .

5. Partial Orthogonalization. In this section, we explore another way of decreasing the condition number, and arrive at a final form of our preconditioner. For computational reasons, we wish to preserve the general structure of the preconditioner from the preceding sections. Instead, we change the spaces $V_{K,i}$, $i = 1, \dots, n_K$ by changing the definition of their basis functions. To stay within the framework of the conforming finite element method, we change the global basis functions and update the global stiffness matrix accordingly. The following concept of orthogonalizing one group of basis functions to another was introduced in [29]; we

modify it here for our purposes.

Let $\{\phi_i\}_{i \in G}$ and $\{\phi_j\}_{j \in H}$ be two disjoint groups of global basis functions, $G \cap H = \emptyset$. By orthogonalizing the group $\{\phi_i\}_{i \in G}$ to $\{\phi_j\}_{j \in H}$ we mean replacing the basis functions ϕ_i by

$$\phi_i^{new} = \phi_i - \sum_{j \in H} s_{ij} \phi_j, \quad i \in G,$$

where the coefficients s_{ij} are chosen from the orthogonality condition

$$a(\phi_i^{new}, \phi_j) = 0, \quad \forall j \in H,$$

and $a(u, v)$ is the energy inner product corresponding to the global stiffness matrix A ,

$$a(u, v) = \tilde{u}^T A \tilde{v}.$$

If $G = \emptyset$ or $H = \emptyset$ then construction is void and nothing happens.

Let the global degrees of freedom be numbered so that the degrees of freedom corresponding to the group G come first, those corresponding to the group H second, and then the rest. In this numbering, we can write the global stiffness matrix in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

and the above orthogonalization means replacing A by

$$A^{new} = X^T A X,$$

where

$$X = \begin{pmatrix} I & 0 & 0 \\ -S & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad S = A_{22}^{-1} A_{21},$$

so that (recall that $A_{kl} = A_{lk}^T$ because A is symmetric),

$$(5.1) \quad A^{new} = \begin{pmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} & 0 & A_{13} - A_{12} A_{22}^{-1} A_{23} \\ 0 & A_{22} & A_{23} \\ A_{31} - A_{32} A_{22}^{-1} A_{21} & A_{32} & A_{33} \end{pmatrix}.$$

If the global stiffness matrix is not stored explicitly, but rather as a collection of local stiffness matrices, then we may temporarily assemble only the submatrices A_{21} and A_{22} above and update the local stiffness matrices accordingly so that the new global stiffness matrix satisfies again (5.1). Cf., Section 6 below.

We can now proceed with the construction of our final preconditioner. After selecting the basis functions spanning the subspaces $V_{K,0}$, we perform a partial orthogonalization between groups of the remaining basis functions as follows.

TABLE 5.1

Condition numbers for element K'_p preconditioned by K'_q and partial orthogonalization. Three-dimensional elasticity, Poisson ratio $\sigma = 0.3$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	71.9	83.9	110.2	119.8	159.7	217.2	256.0
q=2		12.5	115.8	135.2	543.4	552.6	944.6
q=3			116.0	142.2	690.7	649.8	1166.8
q=4				97.6	301.5	525.1	736.6
q=5					381.2	416.9	966.8
q=6						337.2	345.5
q=7							315.1

- (i) For each edge, the edge functions are orthogonalized to all basis functions of all adjacent sides and interiors, and to the basis functions of $V_{K,0}$ on that edge.
- (ii) For each face, the face functions (if any) are orthogonalized to all basis functions of the two adjacent interiors, and to the basis functions of $V_{K,0}$ on that face.

The preconditioner is then constructed in the same way as in Section 3, that is, the decomposition (3.6) is defined by the groups of transformed basis functions for the edges, faces, and interiors.

The new basis functions preserve the "character" of the original ones: The new edge functions are identically zero on all elements except those which contain the edge, and the new side functions are identically zero on all elements except those which contain the side.

This partial orthogonalization process has been motivated by Theorem 4.1. It can be interpreted as the construction of the decomposition (3.6), (3.7) in stages corresponding to the edges and the faces. The components at each stage are uniquely determined by the requirement that their energy is the least possible subject to the topology constraints restricting the possible support of each component.

Performing the partial orthogonalization on the data of a single element, we get a rigorous bound on the condition number for a structure constructed by repeating identical copies of the element. We expect that this bound will be an indication of the performance of the method in the general case.

Numerically computed condition numbers for a cube with the subspace $V_{K,0}$ defined by the serendipity element K'_q are in Table 5.1. Table 5.2 contains condition numbers for the case when $V_{K,0}$ is spanned by the functions of the tensor product element K_q which are also in V_K . We can see that in the latter case, $q = 2$ gives reasonably small condition numbers, which do not grow excessively for $p \leq 8$. The condition numbers can increase with q , which may be related to the fact that the component u_0 is chosen by truncating the expansion of u in the original basis functions rather than taking the function in $V_{K,0}$ with the same value as u at the vertices and the least possible energy. Such a choice of u_0 produces a different method, which does not fall within the present framework of hierarchical

TABLE 5.2

Condition numbers for element K'_p preconditioned by $K_q \cap K'_p$ and partial orthogonalization. Three-dimensional elasticity, Poisson ratio $\sigma = 0.3$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	71.9	83.9	110.2	119.8	159.7	217.2	256.0
q=2		12.5	17.8	35.3	39.9	45.8	56.7
q=3			9.5	12.4	20.2	26.0	108.8
q=4				9.2	10.7	17.7	22.5
q=5					9.1	10.1	15.1
q=6						9.1	9.7
q=7							9.1

TABLE 5.3

Condition numbers for element K'_p preconditioned by $K_q \cap K'_p$ and partial orthogonalization. Three-dimensional elasticity, Poisson ratio $\sigma = 0.45$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	156.7	224.2	289.3	300.0	437.4	505.2	746.0
q=2		15.3	51.0	67.9	86.0	94.1	166.7
q=3			9.8	31.0	42.9	66.7	229.5
q=4				9.5	21.4	38.1	49.7
q=5					9.3	17.7	31.0
q=6						9.3	14.8
q=7							9.3

preconditioning, and it will be studied elsewhere.

We can see from Table 5.1 that for $q = 1$, the condition numbers increase about as p^2 . In the two-dimensional case, however, the condition number for a completely analogous method with $q = 1$ can be proved to grow only as $\log^2 p$ for tensor product elements, and the same growth is observed experimentally for serendipity elements [2]. One reason for this difference is that the discrete Sobolev inequality in two dimensions (cf., [2])

$$(5.2) \quad \|u\|_{L^\infty(K)} \leq \text{const} (1 + \log p)^{1/2} \|u\|_{H^1(K)}, \quad \forall u \in V_K,$$

is no longer true in the three-dimensional case. The inequality (5.2) implies that the energy of the component u_0 for any fixed q grows at most as $\log p$.

Finally, we have tested the sensitivity of the method to increasing the Poisson ratio. As can be expected, the condition numbers deteriorate for the Poisson ratio close to 1/2, see Tables 5.3 and 5.4.

6. Operation Counts for Partial Orthogonalization. In this section, we estimate the computational cost of the transformation (5.1) and give more algorithmic details for the case when the stiffness matrix is kept decomposed in local stiffness matrices. We keep the notation of Section 5. Further, denote by m_G and m_H the cardinality of G and H , respectively. For an element K , let $m_{K,G}$ and

TABLE 5.4

Condition numbers for element K'_p preconditioned by $K_q \cap K'_p$ and partial orthogonalization. Three-dimensional elasticity, Poisson ratio $\sigma = 0.49$.

	p=2	p=3	p=4	p=5	p=6	p=7	p=8
q=1	825.8	1266.1	1170.2	1691.1	2133.1	2995.3	3657.3
q=2		78.6	276.2	295.4	734.0	433.3	722.1
q=3			16.1	69.8	156.4	214.7	919.9
q=4				9.6	48.2	139.8	146.7
q=5					7.3	40.4	112.7
q=6						7.7	32.0
q=7							8.2

$m_{K,H}$ be the number of those functions from G and H which are also basis functions for K , respectively.

Instead of (5.1), each local stiffness matrix is transformed by

$$(6.1) \quad A_K^{new} = X_K^T A_K X_K, \quad X_K = \begin{pmatrix} I & 0 & 0 \\ -S_K & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

where S_K is the corresponding $m_{K,H} \times m_{K,G}$ submatrix of $S = A_{22}^{-1} A_{21}$ after a suitable renumbering of rows and columns.

The computational cost of the transformation (5.1) can be roughly estimated as follows. We give operation counts in flops from LINPACK [12], which measure the number of floating point multiplications.

- (i) Computing $A_{22}^{-1} A_{21}$ by Choleski decomposition of A_{22} and back substitutions for every column of A_{21} ,

$$\left((1/6)m_H^3 + m_H^2 m_G \right) \text{ flops.}$$

- (ii) For every element K , transformation of A_K by (6.1),

$$n_K m_{K,G} m_{K,H} \text{ flops}$$

(using symmetry of A_K).

The total cost of all transformations (6.1) in item (ii) above for one element K is

$$\sum_{G,H} n_K m_{K,G} m_{K,H} \approx \text{const } n_K^3 \text{ flops.}$$

Various bookkeeping operations and assemblies of the submatrices A_{22} and A_{21} contribute an additional computational cost of order n_K^2 per element.

7. Computational Results and Extensions. Based on the calculation of condition numbers in preceding sections, we have selected for a practical test the preconditioning by the element $K_2 \cap K'_p$ with partial orthogonalization as described

TABLE 7.1

Computational results for conjugate gradients with preconditioning by $K_2 \cap K_p^1$, partial orthogonalization, and heuristic modifications for distorted elements. Test problems: I. 9 cubes in a 3 by 3 square, Poisson ratio $\sigma = 0.3$. II. Crankshaft segment from Fig. 7.1, 107 elements, Poisson ratio $\sigma = 0.3$.

Problem	p	Variables	Iterations for $\epsilon = 10^{-4}$	CPU time in seconds on VAX 8700		
				Iterative solver total	Prepro- cessing	Generating matrices
I	4	798	14	23	12	71
I	5	1242	23	57	32	151
I	6	1839	24	148	97	423
I	7	2616	24	412	319	1033
I	8	3600	22	1112	955	2418
II	6	15090	32	2591	1501	8880
II	7	22389	39	6836	4415	20220

in Section 5. Our first test problem consists of 9 identical cubes arranged in a 3 by 3 square, with Dirichlet boundary conditions imposed on the face of one corner cube. The results are summarized in Table 7.1. We report

- the degree p of the elements;
- the total number of degrees of freedom in the finite element model;
- the number of iterations determined by the stopping criterion

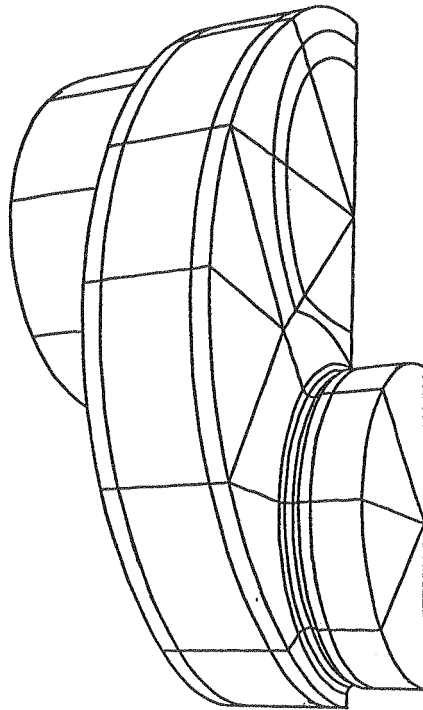
$$\|x_n - x_{n-1}\|_\infty \leq \epsilon \|x_n\|_\infty \quad \text{and} \quad \|Ax_n - b\|_\infty \leq \epsilon \|b\|_\infty,$$

with $\epsilon = 10^{-4}$;

- the total time for the iterative solver, which consists the preprocessing, iterations, and back transformation of the solution to the original degrees of freedom;
- the time for the preprocessing phase, consisting of the transformation (6.1) of the local stiffness matrices, various bookkeeping operations, and the assembly and LU decomposition of the preconditioning matrix C ;
- the time for generating the local stiffness matrices by PROBE.

In practice, however, one needs to solve problems with elements which are curved and often with high aspect ratios. Such problems can be handled successfully by a modification of the present method. The preconditioning matrix C defined in (3.3) is block diagonal, with one full diagonal block for each edge and each face in the finite element model, and one large and sparse block corresponding to the spaces $V_{K,0}$. Poor conditioning of the preconditioned problem in the case of distorted elements can be traced to the fact that such block diagonal matrix C omits some rather big off-diagonal blocks from the transformed global stiffness matrix. For example, for an element K with aspect ratio 100:100:1, the matrix C_K should include the off-diagonal block that corresponds to the two square faces. A method modified so that such off-diagonal blocks are included in C is more robust with respect to high aspect ratios; see [30] for theoretical bounds on the resulting condition numbers in

FIG. 7.1. Crankshaft model with 107 elements (Courtesy of Noetic Technologies Corporation)



the two-dimensional case, and [29] for numerically computed condition numbers in a related method for three dimensions. We use a heuristic procedure to modify the preconditioning matrix C by including suitable off-diagonal blocks. The number of such extra blocks is kept as low as possible with the result that the matrix C is still sparse and its LU decomposition does not present any practical problems. The modified method has been tested on a finite element model of a crankshaft segment with 107 elements, see Fig. 7.1. The results of the computations are included in Table 7.1.

We do not report in Table 7.1 the timing of a direct solver, because the direct solver available to us could not be run for the crankshaft problem at all due to machine limitations. We only note that for the first problem (9 cubes), the time for the direct solver was observed to be lower than the time for the iterative solver, which is to be expected for small problems.

REFERENCES

- [1] I. BABUŠKA, *The p and h - p versions of the finite element method: The state of the art*, in *Finite Elements, Theory and Applications*, D. L. Dwoyer, M. Y. Hussaini, and R. G. Voigt,

- eds., Springer Verlag, New York, 1988.
- [2] I. BABUŠKA, A. W. CRAIG, J. MANDEL, AND J. PITKÄRANTA, *Efficient preconditioning for the p -version finite element method in two dimensions*, SIAM J. Numer. Anal., submitted.
 - [3] I. BABUŠKA AND H. C. ELMAN, *Some Aspects of Parallel Implementation of the Finite Element Method on Message Passing Architecture*, Computer Science Technical Report 2030, University of Maryland, College Park, MD, 1989.
 - [4] I. BABUŠKA, M. GRIEBEL, AND J. PITKÄRANTA, *The problem of selecting the shape functions for a p -type finite element*, Int. J. Numer. Methods Engrg., 28 (1989), pp. 1891–1908.
 - [5] I. BABUŠKA AND M. SURI, *The p - and h - p versions of the finite element method*, International Conference on Spectral and High Order Methods for partial Differential Equations, Como, Italy, June 1989; Comput. Methods Appl. Mech. Engrg., submitted.
 - [6] I. BABUŠKA, B. A. SZABÓ, AND I. N. KATZ, *The p -version of the finite element method*, SIAM J. Numer. Anal., 18 (1981), pp. 515–545.
 - [7] P. E. BJØRSTAD, *Multiplicative and additive Schwarz methods: Convergence in the 2-domain case*, in Domain Decomposition Methods for Partial Differential Equations II, T. Chan, R. Glowinski, G. A. Meurant, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, 1989, pp. 147–159.
 - [8] P. E. BJØRSTAD AND J. MANDEL, *On the Spectra of Sums of Orthogonal Projections with Applications to Parallel Computing*, Preprint, Department of Computer Science, University of Bergen, Allégaten 55, N-5007 Bergen, Norway, 1988. BIT, submitted.
 - [9] J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring, I*, Math. Comp., 47 (1986), pp. 103–134.
 - [10] A. BRANDT, *Multigrid Guide*, Tech. Rep., Weizmann Institute, Rehovot, Israel, 1985.
 - [11] G. BRUSSINO, R. HERBIN, Z. CHRISTIDIS, AND V. SONNAD, *Parallel multilevel finite element method with hierarchical basis functions*, Manuscript, IBM Kingston, NY.
 - [12] J. J. DONGARRA, J. R. BUNCH, C. B. MOLER, AND G. W. STEWART, *LINPACK User's Guide*, SIAM, Philadelphia, 1979.
 - [13] C. C. DOUGLAS AND J. MANDEL, *The domain reduction method: High way reduction in three dimensions and convergence with inexact solvers*, in Multigrid Methods, Proceedings of the Fourth Copper Mountain Conference on Multigrid Methods, J. Mandel, S. F. McCormick, J. E. Dendy, Jr., C. Farhat, G. Lonsdale, S. V. Parter, J. W. Ruge, and K. Stüben, eds., SIAM, Philadelphia, 1989.
 - [14] C. C. DOUGLAS AND W. L. MIRANKER, *Constructive interference in parallel algorithms*, SIAM J. Numer. Anal., 25 (1988), pp. 376–398.
 - [15] C. C. DOUGLAS AND B. F. SMITH, *Using Symmetries and Antisymmetries to Analyze a Parallel Multigrid Algorithm: the Elliptic Boundary Value Problem Case*, Tech. Rep., IBM, 1987. Submitted to SIAM J. Num. Anal.
 - [16] M. DRYJA AND O. B. WIDLUND, *An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions*, Tech. Rep. 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.
 - [17] ———, *On the optimality of an additive iterative refinement method*, in Multigrid Methods, Proceedings of the Fourth Copper Mountain Conference on Multigrid Methods, J. Mandel, S. McCormick, J. E. Dendy, Jr., C. Farhat, G. Lonsdale, S. V. Parter, J. W. Ruge, and K. Stüben, eds., SIAM, Philadelphia, 1989.
 - [18] ———, *Some domain decomposition algorithms for elliptic problems*, 1989, To appear in Proceedings of the Conference on Iterative Methods for Large Linear Systems, Austin, Texas, October 1988, to celebrate the Sixty-fifth Birthday of David M. Young, Jr.
 - [19] S. FORESTI, G. BRUSSINO, AND V. SONNAD, *Multilevel Solution Methods for the p -Version of Finite Elements, Parallel Implementations and Comparison with other Solution Methods*, Tech. Rep. KGN-137, IBM Kingston, NY, 1988.
 - [20] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, John Hopkins University Press, second ed., 1989.
 - [21] D. GOTTLIEB AND R. S. HIRSH, *Parallel Pseudospectral Domain Decomposition Techniques*, ICASE Report 88-15, NASA Langley, February 1988.
 - [22] W. HACKBUSCH, *Multigrid Methods—Theory and Applications*, Springer-Verlag, Berlin, 1986.

- [23] W. HACKBUSCH, *A new approach to robust multi-grid solvers*, in ICIAM '87, Proceedings of the First International Conference on Industrial and Applied Mathematics (Paris, 1987), SIAM, Philadelphia, 1988, pp. 114–126.
- [24] M. R. HESTENES AND E. STIEFEL, *Methods of conjugate gradients for solving linear systems*, J. Res. Nat. Bur. Stand., 49 (1952), pp. 409–436.
- [25] M. KOČVARA AND J. MANDEL, *A multigrid method for three-dimensional elasticity and algebraic convergence estimates*, Appl. Math. Comput., 23 (1987), pp. 121–135.
- [26] P. L. LIONS, *On the Schwarz alternating method. I.*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1988, pp. 1–42.
- [27] Y. MADAY AND R. MUÑOZ, *Spectral element multigrid. II. Theoretical justification*, J. Sci. Comp., 3 (1988), pp. 323–354.
- [28] J. MANDEL, *Iterative solvers by substructuring for the p-version finite element method*, International Conference on Spectral and High Order Methods for Partial Differential Equations, Como, Italy, June 1989; Comput. Methods Appl. Mech. Engrg., submitted.
- [29] ———, *Two-level domain decomposition preconditioning for the p-version finite element method in three dimensions*, Fourth Copper Mountain Conference on Multigrid Methods, Copper Mountain, CO, April 1989; Int. J. Numer. Methods Engrg., to appear.
- [30] J. MANDEL AND G. S. LETT, *Domain decomposition preconditioning for p-version finite elements with high aspect ratios*, Applied Numer. Anal., submitted.
- [31] J. MANDEL AND S. MCCORMICK, *Iterative solution of elliptic equations with refinement: The two-level case*, in Domain Decomposition Methods for Partial Differential Equations II, T. Chan, R. Glowinski, G. A. Meurant, J. Périaux, and O. Widlund, eds., 1989, pp. 81–92.
- [32] J. MANDEL, S. MCCORMICK, AND R. BANK, *Variational multigrid theory*, in Multigrid Methods, S. McCormick, ed., SIAM, Philadelphia, 1987, ch. 5.
- [33] A. QUARTERONI AND G. SACCI-LANDRIANI, *Domain decomposition preconditioners for the spectral collocation method*, J. Sci. Comput., 3 (1988), pp. 45–76.
- [34] E. M. RØNQUIST AND A. T. PATERA, *Spectral element multigrid. I. Formulation and numerical results*, J. Sci. Comput., 2 (1987), pp. 389–406.
- [35] B. A. SZABÓ, *PROBE Theoretical Manual*, Noetic Technologies, St. Louis, MO, 1985.
- [36] A. J. WATHEN, *Realistic eigenvalue bounds for the Galerkin mass matrix*, IMA J. Numer. Anal., 7 (1987), pp. 449–457.
- [37] ———, *Spectral bounds and preconditioning methods using element-by-element analysis for Galerkin finite element equations*, in The Mathematics of Finite Elements and Applications, VI, Uxbridge, 1987, Academic Press, London-New York, 1988, pp. 157–168.
- [38] O. B. WIDLUND, *Iterative substructuring methods: Algorithms and theory for problems in the plane*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1988, pp. 113–128.
- [39] O. C. ZIENKIEWICZ, *The Finite Element Method*, McGraw Hill, London, third ed., 1977.