CHAPTER 4

Local Refinement Via Domain Decomposition Techniques for Mixed Finite Element Methods with Rectangular Raviart-Thomas Elements*

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Abstract. We consider the solution of second-order elliptic equations by Raviart-Thomas mixed finite element methods on locally refined rectangular grids. The local refinement uses "slave" nodes in a manner that assures continuity of normal fluxes at interfaces between refined and unrefined elements. Under these circumstances, the composite elements satisfy the Babuška-Brezzi condition and an error estimate follows. Relative to unrefined grids, this estimate loses half an order of convergence in a strip containing the interfaces, as dictated by approximation theory. The indefinite composite system of algebraic equations is solved by preconditioning the definite Schur complement with a spectrally equivalent matrix and applying the conjugate-gradient method. Numerical computations are presented that verify the expected global orders of convergence and suggest superconvergence phenomena analogous to known results for unrefined grids.

1. Introduction. Mixed finite element methods have been used in a wide variety of applications when high accuracy is desired for both a function and its

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gradient. These methods are applied to both elliptic and parabolic partial differential equations, alone and coupled with other equations. Often the applications are of sufficiently large scale that important local phenomena are not well resolved on a global coarse grid and local grid refinement techniques are needed. Because of the large size of these problems, efficient solution of the algebraic equations resulting from the local refinement procedures is essential. In this paper, domain decomposition ideas are used to develop accurate and efficient techniques for combining local grid refinement and mixed finite element methods.

We consider mixed finite element methods for second-order elliptic equations with Dirichlet boundary conditions. Neumann boundary conditions do not change the results in any significant way. In its mixed form, the problem is to find $(\mathbf{u}, p) \equiv (u_1, u_2, p)$, such that

$$\begin{aligned} &\text{(a)} \quad \mathbf{u} + \alpha \nabla p = 0, \qquad \text{in } \Omega, \\ &\text{(b)} \quad \operatorname{div} \mathbf{u} = f, \qquad \text{in } \Omega, \end{aligned} \tag{1.1}$$

with the boundary condition

$$p = -g, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\partial\Omega$ indicates the boundary of Ω, and $\alpha = \alpha(x)$ is a positive continuous function on $\Gamma = \Omega \cup \partial\Omega$. Here $\mathbf{u}$ and p can be considered to be fluid velocity and pressure, respectively, in a problem describing fluid flow in porous media. (1.1a) is a representation of Darcy's law and (1.1b) is a conservation law.

For the sake of simplicity, we take the domain to be the square $(0,1) \times (0,1)$. The results that we derive in this paper can be easily extended to more general domains and to 3–D elliptic and parabolic problems. We restrict our analysis to the Raviart-Thomas approximating spaces for a rectangular non-uniform partition of the domain induced by local grid refinement. We consider rectangular elements for ease of extension to large, three-dimensional problems.

We introduce the concept of "slave" nodes in the mixed method, construct the corresponding spaces, and give the finite element approximation to the weak saddle-point formulation of the boundary value problem. We show that the constructed finite element spaces on the composite grid satisfy the Babuška-Brezzi condition [1,2,3]. In the particular case of rectangular elements with local refinement, this condition appears to be a simple consequence of the construction of the spaces if "slave" nodes are used on the refinement interface.

We also derive an error estimate for the finite element solution. The lowest-order Raviart-Thomas spaces, combined with numerical integration by a trapezoidal rule for the mass matrices, produce the 5-point finite difference approximation for the pressure equation (see [4]). For regular stars, near the fine- and coarse-grid interface, this approximation produces only an $O(h^{\frac{1}{2}})$ error estimate; thus, one goal here was to check whether the mixed finite element approximation will produce a better error estimate. Unfortunately we are able to prove only $O(h^{r+0.5})$ convergence of the finite element solution to the true solution in an $H(\mathrm{div}; \Omega)$-norm for the velocity and L_2 norm for the pressure, i.e., we have a loss of $h^{0.5}$ compared to the optimal

error estimate for grids without refinement. This reduction of the convergence rate is due to the introduction of "slave" nodes along the interface for coarse/fine-grid domains.

In Section 2, we present notations and definitions which are necessary to describe our methods. We also construct our composite-grid spaces. In Section 3, we show that the composite-element spaces satisfy the Babuška-Brezzi condition and obtain error estimates. We discuss quadratures and approximations which simplify the connections between elements and the corresponding matrices. Optimal preconditioned iterative techniques are described in Section 4. Finally, computational results for local refinement of rectangular elements are presented.

2. Preliminaries. Denote by $(u,v) = \int_\Omega uv\ dx$ the standard $L^2(\Omega)$-inner product. If $\mathbf{u}$ and $\mathbf{v}$ are vector-valued functions, then by $\mathbf{uv}$ we denote the standard inner product.

Let $\mathbf{V} = H(\mathrm{div};\Omega)$ be the set of vector functions $\mathbf{v} = (v_1, v_2)$, $v_i \in L^2(\Omega)$, $i = 1,2$, such that $\nabla \cdot \mathbf{v} \equiv \mathrm{div}\ \mathbf{v} \in W$, with $W = L^2(\Omega)$. We also use the notation $W_2^m(\Omega)$ for the Sobolev space of functions defined on Ω and having their generalized derivatives up to order m summable in $L^2(\Omega)$. The norms on $W_2^m(\Omega)$ are defined in the standard way.

For the elements of the spaces $\mathbf{V}$ and W, we use the following norm notation

$$\begin{aligned} &\text{(a)} \quad ||\mathbf{u}||_{\mathbf{V}}^2 \equiv ||\mathbf{u}||_{H(\mathrm{div};\Omega)}^2 = ||\mathbf{u}||_{0,\Omega}^2 + ||\mathrm{div}\ \mathbf{u}||_{0,\Omega}^2, \quad \mathbf{u} \in \mathbf{V}, \\ &\text{(b)} \quad ||w||_W \equiv ||w||_{0,\Omega}. \end{aligned} \tag{2.1}$$

For $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $p, w \in W$, define the bilinear forms

$$\begin{aligned} &\text{(a)} \quad A(\mathbf{u},\mathbf{v}) = (\alpha^{-1}u_1, v_1) + (\alpha^{-1}u_2, v_2) \\ &\text{(b)} \quad B(\mathbf{u}, w) = (\mathrm{div}\ \mathbf{u}, w) \end{aligned} \tag{2.2}$$

Then, the problem (1.1) is equivalent to solving the saddle-point problem given by

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) - B(\mathbf{v},p) &= \langle g, \mathbf{v}\cdot\mathbf{n}\rangle, \qquad \mathbf{v} \in \mathbf{V}, \\ B(\mathbf{u}, w) &= (f, w), \qquad w \in W, \end{aligned} \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\partial\Omega)$ and $\mathbf{n}$ is the outward normal vector to Ω.

Remark 2.1. In many important practical problems, the system (2.3) (for the pressure and the velocity) is coupled with the equations for the concentrations of various components (see, for example, Douglas, Ewing, and Wheeler [5], Ewing, Russell, and Wheeler [6]). In this case, it is natural to combine the mixed method for the pressure with the standard finite element method for the concentrations [5,7], or with methods tailored to transport-dominated equations [6].

Our goal is to construct and study the mixed finite element approximations to the problem (2.3) on rectangular grids with local refinement. We describe an approach using the concept of "slave nodes". This carries over the ideas of the two-grid approach widely used with standard finite element and finite difference methods (see Bramble, Ewing, Pasciak, and Schatz [8], McCormick [9], Bank and Dupont [10], Ewing, Lazarov, and Vassilevski [11], etc.) to the mixed finite element method.

First, we introduce some notation. We make use of the following set of polynomials of two variables (x, y)

$$Q(r,m) = \left\{ g(x,y) : g(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{m} \alpha_{ij} x^i y^j \right\}$$

(polynomials of degree at most r in the variable x and degree at most m in the variable y).

In the interval $[-1,1]$, we introduce the Gauss and Lobatto points. Let $L_{r+1}(x)$, $x \in [-1,1]$ be the Legendre polynomial of degree $r+1$; then the Gauss points $\hat{g}_i$, $i = 1, \ldots, r+1$ are zeros of $L_{r+1}(x)$, i.e., $L_{r+1}(\hat{g}_i) = 0$, $i = 1, \ldots, r+1$; the Lobatto points $\hat{\ell}_i$, $i = 0, 1, \ldots, r+1$ are zeros of $L'_{r+1}(x)$ and ± 1; i.e., $\hat{\ell}_0 = -1$, $\hat{\ell}_{r+1} = 1$, $L'_{r+1}(\hat{\ell}_i) = 0$, $i = 1, \ldots, r$.

Let $\hat{e} = [-1,1]^2$ be the reference element. In $\hat{e}$, we introduce the following sets of points:

$$\begin{aligned}
\hat{G}_0 &= \{(\hat{g}_i, \hat{g}_j) : i,j = 1, \ldots, r+1\}, \\
\hat{G}_1 &= \left\{(\hat{\ell}_i, \hat{g}_j) : i = 0, 1, \ldots, r+1, \quad j = 1, \ldots, r+1\right\}, \\
\hat{G}_2 &= \left\{(\hat{g}_i, \hat{\ell}_j) : i = 1, \ldots, r+1, \quad j = 0, 1, \ldots, r+1\right\}.
\end{aligned}$$

We use these sets to define nodal bases for the Raviart-Thomas finite element spaces. First, we introduce the coarse-grid partition denoted by $\tilde{T}_h$ of the rectangle $\Omega = (0,1) \times (0,1)$; the subscript indicates that we have a family of partitions depending on the mesh parameter h.

Let Δ_x and Δ_y be partitions of $I = [0,1]$, with local mesh sizes $h_{c,i}$, $h'_{c,j}$, $i = 1, \cdots, N_x$, $j = 1, \cdots, N_y$, respectively. Let $h_c = \max_{ij}(h_{c,i}, h'_{c,j})$. Thus, Ω has been partitioned into coarse finite elements $\tilde{e} = \tilde{e}_{k\ell} = [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell]$, $k = 1, \ldots, N_x$, $\ell = 1, \ldots, N_y$. We set $\text{meas}(\tilde{e}_{k\ell}) = h_{c,k-1} h'_{c,\ell-1}$ and suppose that there exist constants c_0 and c_1, independent of h_c, such that

$$c_0 h^2 \leq \text{meas } \tilde{e}_{k\ell} \leq c_1 h^2, \quad k = 1, \ldots, N_x, \quad \ell = 1, \ldots, N_y, \tag{2.4}$$

showing that the partition $\tilde{T}_h$ is in general nonuniform but regular (see, Ciarlet [7]).

Set

$$M_j^m(\Delta_x) = \left\{ g \in C^j(I) : g\big|_{[x_{k-1}, x_k]} \in P(m), \quad k = 1, \ldots, N_x \right\} \tag{2.5}$$

where $P(m)$ is the set of all polynomials of degree at most m. We define $M_j^m(\Delta_y)$ similarly. If $j = -1$, this means that $g(x)$ might be discontinuous at the points $x_1, \ldots, x_{N_x-1}$. Then we establish the following notation, associated with the coarse-grid partition $\tilde{T}_h$:

$$\begin{aligned}
&\text{(a)} \quad \tilde{V}_h^{r,1} = M_0^{r+1}(\Delta_x) \otimes M_{-1}^r(\Delta_y) \\
&\text{(b)} \quad \tilde{V}_h^{r,2} = M_{-1}^r(\Delta_x) \otimes M_0^{r+1}(\Delta_y) \\
&\text{(c)} \quad \tilde{\mathbf{V}}_h^r = \tilde{V}_h^{r,1} \times \tilde{V}_h^{r,2} \\
&\text{(d)} \quad \tilde{W}_h^r = M_{-1}^r(\Delta_x) \otimes M_{-1}^r(\Delta_y).
\end{aligned} \tag{2.6}$$

Note that, for any $\mathbf{v} \in \tilde{\mathbf{V}}_h^r$, we have div $\mathbf{v} \in \tilde{W}_h^r$. The spaces $\tilde{\mathbf{V}}_h^r$ and $\tilde{W}_h^r$ are usually referred to as Raviart-Thomas spaces of order r [12]. Now, we introduce nodal bases for these spaces.

For each element $\tilde{e} \in \tilde{T}_h$, there exists an affine mapping $F : \hat{e} \to \tilde{e}$. The map F introduces the sets $G_i = G_i(\tilde{e}) = F(\hat{G}_i)$, $i = 0, 1, 2$ on each $\tilde{e} \in \tilde{T}_h$. Then the degrees of freedom in each element associated with the spaces $\tilde{W}_h^r$, $\tilde{V}_h^{r,1}$, and $\tilde{V}_h^{r,2}$ are the values of the functions at G_0, G_1, and G_2, respectively. Then the interpolants of given continuous functions w, v_1, v_2 are defined by the following: on each $\tilde{e} \in \tilde{T}_h$,

$$\begin{aligned}
&\tilde{I}w \in Q(r,r) \quad \text{and} \quad \tilde{I}w(x,y) = w(x,y), \quad \text{for } (x,y) \in G_0, \\
&\tilde{I}v_1 \in Q(r+1,r) \quad \text{and} \quad \tilde{I}v_1(x,y) = v_1(x,y), \quad \text{for } (x,y) \in G_1, \\
&\tilde{I}v_2 \in Q(r,r+1) \quad \text{and} \quad \tilde{I}v_2(x,y) = v_x(x,y), \quad \text{for } (x,y) \in G_2.
\end{aligned} \tag{2.7}$$

Obviously $\tilde{I}w \in \tilde{W}_h^r$, $\tilde{I}v_i \in \tilde{V}_h^{r,i}$, $i = 1, 2$.

Remark 2.2. The bases of $\tilde{W}_h^r$, $\tilde{V}_h^{r,1}$, and $\tilde{V}_h^{r,2}$ introduced by the nodal values of the functions at the sets G_0, G_1, and G_2, respectively, on each $\tilde{e} \in \tilde{T}_h$, represent only one possible choice of bases in these spaces. Actually, they do not necessarily appear in a particular implementation of the mixed method. However, they play an essential role in our construction of the composite-grid approximation.

Now we construct the corresponding finite element spaces over the partition with local refinement. Let Ω_1, a subdomain of Ω, be a union of a certain number of coarse finite elements. We partition the elements in Ω_1, introducing a finer mesh as shown in Figure 1. In order to simplify our notation, we suppose that the refinement in each coarse element is uniform and the maximal fine grid step size is h_f. This partition is regular and consistent, i.e., any two adjacent elements in Ω_1 have the same partition along their common side (see Figure 1).

Therefore we have partitioned the domain Ω into coarse and fine elements. We call this partition a composite partition (or composite grid) and denote it by T_h.

Now, we construct the composite-grid finite element spaces W_h^r, $V_h^{r,1}$, and $V_h^{r,2}$. Since the construction of the space W_h^r does not require any continuity of the

functions across the finite element boundaries, an obvious definition of W_h^r is

$$W_h^r = \left\{ w(x,y) : w(x,y)\Big|_e \in Q(r,r) \text{ for any } e \in T_h \right\}.$$

Then the Gauss nodes $G_0(e)$ in each element $e \in T_h$ will introduce a nodal basis for the space W_h^r that is convenient for computations.

Since the velocity $\mathbf{u}$ should have certain continuity properties across the element interfaces, the construction of the space $\mathbf{V}_h^r$ is much more complicated. Here we use the concept of "slave" nodes widely used in the standard finite element method on grids with local refinement (see, e.g., [8,13]), where a natural and compact construction is given by Dryja and Widlund [13]. We use their approach in order to construct the finite element space $\mathbf{V}_h^r$ for the mixed method.

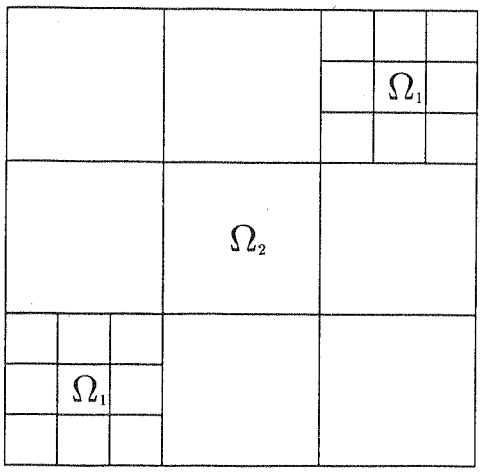

Figure 1. Grid with local refinement.

Let $\Gamma_x(\Gamma_y)$ denote the vertical (horizontal) part of the boundary between Ω_1 and $\Omega_2 = \Omega \backslash \Omega_1$ (the parts of the boundary $\partial\Omega$ are excluded). Now, define the space $V_h^{r,1}(\Omega_1)$ as the set of functions vanishing in Ω_2, which are continuous in the variable x for each fixed y and are polynomials in $Q(r+1,r)$ on each element. Similarly, $V_h^{r,2}(\Omega_1)$ is the set of functions vanishing in Ω_2, which are continuous in the variable y for each fixed x and are polynomials in $Q(r,r+1)$ on each element. The continuity requirement for the elements of $V_h^{r,1}(\Omega_1)$ and $V_h^{r,2}(\Omega_1)$ will produce functions which will be zero on Γ_x and Γ_y, respectively.

Then the spaces $V_h^{r,1}$, $i = 1,2$ on the composite partitioning T_h can be defined as follows:

$$V_h^{r,i} = \tilde{V}_h^{r,i} + V_h^{r,i}(\Omega_1), \quad i = 1,2.$$

Note that this is not a direct sum of finite element spaces. Then $\mathbf{V}_h^r = V_h^{r,1} \times V_h^{r,2}$ is the composite-grid finite element space for the velocity.

In order to better understand this construction, we introduce nodal bases for $V_h^{r,i}$, $i = 1,2$. If we introduce the nodes $G_i(e)$, $i = 1,2$ in each fine-grid element, then the corresponding Lagrangian polynomials on each element will form a basis for $V_h^{r,i}(\Omega_1)$ over e. Figure 2 illustrates this basis at the fine-grid elements which are adjacent to the boundary of fine/coarse-grid regions.

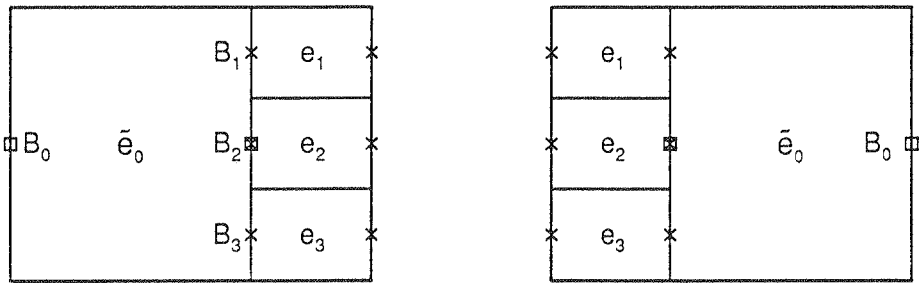

Figure 2. RT elements of order $r = 0$.

For the lowest-order RT finite elements ($r = 0$), the unknown values of the first component of the velocity v_1 in the coarse-grid finite element $\tilde{e}_0$ are $v_1(B_0)$ and $v_1(B_2)$. From the fine-grid elements, we have as unknowns the values of v_1 at the points B_1, B_2, and B_3. The continuity requirement will be satisfied if B_1 and B_3 are "slave" nodes, i.e., $v_1(B_1) = v_1(B_3) = v_1(B_2)$, which is in agreement with the construction of the space $V_h^{r,1}$. The second component of the velocity vector, v_2, is treated in a similar way.

3. Finite Element Method. The finite element solution of the problem (2.3) is defined in the following way: find $\mathbf{U} \in \mathbf{V}_h^r$ and $P \in W_h^r$ such that

$$\begin{aligned} A(\mathbf{U}, \mathbf{v}) - B(\mathbf{v}, P) &= \langle g, \mathbf{v} \cdot \mathbf{n} \rangle \\ B(\mathbf{U}, w) &= (f, w) \end{aligned} \tag{3.1}$$

are satisfied for any $\mathbf{v} \in \mathbf{V}_h^r$ and any $w \in W_h^r$.

The stability of problem (3.1) and the uniqueness of its solution is established under the well-known Babuška-Brezzi condition [1,2], which reads as follows: first, if $B(\mathbf{v}, w) = 0$ for all $w \in W_h^r$, then div $\mathbf{v} = 0$ (i.e., given W_h^r, $\mathbf{V}_h^r$ cannot be too large). Second, there exists a constant $\beta > 0$, independent of h, such that

$$\inf_{w \in W_h^r} \sup_{\mathbf{v} \in \mathbf{V}_h^r} \frac{B(\mathbf{v}, w)}{||w||_W ||\mathbf{v}||_{\mathbf{V}}} \geq \beta \tag{3.2}$$

(i.e., given W_h^r, $\mathbf{V}_h^r$ cannot be too small). Since div $\mathbf{V}_h^r \subset W_h^r$, it is evident that our spaces satisfy the first condition.

Falk and Osborn [3] showed that the condition (3.2) is satisfied if, for any $w \in W_h^r$, there exists $\mathbf{v} \in \mathbf{V}_h^r$ such that

$$\text{div } \mathbf{v} = w \text{ in } \Omega \text{ and } ||\mathbf{v}||_{\mathbf{V}} \leq C||w||_W, \tag{3.3}$$

with a constant independent of h.

It is well known that the finite element spaces of Raviart-Thomas satisfy the condition (3.2). In our case, these spaces were augmented by adding some new functions in the refined region Ω_1. We show that the finite element spaces $\mathbf{V}_h^r$ and W_h^r, defined on the composite partitioning of the domain Ω, satisfy condition (3.3).

Lemma 3.1. The spaces $\mathbf{V}_h^r$ and W_h^r, defined on the composite partitioning of the domain Ω, satisfy the Babuška-Brezzi condition.

Proof. We show that the condition (3.3) is satisfied, for example, for the refined region in the top right corner of Figure 1. Let

$$v_1(x,y) = \frac{1}{2}\int_0^x w(s,y)ds, \qquad v_2(x,y) = \frac{1}{2}\int_0^y w(x,s)ds.$$

By $v_1(x,y) \in V_h^{r,1}$; since w is a piecewise polynomial in $Q(r,r)$, then after integration in x, it will be continuous as a function of x and will be a piecewise polynomial in $Q(r+1,r)$. Similarly, $v_2(x,y) \in V_h^{r,2}$ and, therefore, $\mathbf{v} = (v_1, v_2) \in \mathbf{V}_h^r$. Clearly $\operatorname{div} \mathbf{v} = w$.

Now, we show that $||\mathbf{v}||_{\mathbf{V}} \le c||w||_{0,\Omega}$. By the Cauchy inequality, we have

$$\begin{aligned}
||\mathbf{v}||_{\mathbf{V}}^2 &= ||v_1||_{0,\Omega}^2 + ||v_2||_{0,\Omega}^2 + ||\operatorname{div}\mathbf{v}||_{0,\Omega}^2 \\
&= \frac{1}{4}\left\{\left\|\int_0^x w(s,y)ds\right\|_{0,\Omega}^2 + \left\|\int_0^y w(x,s)ds\right\|_{0,\Omega}^2\right\} + ||w||_{0,\Omega}^2 \\
&\le \frac{5}{4}||w||_{0,\Omega}^2,
\end{aligned}$$

where we use

$$\begin{aligned}
\left\|\int_0^x w(s,y)ds\right\|_{0,\Omega}^2 &\le \int_0^1\int_0^1\int_0^x w(s,y)^2 ds\, x\, dx\, dy \\
&\le \int_0^1\int_0^1\int_0^x w(s,y)^2 ds\, dy\, x\, dx \\
&\le \int_0^1 ||w||_{0,\Omega}^2\, x\, dx = \frac{1}{2}||w||_{0,\Omega}^2.
\end{aligned}$$

A similar argument will treat the refinement in the lower left corner of Figure 1. A slightly more general treatment is required for the general case. Thus the proof of Lemma 3.1 in the case of Figure 1 is completed.

As a corollary of Lemma 3.1, by the main result of Brezzi's paper [2], we get the stability of the discrete problem and the estimate for the error of the method.

Theorem 3.1. Problem (3.1) has a unique solution, $(\mathbf{U}, P) \in \mathbf{V}_h^r \times W_h^r$, and there is a constant C, dependent only on β, such that

$$||\mathbf{u}-\mathbf{U}||_{\mathbf{V}} + ||p-P||_{0,\Omega} \le C\left\{\inf_{\mathbf{v}\in\mathbf{V}_h^r} ||\mathbf{u}-\mathbf{v}||_{\mathbf{V}} + \inf_{w\in W_h^r} ||p-w||_{0,\Omega}\right\}. \tag{3.4}$$

We use this theorem in order to obtain an error estimate of our finite element method on the composite-grid partitioning T_h. Since $\tilde{\mathbf{V}}_h^r \subset \mathbf{V}_h^r$ and $\tilde{W}_h^r \subset W_h^r$, then by the standard approximation theory [3], we have the following obvious estimate

$$\begin{aligned}\inf_{\mathbf{v}\in\mathbf{V}_h^r} ||\mathbf{u}-\mathbf{v}||_{\mathbf{V}} + \inf_{w\in W_h^r} ||p-w||_{0,\Omega} &\leq \inf_{\mathbf{v}\in\tilde{\mathbf{V}}_h^r} ||\mathbf{u}-\mathbf{v}||_{\mathbf{V}} + \inf_{w\in\tilde{W}_h^r} ||p-w||_{0,\Omega} \\ &\leq Ch_c^{r+1}\left(||\mathbf{u}||_{r+2,\Omega} + ||p||_{r+1,\Omega}\right).\end{aligned}$$

This estimate is too pessimistic since it does not take into account the grid refinement in the subdomain Ω_1 (where we have augmented our finite element space). Our goal is to produce an error estimate in mesh-dependent norms, which takes the refinement procedure into account.

The second term in the right-hand side of (3.4) can be estimated in the following way (for $r > 0$)

$$\inf_{w\in W_h^r} ||p-w||_{0,\Omega} \leq ||p-Ip||_{0,\Omega} \leq C\left(h_c^{r+1}||p||_{r+1,\Omega_2} + h_f^{r+1}||p||_{r+1,\Omega_1}\right), \tag{3.5}$$

where Ip is the interpolant of p using $(r+1)^2$ Gauss points in each finite element, defined by (2.7). The estimate (3.5) is valid also for $r = 0$ but instead of the interpolant Ip, we have to use the L_2-projection of p.

The first term in the right-hand side in (3.4) is more difficult to estimate since the norm of the space $\mathbf{V}$ involves L_2-norms of $\mathbf{v}$ and div $\mathbf{v}$. If we again use the interpolant $I\mathbf{u}$, then

$$\inf_{\mathbf{v}\in\mathbf{V}_h^r} ||\mathbf{u}-\mathbf{v}||_{\mathbf{V}}^2 \leq ||\mathbf{u}-I\mathbf{u}||_{0,\Omega}^2 + ||\text{div}\,(\mathbf{u}-I\mathbf{u})||_{0,\Omega}^2. \tag{3.6}$$

The first term in the right-hand side of (3.6) gives a similar error as the term with p, estimated by (3.5). Then, we have the desired optimal error estimate

$$||\mathbf{u}-I\mathbf{u}||_{0,\Omega} \leq C\left(h_c^{r+1}||\mathbf{u}||_{r+1,\Omega_2} + h_f^{r+1}||\mathbf{u}||_{r+1,\Omega_1}\right). \tag{3.7}$$

The main difficulties are connected with the estimation of the error of the second term in the right-hand side of (3.6). Using the same technique, based on the Bramble-Hilbert lemma argument, we can prove the following estimate

$$||\text{div}\,(\mathbf{u}-I\mathbf{u})||_{0,\Omega} \leq C\left(h_c^{r+1}||\mathbf{u}||_{r+2,\Omega_2} + h_f^{r+1}||\mathbf{u}||_{r+2,\Omega_1} + h_f^{r+0.5}||\mathbf{u}||_{r+2,\delta}\right), \tag{3.8}$$

where δ is a strip of finite width around the interface between Ω_1 and Ω_2.

Then, substituting the estimates (3.5)–(3.8) into (3.4), we get the following error estimate for the mixed method with local refinement.

Theorem 3.2. Let the solution to the problem (1.1) and (1.2) satisfy the following regularity conditions: $u_i \in W_2^{r+2}(\Omega)$, $i = 1, 2$ and $p \in W_2^{r+1}(\Omega)$. Then the finite element solution $(\mathbf{U}, P)$ of the problem (3.1) satisfies the following error estimate

$$\begin{aligned}||\mathbf{u}-\mathbf{U}||_{\mathbf{V}} + ||p-P||_{0,\Omega} \leq\ & C\left(h_c^{r+1}||p||_{r+1,\Omega_2} + h_f^{r+1}||p||_{r+1,\Omega_1}\right. \\ & \left. + h_c^{r+1}||\mathbf{u}||_{r+2,\Omega_2} + h_f^{r+1}||\mathbf{u}||_{r+2,\Omega_1} + h_f^{r+0.5}||\mathbf{u}||_{r+2,\delta}\right)\end{aligned} \tag{3.9}$$

where the constant C does not depend on h.

The appearance of the $h^{0.5}$ convergence rate for $r = 0$ is due to the construction of the finite element space on the composite grid using the concept of "slave" nodes. In order to obtain an optimal order for the composite-grid finite element solution, one should look for a finite element partitioning of the subdomain, where the coarse grid gradually is refined into a fine one using triangular or trapezoidal finite elements.

In practice, the bilinear form $A(\mathbf{U}, \mathbf{v})$ in (3.1) is often evaluated using some quadrature rule. If the coefficient $\alpha(x, y)$ is sufficiently smooth and the quadrature has high-order accuracy (e.g., on each element it is exact for polynomials in $Q(2r+3, 2r+1)$ while integrating $(\alpha^{-1}u_1, v_1)$ and exact for polynomials in $Q(2r+1, 2r+3)$ while integrating the term $(\alpha^{-1}u_2, v_2)$), then we can easily obtain the basic result of Theorem 3.2. Therefore, the convergence rate for sufficiently accurate quadratures will in principle be the same as in the case of exact evaluation of the form $A(\mathbf{U}, \mathbf{v})$.

There is one very important class of quadratures for evaluating the form $A(\mathbf{U}, \mathbf{v})$ which produces the so-called "lumped mass" approximation. This means that the matrix corresponding to this bilinear form is diagonal. Therefore, the velocity vector $\mathbf{U}$ can be eliminated from the algebraic problem and we can get a discrete problem only in terms of P.

Remark 3.1. It is possible to confine the lumped mass procedure to only some of the finite elements, for example, to those adjacent to the coarse/fine interface.

Remark 3.2. Using the lumped mass approximation, we obtain the following algebraic problem

$$\begin{vmatrix} \hat{M}_1 & 0 & N_1 \\ 0 & \hat{M}_2 & N_2 \\ N_1^T & N_2^T & 0 \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \\ P \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ b_3 \end{vmatrix} \tag{3.10}$$

where the matrices $\hat{M}_i$ are diagonal. Then the elimination of the velocity is straightforward and we can get an algebraic equation only for the pressure P

$$\left(N_1^T \hat{M}_1^{-1} N_1 + N_2^T \hat{M}_2^{-1} N_2\right) P = b_3 \tag{3.11}$$

which is a direct approximation of the elliptic equation. In the case of lowest-order Raviart-Thomas finite elements, i.e., $r = 0$, this is a standard 5-point approximation on a cell-centered grid with local refinement (see, e.g., Ewing, Lazarov, and Vassilevski [11]).

4. Iterative Methods for Solving the Composite Grid System. In order to simplify our notations, we introduce standard notations for the systems with consistent mass approximation and with lumped mass approximation.

$$A\mathbf{Y} = \mathbf{b} \quad \text{and} \quad \hat{A}\mathbf{Y} = \mathbf{b}, \quad \text{where } \mathbf{Y} = \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix}. \tag{4.1}$$

Together with systems (4.1), we consider the coarse-grid approximation to problem (2.3) defined in the following way: find $\tilde{U} \in \tilde{V}_h^r$ and $\tilde{P} \in \tilde{W}_h^r$ such that

$$\begin{aligned} A(\tilde{\mathbf{U}}, \mathbf{v}) - B(\mathbf{v}, \tilde{P}) &= \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, \qquad \forall\, \mathbf{v} \in \tilde{V}_h^r, \\ B(\tilde{\mathbf{U}}, w) &= (f, w), \qquad \forall\, w \in \tilde{W}_h^r. \end{aligned} \tag{4.2}$$

The system of linear equations corresponding to (4.2), for the unknown vector $\tilde{\mathbf{Y}} = (\tilde{\mathbf{U}}, \tilde{P})$ of approximate values of the solution $(\mathbf{u}, p)$ at the coarse-grid points from $\tilde{T}_h$, is denoted by

$$\tilde{A}\tilde{\mathbf{Y}} = \tilde{\mathbf{b}}, \quad \tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{U} \\ \tilde{P} \end{bmatrix}. \tag{4.3}$$

Our goal is to construct a preconditioner B based on the BEPS [3] preconditioning technique for solving the composite-grid system (4.1).

First, we denote by ω_1 all fine-grid points associated with the values of the pressure P in the refined region Ω_1. We denote the remaining points, corresponding to the values of P in the unrefined portion of the composite-grid partitioning, by ω_2. Then, we have the following partitioning of the vector P

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{matrix} \}\omega_1 \\ \}\omega_2 \end{matrix}. \tag{4.4}$$

Using the representation $\mathbf{Y} = \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix}$ and $\tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{U}} \\ \tilde{P} \end{bmatrix}$, the matrices A, $\hat{A}$, and $\tilde{A}$ can be written in the factorized form

$$\begin{aligned} A &= \begin{vmatrix} M & N \\ N^T & 0 \end{vmatrix} = \begin{vmatrix} M & 0 \\ N^T & -C \end{vmatrix} \begin{vmatrix} I & M^{-1}N \\ 0 & I \end{vmatrix}, \quad C = N^T M^{-1} N, \\ \hat{A} &= \begin{vmatrix} \hat{M} & N \\ N^T & 0 \end{vmatrix} = \begin{vmatrix} \hat{M} & 0 \\ N^T & -\hat{C} \end{vmatrix} \begin{vmatrix} I & \hat{M}^{-1}N \\ 0 & I \end{vmatrix}, \quad \hat{C} = N^T \hat{M}^{-1} N, \\ \tilde{A} &= \begin{vmatrix} \tilde{M} & \tilde{N} \\ \tilde{N}^T & 0 \end{vmatrix} = \begin{vmatrix} \tilde{M} & 0 \\ \tilde{N}^T & -\tilde{C} \end{vmatrix} \begin{vmatrix} I & \tilde{M}^{-1}\tilde{N} \\ 0 & I \end{vmatrix}, \quad \tilde{C} = \tilde{N}^T \tilde{M}^{-1} \tilde{N}. \end{aligned} \tag{4.5}$$

Obviously, C, $\hat{C}$, and $\tilde{C}$ are the corresponding Schur complements, and the system $CP = b$ represents an approximation to the elliptic equation for the pressure P. Note that since $\hat{M}$ is a diagonal matrix, then the matrix $\hat{C}$ is explicitly defined. The other two matrices are defined only implicitly. The coupling of the unknowns of the vector $\tilde{U}_1(\tilde{U}_2)$ is only along the lines $y = \text{const}$ ($x = \text{const}$). Therefore, the inversion of the matrix $\tilde{M}$ can be performed efficiently with arithmetic work proportional to the number of unknowns in $\tilde{U}_1$ and $\tilde{U}_2$ (we have to solve $(2r+3)$-diagonal systems).

The inversion of the composite-grid matrix is much more complicated, since the unknowns in the refined regions are coupled with the unknowns in the coarse-grid region as shown in Figure 3.

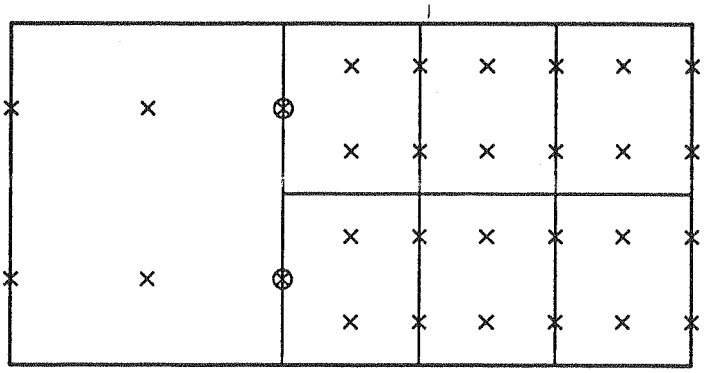

Figure 3.

All unknowns U_1 are coupled via the values at the nodes marked by "$\otimes$". Nevertheless, we can also design an efficient algorithm for solving the system $MU = d$, and therefore, computing $M^{-1}d$.

Remark 4.1. Matrices A, $\hat{A}$, and $\tilde{A}$ are invertible and symmetric, but not definite. Their Schur complements, C, $\hat{C}$, and $\tilde{C}$, respectively, are symmetric and positive definite. Actually we may consider that $A\mathbf{Y} = -CP$ in the subspace of $\mathbf{Y}$ satisfying $MU + NP = 0$. Therefore, in the subspace of vectors satisfying the condition $MU + NP = 0$, we can use the preconditioned conjugate-gradient method. In other words, we have to find a preconditioner B for the matrix C.

The preconditioner B for matrix C will be based on the general idea of the BEPS-preconditioning technique proposed by Bramble, Ewing, Pasciak, and Schatz [8]. We use its algebraic formulation described in [11]. The partitioning of the vector P by (4.4) implies 2×2 block partitioning of the matrices C and $\hat{C}$:

$$C = \left|\begin{array}{cc|l} C_{11} & C_{12} & \}\omega_1 \\ C_{21} & C_{22} & \}\omega_2 \end{array}\right| \qquad \hat{C} = \left|\begin{array}{cc} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{array}\right|. \tag{4.6}$$

Since $\hat{C}$ is explicitly known, then the corresponding blocks $\hat{C}_{ij}$, $i, j = 1, 2$ are known explicitly, too.

Since the nodes ω_2 are part of the coarse-grid points $\tilde{\omega}$, we can partition the vector $\tilde{P}$ in the form

$$\tilde{P} = \left|\begin{array}{c|l} \tilde{P}_1 & \}\tilde{\omega}\backslash\omega_2 \\ \tilde{P}_2 & \}\omega_2 \end{array}\right|, \quad \text{and therefore} \quad \tilde{C} = \left|\begin{array}{cc|l} \tilde{C}_{11} & \tilde{C}_{12} & \}\tilde{\omega}\backslash\omega_2 \\ \tilde{C}_{21} & \tilde{C}_{22} & \}\omega_2 \end{array}\right|. \tag{4.7}$$

Then the preconditioner B for the matrix C can be chosen in the following form

$$B = \left|\begin{array}{cc} \hat{C}_{11} & 0 \\ \hat{C}_{21} & \tilde{\sigma}_{22} \end{array}\right| \left|\begin{array}{cc} I & \hat{C}_{11}^{-1}\hat{C}_{12} \\ 0 & I \end{array}\right|, \quad \tilde{\sigma}_{22} = \tilde{C}_{22} - \tilde{C}_{21}\tilde{C}_{11}^{-1}\tilde{C}_{12}. \tag{4.8}$$

Now, we show that this choice of B leads to an algorithm which uses the solution of an approximation of the elliptic equation on a standard uniform partitioning (coarse for the whole domain Ω or fine in the subdomain Ω_1).

We define B^{-1} by the following algorithm (and later we show that this really produces B of (4.8)). From the system $BP = b$:

(i) Solve the system

$$\hat{C}_{11} P_1^{(f)} = b_1;$$

(ii) Compute the residual

$$r = \begin{vmatrix} b_1 \\ b_2 \end{vmatrix} - \hat{C} \begin{vmatrix} P^{(f)} \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ b_2 - \hat{C}_{21} \quad P_1^{(f)} \end{vmatrix};$$

(iii) Restrict the residual r onto the coarse grid $\tilde{\omega}$, i.e.,

$$\tilde{r} = \begin{vmatrix} 0 \\ b_2 - \hat{C}_{21} \quad P_1^{(f)} \end{vmatrix} \begin{matrix} \}\tilde{\omega}\backslash\omega_2 \\ \}\omega_2 \end{matrix}$$

and solve $\tilde{C}\tilde{P} = \tilde{r}$;

(iv) Since

$$\tilde{P} = \begin{vmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{vmatrix} \begin{matrix} \}\tilde{\omega}\backslash\omega_2 \\ \}\tilde{\omega}_2 = \omega_2 \end{matrix},$$

find the harmonic extension by solving the system

$$\hat{C}_{11} P_1^{(h)} = -\hat{C}_{12} \tilde{P}_2;$$

(v) Then,

$$P = B^{-1} b = \begin{bmatrix} P_1^{(f)} + P_1^{(h)} \\ \tilde{P}_2 \end{bmatrix}.$$

Using the same arguments as in [11], we can show that steps (iii) and (iv), which solve the coarse-grid problem and restrict the solution to the coarse-grid points in Ω_2, are equivalent to solving a problem with $\tilde{\sigma}_{22}$. Therefore, the action of the preconditioner B is fully described. As we mentioned above, one step of the iteration with B includes solving two problems in the fine-grid region (with lumped mass approximation) and one problem on the coarse grid $\tilde{\omega}$ in the whole domain Ω. Efficient solution of these problems will be discussed elsewhere.

Using the techniques of [11], an intricate analysis shows that the preconditioner B is optimal; i.e., it is spectrally equivalent to the matrix C and, therefore, to the matrix A in the subspace $MU + NP = 0$, with constant independent of the mesh size. Full details will appear later.

5. Computational Results. We have performed extensive computations on the lowest-order Raviart-Thomas spaces in order to determine convergence rates under various refinement schemes. In this section we will describe our results and

make several observations about their relation to theoretical error estimates obtained elsewhere.

The lowest-order Raviart-Thomas spaces have discontinuous constants as pressure approximations. The x-component of the approximate velocity is a continuous linear function in x tensored with a constant in the y direction; the corresponding y-component, U_y, has the roles of the x and y variables reversed in the description of U_x.

For grids without local refinement, the error estimate in the same norms as those in the estimate (3.9) is $O(h)$ for the lowest-order Raviart-Thomas spaces. Along special lines through the cell centers and parallel to the coordinate axes, superconvergence results for the velocity components in $L^p(\Omega)$, $1 \leq p < \infty$, of $O(h^2)$ were obtained in [14]; [14] also contained a corresponding result for $L^\infty(\Omega)$ of $O(h^2 \ln h^{-1})$. Computational results illustrating these asymptotic convergence results were presented in [15]. The computational results given below were produced by codes which are very similar to those described in detail in [15]. Therefore, details will not be given here.

For testing purposes, we considered the problem (1.1) with $\Omega = [0,1] \times [0,1]$ and with f given by point sources and sinks, $f = \delta(0,0) - \delta(1,1)$. If $\alpha \equiv 1$, we have analytic series solutions for both p and $\mathbf{u}$. In this case one can compare the computed approximation to the analytic solution for various mesh sizes and local grid refinements to obtain estimates of the asymptotic convergence rates for our computational methods. Due to the Dirac delta sources, p has a singularity of order $\ln r$ and $\mathbf{u}$ has a singularity of order $1/r$, where r is the distance away from the point source or sink. Since $1/r$ is not in $L^2(\Omega)$, in order to obtain non-trivial convergence rates for the velocity approximation, we delete small squares with sides of length 0.125 around the sources and compute the convergence in the rest of the domain to obtain "interior" estimates.

Computational results are presented in Tables 1 and 2 in the form of convergence rates of the form

$$\text{error} = Kh^\alpha \tag{5.1}$$

for various levels of refinement (1 is no refinement; otherwise the given number is h_c/h_f). In Table 1, convergence rates in the L^2 norms are computed using quadrature points along the lines of superconvergence described in [14]. As expected, since $1/r$ is not square-integrable in the unit square, there is essentially no convergence for the velocity in the full region. Although there are currently no theoretical results yielding superconvergence in a regime of local refinement as described in this paper, we see from Table 1 that along the lines of superconvergence for the elements without refinement, we get approximately $O(h^{3/2})$ convergence for the L^2 norm of the velocity errors in the interior regions. This type of superconvergence will be discussed in more detail in a forthcoming paper. We also see superconvergence for the pressure errors at the cell centers of the form $O(h^2)$, approximately. Unlike many superconvergence results, the results from [14] are truly local, cell-by-cell results which do not require uniform gridding. We feel these local properties are also being reflected in the local grid refinement computations.

TABLE 1
Convergence Estimates Using 1×2 Gauss Points
$||p - P|| \leq K h_f^{\alpha}$ $\quad$ $||u - U|| \leq K h_f^{\alpha}$

Refinement		Full Region		Corners Removed	
h_c/h_f		$\|\|p-P\|\|$	$\|\|u-U\|\|$	$\|\|p-P\|\|$	$\|\|u-U\|\|$
1	α	1.01	0.00	1.89	1.96
	K	0.01	0.01	0.01	0.52
2	α	1.00	0.02	1.93	1.60
	K	0.02	0.01	0.02	0.15
3	α	1.00	0.02	1.93	1.58
	K	0.02	0.01	0.01	0.13
4	α	1.00	0.02	1.93	1.58
	K	0.01	0.01	0.01	0.13
5	α	1.01	0.03	1.91	1.57
	K	0.01	0.01	0.01	0.12
6	α	1.01	0.03	1.90	1.56
	K	0.01	0.01	0.01	0.12

TABLE 2
Convergence Estimates Using 2×2 Gauss Points
$||u - U|| \leq K h_f^{\alpha}$

Refinement		Full Region	Corners Removed
2	α	0.01	0.98
	K	0.06	0.15
3	α	0.01	0.94
	K	0.06	0.12
4	α	0.01	0.92
	K	0.06	0.11
5	α	0.01	0.89
	K	0.06	0.10
6	α	0.01	0.89
	K	0.06	0.10
7	α	0.01	0.90
	K	0.06	0.10

In order to reinforce the nature and location of the superconvergence computations we also computed the L^2 norm of the velocity errors using 2×2 Gauss-quadrature rules where superconvergence is not present in the uniform-grid cases. As expected, the convergence rates dropped in the interior region to approximately $O(h)$. This is still higher than the $O(h^{1/2})$ results obtained in Theorem 3.2, but there are important differences. The estimate in Theorem 3.2 is for the $H(\mathrm{div}; \Omega)$ norm for the velocity errors, while the computations are formally for the L^2 norm. However, in the interior region for this problem, div $\mathbf{u}$ and div $\mathbf{U}$ are both equal to zero and the $H(\mathrm{div}; \Omega)$ norm reduces to the L^2 norm in this case. The approximation-theory result is $O(h)$ in the L^2 norm, so we hope to be able to improve the $O(h^{1/2})$ result to $O(h)$ in this case. This will also be the subject of a later report.

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