CHAPTER 16

Multigrid Domain Decomposition Methods

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Abstract - The paper outlines the basic items and some general results of the multi-level domain decomposition methods (DD-methods) with the alternating Neumann-Dirichlet boundary conditions, which are based on a symmetric representation of the DD-preconditioner. For the model diffusion problem a close relation is established between the multi-level DD-methods involving a partitioning of grids into small substructures and the algebraic multigrid methods. For this reason the methods constructed have been called multigrid domain decomposition methods (MGDD-methods). Explicit estimates of condition numbers are established for the constructed two-grid methods and multigrid methods with inner Tchebyshev iterative procedures. The paper contains estimates of the arithmetic complexity of the methods suggested and the results of the numerical experiments.

1. INTRODUCTION

In this paper we will outline multi-level domain decomposition methods with alternating Neumann-Dirichlet boundary conditions and establish their remarkable relation with algebraic multigrid methods. To be more precise, we will prove that the multi-level domain decomposition methods involving a partitioning of grids into small substructures will lead to special multigrid methods and, hence, the intersection of these important classes of iterative methods is non-empty.

An impetus to these investigations was given by the publication

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[6] which suggested a new symmetric representation for preconditioners in the Neumann-Dirichlet domain decomposition methods [4,8,13,14]. The first results were announced by the author in [9]. The theory of multi-level DD-methods was outlined in more detail in [10]. The multigrid versions of these methods for the model diffusion equation were described in detail in the recent paper [11] which contains all the arguments. The author called them multigrid domain decomposition methods (MGDD-methods). The theory of MGDD-methods with inner Tchebyshev iterative procedures has much in common with the theory of algebraic multigrid methods contained in [2]. In addition, the superelement approach used here to estimate condition numbers for two-grid methods coincides with that used before in [1,3,5,12].

2. MULTI-LEVEL DOMAIN DECOMPOSITION METHOD

Let Ω be a bounded r-dimensional ($r = 2,3$) domain with the piecewise-linear boundary $\partial\Omega$ and Γ_0 be a closed poligonal subset of $\partial\Omega$.

Define a bilinear form

$$a(u,v) = \int_{\Omega} [a\nabla u \circ \nabla v + buv] d\Omega + \int_{\Gamma_1} \sigma uv \, d\Gamma \qquad (2.1)$$

where $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, a linear form

$$l(v) = \int_{\Omega} fv \, d\Omega \qquad (2.2)$$

where $f \in L_2(\Omega)$ is a given function and a space

$$V = \{v:\ v \in H' \ ,\quad v = 0 \quad \text{on} \quad \Gamma_0\} \qquad (2.3)$$

where $H' \equiv H'(\Omega)$ is a Sobolev space. We assume that a is a bounded piecewise-smooth function satisfying the condition $\inf_{\Omega} a > 0$, b and σ are bounded piecewise-smooth nonnegative functions, and the form $a(u,v)$ is positive definite.

Let us consider the following variational problem: find $u \in V$ such that

$$a(u,v) = l(v) \qquad \forall v \in V \ . \qquad (2.4)$$

Let us construct a triangular (tetrahedral) partitioning Ω_h of the domain Ω and define the space V_h as a set consisting of functions wich are continuous in Ω, linear in each of triangles (tetrahedrons) and vanishing on Γ_0. We assume that $V_h \subset V$. Then the dimension of V_h is equal to N, where N is the number of vertices of the triangles (tetrahedrons) belonging to $\Omega \cup \Gamma_1$.

Consider the following finite element problem: find $u_h \in V_h$ such that

$$a(u_h, v) = l(v) \quad \forall v \in V_h . \tag{2.5}$$

With respect to the natural basis of V_h this problem leads to the system of linear algebraic equations

$$Au = f \tag{2.6}$$

with a symmetric positive definite $N{\times}N$ matrix A and a vector $f \in \mathbb{R}^N$.

Remark. In case of the Neumann problem where $\Gamma_0 = \emptyset$, $b \equiv 0$ in Ω and $\sigma \equiv 0$ on Γ_1, the matrix A is singular, and by virtue of the condition $\int_\Omega f \, d\Omega = 0$ system (2.6) is compatible.

Partition the set of triangles (tetrahedrons) of Ω_h into two non-intersecting subsets and construct domains Ω_1 and Ω_2 consisting of triangles (tetrahedrons) from these subsets. Then partition the set of vertices of the triangles (tetrahedrons) into two groups; the second group includes the vertices belonging to $\bar{\Omega}_2$, and the first group includes all the remaining vertices. Similarly, partition the components of the vectors from $\mathbb{R}^N$ into two groups. Then (2.6) can be rewritten as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{2.7}$$

where A_{ij} are $N_i{\times}N_j$ submatrices of A and N_i are equal to the number of components in the ith group, $i = 1,2$.

Let us consider the representation

$$A = F \begin{bmatrix} A_{11} & 0 \\ 0 & S_{22} \end{bmatrix} F^T \tag{2.8}$$

where $S_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$,

$$F = \begin{bmatrix} E_{11} & 0 \\ A_{21}A_{11}^{-1} & E_{22} \end{bmatrix} \tag{2.9}$$

and E_{ii} are identity $N_i \times N_i$ matrices, $i = 1,2$. The matrix S_{22} is called the Schur complement for the matrix A.

Prescribe a symmetric positive definite $N_2 \times N_2$ matrix B_{22} and an $N \times N$ matrix

$$B = F\begin{bmatrix} A_{11} & 0 \\ 0 & B_{22} \end{bmatrix} F^{\mathrm{T}} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & B_{22}+A_{21}A_{11}^{-1}A_{12} \end{bmatrix} . \qquad (2.10)$$

Assume the eigenvalues of the matrix $B_{22}^{-1}S_{22}$ to belong to a segment $[\hat{\alpha};\hat{\beta}]$, where $0 < \hat{\alpha} < \hat{\beta}$. Then the following statement is valid.

Lemma 2.1. The eigenvalues of the matrix $B^{-1}A$ belong to the segment $[\alpha;\beta]$, where $\alpha = \min\{1;\hat{\alpha}\}$ and $\beta = \max\{1;\hat{\beta}\}$.

Denote by $a_2(u,v)$ the restriction of the form $a(u,v)$ onto the subdomain Ω_2 and define the $N_2 \times N_2$ matrix B_{22} by using the relation

$$(B_{22}u_2, v_2) = a_2(u_h, v_h) \quad \forall u_h, v_h \in V_h \quad (u, v \in \mathbb{R}^N) \qquad (2.11)$$

where $u^{\mathrm{T}} = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}]$, $v^{\mathrm{T}} = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}]$ and $u_2, v_2 \in \mathbb{R}^{N_2}$. As is known [4,13], in this case we have $\hat{\alpha} = 1$ and $\hat{\beta} > 1$. Hence, in Lemma 2.1 we have $\alpha = 1$ and $\beta = \hat{\beta}$.

Remark. Here and henceforth, we make use of the one-to-one correspondence between the functions from V_h and the vectors from $\mathbb{R}^N$.

The matrix B from (2.10) with the matrix B_{22} from (2.11) will be called a one-level DD-preconditioner for the matrix A; this preconditioner is supposed to arise in domain decomposition methods with the alternating Neumann-Dirichlet boundary conditions. The DD-preconditioner in form (2.10) was suggested in [6].

Denote by $\Omega^{(k)}$ a subdomain of Ω composed of the triangles (tetrahedrons) belonging to Ω_h, by $a^{(k)}(u,v)$ the restriction of $a(u,v)$ onto this subdomain and by $A^{(k)}$ the stiffness matrix for the subdomain $\Omega^{(k)}$ which is considered as a superelement of Ω_h. The latter means that the $N^{(k)} \times N^{(k)}$ matrix $A^{(k)}$ is determined by using the relation

$$(A^{(k)}u^{(k)}, v^{(k)}) = a^{(k)}(u_h, v_h) \quad \forall u_h, v_h \in V_h \qquad (2.12)$$

where $u^{(k)}, v^{(k)} \in \mathbb{R}^{N^{(k)}}$ are subvectors of the vectors $u, v \in \mathbb{R}^N$.

Let us construct a one-level DD-preconditioner $B^{(k)}$ for the matrix $A^{(k)}$. To this end, partition $\Omega^{(k)}$ into two non-overlapping grid subdomains $\Omega_1^{(k)}$ and $\Omega_2^{(k)}$, denote by $a_2^{(k)}(u,v)$ the restriction of $a(u,v)$ onto $\Omega_2^{(k)}$ and determine the $N_2^{(k)} \times N_2^{(k)}$ matrix $B_{22}^{(k)}$ by using the relation

$$(B_{22}^{(k)} u_2^{(k)}, v_2^{(k)}) = a_2^{(k)}(u_h, v_h) \quad \forall u_h, v_h \in V_h \tag{2.13}$$

where $u_2^{(k)}, v_2^{(k)} \in \mathbb{R}^{N_2^{(k)}}$. Then the one-level DD-preconditioner for the $N^{(k)} \times N^{(k)}$ matrix

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \tag{2.14}$$

with $N_i^{(k)} \times N_j^{(k)}$ submatrices $A_{ij}^{(k)}$, $i,j = 1,2$, is prescribed by the formula

$$B^{(k)} = F_k \begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & B_{22}^{(k)} \end{bmatrix} F_k^{\mathrm{T}} \tag{2.15}$$

where

$$F_k = \begin{bmatrix} E_{11}^{(k)} & 0 \\ A_{21}^{(k)} [A_{11}^{(k)}]^{-1} & E_{22}^{(k)} \end{bmatrix} \tag{2.16}$$

and $E_{ii}^{(k)}$ are identity $N_i^{(k)} \times N_i^{(k)}$ matrices, $i = 1,2$.

In what follows, we shall assume the quadratic forms $a_2^{(k)}(v,v)$ to be positive definite.

Assume that $\Omega^{(1)} = \Omega_h$ and $A^{(1)} = A$. Then successively for $k = 1,2,\dots,p$ prescribe partitionings of $\Omega^{(k)}$ into non-overlapping subdomains $\Omega_1^{(k)}$ and $\Omega_2^{(k)}$, for the stiffness matrices $A^{(k)}$ from (2.12) determine the one-level DD-preconditioners $B^{(k)}$ form (2.15) and set $\Omega^{(k+1)} = \Omega_2^{(k)}$. Finally, using the recurrent formulae

$$B_k = F_k \begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & B_{k+1} \end{bmatrix} F_k^{\mathrm{T}}, \quad k = p-1,\dots,1 \tag{2.17}$$

where $B_p = B^{(p)}$, determine the $N \times N$ matrix

$$B = B_1 \tag{2.18}$$

as a p-level DD-preconditioner for the matrix A of system (2.6).

Lemma 2.2. Let the eigenvalues of the matrices $[B^{(k)}]^{-1}A^{(k)}$ belong to segments $[1;\beta_k]$, $k = 1,\dots,p$. Then the eigenvalues of the matrix $B^{-1}A$ belong to the segment $[1;\beta]$, where $\beta = \prod_{k=1}^{p} \beta_k$.

Corollary. Under the assumptions made, we have

$$\text{Cond } B^{-1}A \leqslant \prod_{k=1}^{p} \text{Cond } [B^{(k)}]^{-1}A^{(k)} . \tag{2.19}$$

Let us present Ω_h as a union of $(p+1)$ non-overlapping subdomains $\Omega_1^{(1)}, \Omega_1^{(2)}, \dots, \Omega_1^{(p)}$ and $\Omega_2^{(p)}$ and assume that the corresponding partitioning of A into blocks is the block tridiagonal $(p+1)\times(p+1)$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & & 0 \\ A_{21} & \ddots & \ddots & \\ & \ddots & \ddots & A_{p,p+1} \\ 0 & & A_{p+1,p} & A_{p+1,p+1} \end{bmatrix} \tag{2.20}$$

with $n_i\times n_i$ submatrices A_{ij}, $i,j = 1,\dots,p+1$. Here, $n_i = N_1^{(i)}$, $i = 1,\dots,p$, and $n_{p+1} = N_2^{(p)}$. The corresponding p-level DD-preconditioner for the matrix A is prescribed by the formula

$$B = F[B_{11} \oplus B_{22} \oplus \dots \oplus B_{pp} \oplus B_{22}^{(p)}]F^{\mathrm{T}} \tag{2.21}$$

where $B_{ii} = A_{11}^{(i)}$, $i = 1,\dots,p$,

$$F = \begin{bmatrix} E_{11} & & & 0 \\ A_{21}B_{11}^{-1} & \ddots & & \\ & \ddots & \ddots & \\ 0 & & A_{p+1,p}B_{pp}^{-1} & E_{p+1,p+1} \end{bmatrix} \tag{2.22}$$

$E_{ii} = E_{11}^{(i)}$, $i = 1,\dots,p$, and $E_{p+1,p+1} = E_{22}^{(p)}$.

3. MULTI-LEVEL DD-PRECONDITIONER WITH INNER CHEBYSHEV ITERATIVE PROCEDURE

For the sake of simplicity, let us consider only the case of block tridiagonal matrices.

Prescribe a symmetric $N_2^{(p)} \times N_2^{(p)}$ matrix $H_{22}^{(p)}$ and assume the eigenvalues of the matrix $H_{22}^{(p)} B_{22}^{(p)}$ to belong to a segment $[d_1; d_2]$, $0 < d_1 < d_2$. Then following [9,11] for a fixed integer $s > 0$ define symmetric positive definite $N_2^{(p)} \times N_2^{(p)}$ matrices

$$R_{22}^{(p)} = [E_{22}^{(p)} - \prod_{j=1}^{s} (E_{22}^{(p)} - \tau_j H_{22}^{(p)} B_{22}^{(p)})][B_{22}^{(p)}]^{-1} \tag{3.1}$$

and

$$\hat{B}_{22}^{(p)} = [R_{22}^{(p)}]^{-1} \tag{3.2}$$

where the parameters τ_j, $j = 1, \ldots, s$, are chosen such that the polynomial

$$Z_s(t) = \prod_{j=1}^{s} (1 - \tau_j t) \tag{3.3}$$

is least deviating from zero on the segment $[d_1; d_2]$. As the solution to this problem is given in terms of Tchebyshev polynomials [15], the corresponding procedure will be called a Tchebyshev one.

It is obvious that for given g the vector $w = R_{22}^{(p)} g$ can be computed as a result of the following iterative procedure: $w^{(0)} = 0$,

$$w^j = w^{j-1} - \tau_j H_{22}^{(p)} (B_{22}^{(p)} w^{j-1} - g) , \quad j = 1, \ldots, s \tag{3.4}$$

that is $w = w^{(s)}$. Call the matrix

$$\hat{B} = F[B_{11} \oplus B_{22} \oplus \ldots \oplus B_{pp} \oplus \hat{B}_{22}^{(p)}]F^{T} \tag{3.5}$$

a p-level DD-preconditioner with the inner Tchebyshev iterative procedure for the matrix A from (2.20).

The theory of Tchebyshev methods implies [11,15] that the eigenvalues of the matrix $[\hat{B}_{22}^{(p)}]^{-1} B_{22}^{(p)}$ belong to the segment

$$\left[\frac{(1-q^s)^2}{1+q^{2s}} ; \frac{(1+q^s)^2}{1+q^{2s}} \right]$$

where $q = (\sqrt{\nu} - 1)/(\sqrt{\nu} + 1)$ and $\nu = d_2/d_1$.

This fact and Lemmas 2.1 and 2.2 imply the following statement.

Lemma 3.1. The eigenvalues of the matrix $\hat{B}^{-1}A$ belong to the segment

$$\left[\frac{(1-q^s)^2}{1+q^{2s}};\frac{(1+q^s)^2}{1+q^{2s}}\beta\right] .$$

Corollary. Under the assumptions made, we have

$$\text{Cond}\ \hat{B}^{-1}A \leqslant \frac{(1\ +\ q^s)^2}{(1\ -\ q^s)^2}\ \text{Cond}\ B^{-1}A\ . \tag{3.6}$$

4. TWO-DIMENSIONAL MGDD-PRECONDITIONER

Let Ω be a two-dimensional domain with the boundary $\partial\Omega$, which is a union of a certain number $m \geqslant 1$ of unit squares with sides parallel to the coordinate axes x_1 and x_2, as is shown, for example, in Fig.1. To be precise, we set $\bar{\Omega} = \bigcup_{i=1}^{m} \bar{G}_i$, where G_i are pairwise-disjoint squares with boundaries ∂G_i, $i = 1,\dots,m$. We define Γ_0 as a closed subset of $\partial\Omega$ consisting of the sides of the squares G_i and set $\Gamma_1 = \partial\Omega\backslash\Gamma_0$.

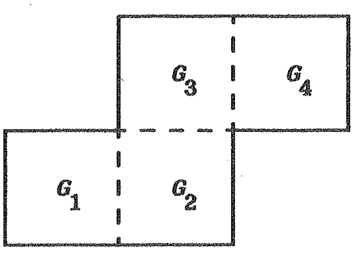

Figure 1. A case of domain Ω (m = 4).

Consider the following variational problem: for given $f \in L_2(\Omega)$ find $u \in V$ such that

$$\int_\Omega a\nabla u\circ\nabla v\ d\Omega = \int_\Omega fv\ d\Omega \quad \forall v \in V \tag{4.1}$$

where a is a positive function constant in each square G_i.

Let us choose a certain positive $t \geqslant 1$ and for the values $l = 0,1,\dots,t$ determine grid domains $\hat{\Omega}_h^{(l)}$ as unions of pairwise-disjoint squares $G_i^{(l)}$, $i=1,\dots,m_l$, with side lengths equal to $h_l = 2^{-l}$, where $m_l = 4^l m$. Then again for the values $l = 0,1,\dots,t$ partition each square $G_i^{(l)}$, $i = 1,\dots,m_l$, into two triangles and denote by $\Omega_h^{(l)}$ a union of such triangles. The grid domains $\hat{\Omega}_h^{(l)}$ and $\Omega_h^{(l)}$ thus constitute the domain Ω decomposed into squares and right-angled isosceles triangles, respectively. For the domain Ω shown in Fig.1, the grid domains $\hat{\Omega}_h^{(l)}$ and $\Omega_h^{(l)}$ are shown in Fig.2.

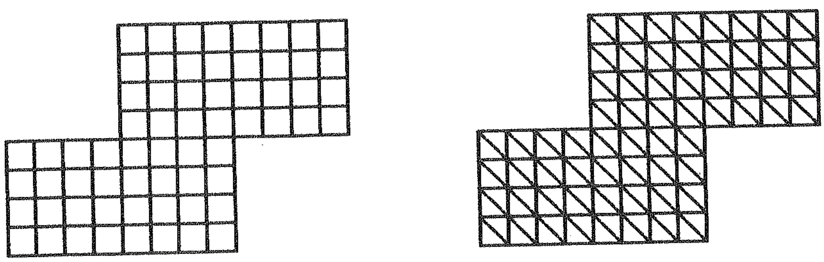

Figure 2. Grid domains $\hat{\Omega}_h^{(l)}$ and $\Omega_h^{(l)}$ for value $l = 2$.

Now we define a sequence of grids $\Gamma_h^{(l)} = \bigcup_{i=1}^{m_l} \partial G_i^{(l)}$, $l=0,1,\dots,t$, where $\partial G_i^{(l)}$ are boundaries of the squares $G_i^{(l)}$, and also a sequence of grids $\Gamma_h^{(l-1/2)}$ which are constructed using the grids $\Gamma_h^{(l-1)}$ by introducing additional nodes at the mid-sides of the squares $G_i^{(l-1)}$, $l = 1,\dots,t$, as is shown, for example, in Fig.3.

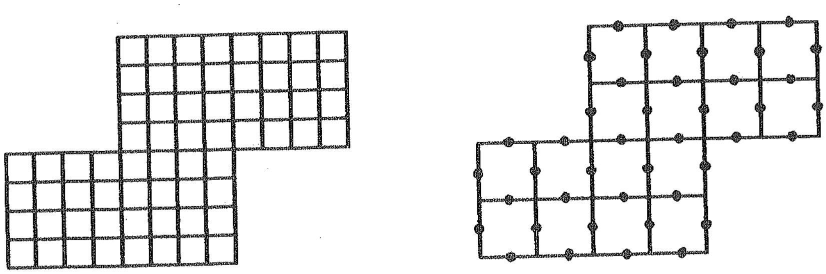

Figure 3. Grids $\Gamma_h^{(l)}$ and $\Gamma_h^{(l-1/2)}$ for value $l = 2$.

The grids $\Gamma_h^{(l)}$ and $\Gamma_h^{(l-1/2)}$ thus consist of segments of straight lines, whose length is h_l.

Determine a sequence of spaces $V_h^{(l)}$ as a set of functions continuous in Ω, linear in each triangle from $\Omega_h^{(l)}$, $l = 0,1,\ldots,t$, and vanishing on Γ_0.

Let us consider the finite element problem: find $u^h \in V_h$ such that

$$\int_\Omega a\nabla u_h \circ \nabla v \, d\Omega = \int_\Omega fv \, d\Omega \quad \forall v \in V_h \tag{4.2}$$

which leads to the system of linear algebraic equations

$$Au = f \tag{4.3}$$

with the symmetric, positive definite $N\times N$ matrix A and the vector $f \in \mathbb{R}^N$. Here, $V_h = V_h^{(t)}$, and N denotes the dimension of the space V_h.

Let us fix a value $t \geqslant 1$ and partition the nodes of the grid $\Gamma_h^{(t)}$ belonging to $\Omega \cup \Gamma_1$ into three groups. To the third group we refer the nodes which are at the same time nodes of the grid $\Gamma_h^{(t-1)}$, to the second group we refer the nodes which are nodes of $\Gamma_h^{(t-1/2)}$ but not included into the third group [these are midpoints of sides of the squares $G_i^{(t-1)}$], and to the first group we refer all the remaining nodes [these are the centres of the squares $G_i^{(t-1)}$]. In Fig.4 the nodes of the first group are denoted by rhombs, while the nodes of the second and third groups are denoted by boxes and circles, respectively.

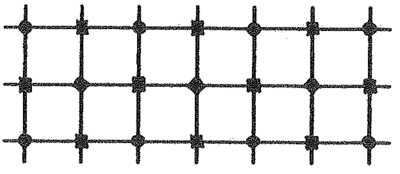

Figure 4. Grid $\Gamma_h^{(t)}$: ◆ are nodes of the first group
■ are nodes of the second group
● are nodes of the third group .

According to the given partition of nodes of the grid $\Gamma_h^{(t)} \setminus \Gamma_0$ the matrix A of system (4.3) can be presented in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} . \tag{4.4}$$

Note that A_{11}, A_{22} and A_{33} are diagonal matrices of orders m_{t-1}, n_{t-1} and N_{t-1}, respectively, where $n_{t-1} = N_t - N_{t-1} - m_{t-1}$.

The following statement plays a very important role in all further arguments. It can be established by straightforward computations.

Lemma 4.1. The matrix A of system (4.2) can be defined by the relation

$$(Av, w) = h_t \sum_{i=1}^{m_t} \frac{a_i^{(t)}}{2} \int_{\partial G_i^{(t)}} \frac{dv_h}{ds} \cdot \frac{dw_h}{ds} \, ds \tag{4.5}$$

which is assumed to hold for any $v, w \in \mathbb{R}^N$ $(v_h, w_h \in V_h)$.

In (4.5) the quantities $a_i^{(t)}$ are restrictions of a onto the squares $G_i^{(t)}$, $i = 1,\dots,m_t$.

Representation (4.5) makes it possible to apply to constructing the preconditioner B for the matrix A the technique of domain decomposition methods (DD-methods) with alternating Neumann-Dirichlet boundary conditions but in a very specific formulation. Introduce an additional notation $Y = \Gamma_h^{(t)}$ and conditionally call the set Y a 'domain'. Then decompose Y into two 'subdomains': $Y_2 = \bigcup_{i=1}^{t-1} \partial G_i^{(t-1)}$ and $Y_1 = Y \setminus Y_2$. The structure of the sets Y_1 and Y_2 is shown in Fig.5 (the arrow-heads at the end points of the segments indicate that these end points do not belong to this 'subdomain').

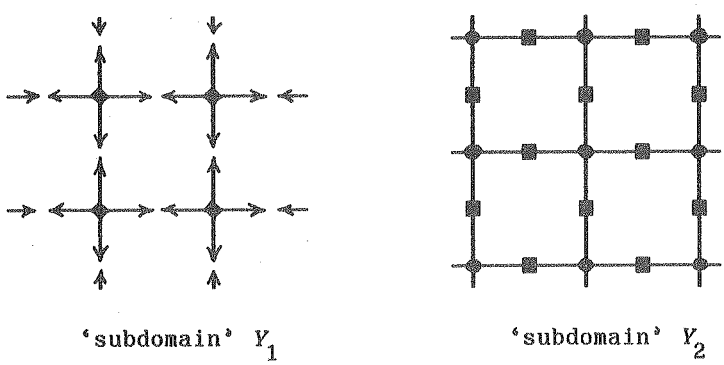

Figure 5. Decomposition of Y into 'subdomains' Y_1 and Y_2 .

It can be easily seen that the 'subdomain' Y_1 contains only $\tilde{N}_1 \equiv m_{t-1}$ nodes of the first group. In addition, the following statement is obvious.

Lemma 4.2. The matrix A_{11} in (4.4) can be defined by the relation

$$(A_{11}v_1, w_1) = h_t \int_{Y_1} a \frac{dv_h}{ds}\cdot\frac{dw_h}{ds}\, ds \equiv h_t \sum_{i=1}^{m_{t-1}} \frac{a_i^{(t-1)}}{2} \int_{G_i^{(t-1)}\cap Y_1} \frac{dv_h}{ds}\cdot\frac{dw_h}{ds}\, ds \tag{4.6}$$

which is assumed to hold for any $v_h, w_h \in V_h$ under the additional condition $v_h = w_h = 0$ on Y_2.

The last lemma directly enables us to construct DD-preconditioners with alternating Neumann-Dirichelt boundary conditions with respect to the 'domain' Y following, for example, [4,8,13,14]. Using the relation

$$(\tilde{B}_{22}v, w) = h_t \int_{Y_1} \tilde{a} \frac{dv_h}{ds}\cdot\frac{dw_h}{ds}\, ds \equiv h_t \sum_{i=1}^{m_{t-1}} \frac{a_i^{(t-1)}}{2} \int_{\partial G_i^{(t-1)}} \frac{dv_h}{ds}\cdot\frac{dw_h}{ds}\, ds \tag{4.7}$$

which is assumed to hold for all $v, w \in \mathbb{R}^{\tilde{N}_2}$ or (this is the same) for all $v_h, w_h \in V_h$, let us define the $\tilde{N}_2\times\tilde{N}_2$ matrix $\tilde{B}_{22}$, where $\tilde{N}_2 = N - \tilde{N}_1$. Here, by $\tilde{a} = \tilde{a}(x)$ we denote the mean value of the piecewise-constant function a over all closed unit squares $\bar{G}_i$, $i = 1,\dots,m$, to which the given point $x = (x_1, x_2)$ belongs.

Write down the matrix A in the new block form

$$A = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \tag{4.8}$$

with $\tilde{N}_i\times\tilde{N}_j$ submatrices $\tilde{A}_{ij}$, $i, j = 1,2$. It is obvious that

$$\tilde{A}_{11} = A_{11}, \quad \tilde{A}_{12}^{T} = \tilde{A}_{21} = \begin{bmatrix} A_{21} \\ 0 \end{bmatrix}, \quad \tilde{A}_{22} = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \tag{4.9}$$

Lemma 4.3. The matrix $\tilde{B}_{22}$ in (4.7) can be written in the following block form:

$$\tilde{B}_{22} = \begin{bmatrix} B_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \tag{4.10}$$

where B_{22} is a diagonal $n_{t-1}\times n_{t-1}$ submatrix.

According to Section 2 and [6] define the symmetric, positive definite $N \times N$ matrix

$$B = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{B}_{22}+\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & B_{22}+A_{21}A_{11}^{-1}A_{12} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} \tag{4.11}$$

as a one-level DD-preconditioner for the matrix A.

Lemma 4.4. The eigenvalues of the matrix $B^{-1}A$ belong to the segment $[1;3]$.

Remark. The condition number of the matrix $B^{-1}A$ (with respect to the energetic norm generated by the matrix A) is thus estimated from above by a constant independent of the grid step size $h = h_t$, the values of the diffusion coefficient a in subdomains G_i, $i = 1,\dots,m$, and the structure of the boundary conditions, i.e. the structure of the set Γ_0.

Consider the $N_{t-1}\times N_{t-1}$ matrix

$$B_{33} = A_{33} - A_{32}B_{22}^{-1}A_{23} \tag{4.12}$$

where the matrix B_{22} is defined in (4.7), (4.10), and introduce another notation $B_{11} = A_{11}$. Then the matrix B from (4.11) can be written in the following form:

$$B = \begin{bmatrix} B_{11} & A_{12} & 0 \\ A_{21} & B_{22}+A_{23}B_{33}^{-1}A_{32} & A_{23} \\ 0 & A_{32} & B_{33}+A_{32}B_{22}^{-1}A_{23} \end{bmatrix} = F\begin{bmatrix} B_{11} & & \\ 0 & B_{22} & \\ 0 & & B_{33} \end{bmatrix}F^{T} \tag{4.13}$$

where

$$F = \begin{bmatrix} E_{11} & 0 & 0 \\ A_{21}B_{11}^{-1} & E_{22} & 0 \\ 0 & A_{32}B_{22}^{-1} & E_{33} \end{bmatrix}. \tag{4.14}$$

This representation of the matrix B coincides with representation of the two-level preconditioner for the matrix A in (4.4).

All consequent arguments are based on the following result which can be proved by direct verification.

Lemma 4.5. The following equality is valid:

$$B_{33} = \frac{1}{2}A_{t-1} \ . \tag{4.15}$$

Thus, the matrix B_{33} from (4.13) is, up to a constant multiplier, the stiffness matrix of the finite element method for the space $V_h^{(t-1)}$. It is precisely due to this fact, the iterative methods with the constructed preconditioner B for the matrix A can be called two-grid methods. On the other hand, the matrix B has been constructed on the basis of the decomposition of the 'domain' Y into the 'subdomains' Y_1 and Y_2, and hence, we refer the iterative methods with B used as a preconditioner to domain decomposition methods. Combining these two standpoints we call the above-constructed matrix B a two-grid DD-preconditioner of the matrix A of system (4.3).

Using the assumptions and results of the previous sections let us define $N_l{\times}N_l$ matrices

$$A_l = \begin{bmatrix} A_{11}^{(l)} & A_{12}^{(l)} & 0 \\ A_{21}^{(l)} & A_{22}^{(l)} & A_{23}^{(l)} \\ 0 & A_{32}^{(l)} & A_{33}^{(l)} \end{bmatrix} \tag{4.16}$$

and put them into correspondence with two-grid DD-preconditioners

$$B_l = F_l \begin{bmatrix} B_{11}^{(l)} & 0 & 0 \\ 0 & B_{22}^{(l)} & 0 \\ 0 & 0 & B_{33}^{(l)} \end{bmatrix} F_l^{\mathrm{T}} \tag{4.17}$$

where $B_{33}^{(l)} = \frac{1}{2}A_{l-1}$ and $B_{11}^{(l)} = A_{11}^{(l)}$, $l = 1,2,\ldots,t$. Then choose a positive integer $s \geqslant 1$, set $H_{33}^{(2)} = \hat{B}_1^{-1} \equiv B_1^{-1}$, and for the values $l = 2,\ldots,t$, successively define (using the results of Section 3) $N_{l-1}{\times}N_{l-1}$ matrices

$$R_{33}^{(l)} = \left[E_{33}^{(l)} - \prod_{j=1}^{s} (E_{33}^{(l)} - \tau_j^{(l)} H_{33}^{(l)} A_{l-1})\right] A_{l-1}^{-1} \tag{4.18}$$

$$\hat{B}_{33}^{(l)} = \frac{1}{2}[R_{33}^{(l)}]^{-1} \tag{4.19}$$

and, finally, $N_l{\times}N_l$ matrices

$$\hat{B}_l = F_l \begin{bmatrix} B_{11}^{(l)} & 0 & 0 \\ 0 & B_{22}^{(l)} & 0 \\ 0 & 0 & \hat{B}_{33}^{(l)} \end{bmatrix} F_l^T , \quad H_{33}^{(l+1)} = \hat{B}_l^{-1} . \tag{4.20}$$

Lemma 4.4 implies that the eigenvalues of matrices $B_l^{-1}A_l$ and, specifically, the eigenvalues of the matrix $H_{11}^{(2)}A_1 \equiv B_1^{-1}A_1$, belong to the segment $[1;3]$. Choose an integer $l \geq 2$ and assume that the eigenvalues of the matrix $H_{33}^{(l)}A_{l-1}$ belong to the segment $[\alpha_{l-1};\beta_{l-1}]$, $0 < \alpha_{l-1} < \beta_{l-1}$, where $\alpha_1 = 1$ and $\beta_1 = 3$. Then following Section 3 choose parameters $\tau_j^{(l)}$, $j = 1,\dots,s$, in formula (4.19) as roots of the corresponding Tchebyshev polynomial.

Let us set

$$B = \hat{B}_t \tag{4.21}$$

and call this matrix a multigrid DD-preconditioner (MGDD-preconditioner) for the matrix A in (4.4).

The above-outlined arguments imply

Lemma 4.6. The following inequality is valid:

$$\text{Cond } B^{-1}A \leq \nu_t \tag{4.22}$$

where $\nu_1 = 3$ and

$$\nu_l = 3\left[\frac{(\sqrt{\nu_{l-1}} + 1)^s + (\sqrt{\nu_{l-1}} - 1)^s}{(\sqrt{\nu_{l-1}} + 1)^s - (\sqrt{\nu_{l-1}} - 1)^s}\right]^2 , \quad l = 2,\dots,t . \tag{4.23}$$

The simplest calculations lead to the following statements.

Lemma 4.7. If $s = 2$, the following estimate is valid:

$$\text{Cond } B^{-1}A \leq 3 + 2\sqrt{3} . \tag{4.24}$$

Lemma 4.8. If $s = 3$, the following estimate is valid:

$$\text{Cond } B^{-1}A \leq 1 + \tfrac{4}{3}\sqrt{3} . \tag{4.25}$$

5. THREE-DIMENSIONAL MGDD-PRECONDITIONER

Let Ω be a connected three-dimensional domain with the boundary $\partial\Omega$ which is a union of a certain number m of unit cubes whose vertices have only integral-valued coordinates. In other words, we set $\bar{\Omega} = \bigcup_{i=1}^{m} \bar{G}_i$, where G_i are pairwise-disjoint unit cubes with the boundaries ∂G_i, $i = 1,\dots,m$. Similarly to Section 4 we define Γ_0 as a closed subset $\partial\Omega$ consisting of sides of the cubes G_i and set $\Gamma_1 = \partial\Omega\backslash\Gamma_0$.

Choose a positive $t \geq 1$ and for the values $l = 0,1,\dots,t$ find grid domains $\hat{\Omega}_h^{(l)}$ as unions of pairwise-disjoint cubes $\bar{G}_i^{(l)}$, $i = 1,\dots,m_l$, with the edge length $h_l = 2^{-l}$, where $m_l = 8^l m$. Then partition each cube $G_i^{(l)}$, $i = 1,\dots,m$, into tetrahedrons in such a way that the resultant tetrahedron partitioning of the domain Ω permits the application of the finite element method with piecewise-linear basis functions. Denote such tetrahedron partitions of the domain Ω by $\Omega_h^{(l)}$, $l = 0,1,\dots,t$.

Let us consider variational problem (4.1) assuming the function a to be a positive constant in each cube G_i. To approximate this problem, we make use of finite element method (4.2), which with the natural basis used leads to the system of linear algebraic equations

$$Au = f \tag{5.1}$$

with the symmetric, positive definite $N\times N$ matrix A, where N is equal to the dimension of the space $V_h \equiv V_h^{(t)}$. Then we assume that the utilized tetrahedron partitioning of the domain Ω is such that system (5.1) is a classical seven-point difference scheme. It means that in the case where Ω is a unit cube and $\partial\Omega = \Gamma_0$, the matrix of system (5.1) can be written in terms of tensor products as

$$A = K \otimes M \otimes M + M \otimes K \otimes M + M \otimes M \otimes K \tag{5.2}$$

where

$$K = ah_t \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix} \tag{5.3}$$

is a $(2^t-1)\times(2^t-1)$ matrix, and M is an identity $(2^t-1)\times(2^t-1)$

matrix. Likewise, if $\partial\Omega = \Gamma_1$, then

$$K = ah_t \begin{bmatrix} 1 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1/2 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1/2 \end{bmatrix} \tag{5.4}$$

are $(2^t+1)\times(2^t+1)$ matrices. The assumption made will be exploited for the alternative technique of determining the matrix A.

Let us fix a value of $t \geqslant 1$ and partition the nodes of the grid domain $\Omega_h^{(t)}$ belonging to $\Omega \cup \Gamma_1$ into four groups. To the fourth group we refer the vertices of the cubes $G_i^{(t-1)}$, to the third one, the centres of the edges of these cubes, to the second one, the centres of the sides, and to the first one, all the remaining nodes which are at the same time the centres of the cubes $G_i^{(t-1)}$, $i = 1,\dots,m_{t-1}$. According to such partitioning of the nodes the matrix A of system (7.3) can be presented in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \tag{5.5}$$

with $N_i \times N_j$ submatrices A_{ij}, $i,j = 1,2,3,4$. Note that in the case where $a \equiv 1$ and $\partial\Omega = \Gamma_0$, we have $A_{ii} = h_t 6 E_{ii}$, where E_{ii} are identity $N_i \times N_i$ matrices, $i = 1,2,3,4$.

Similarly to Section 4 denote by $\Gamma_h^{(l)}$ a set made up of all the edges of the cubes $G_i^{(l)}$, $i = 1,\dots,m_l$, and define the sets of elements

$$Z_i^{(l)} = \bar{G}_i^{(l)} \cap \Gamma_h^{(l)}, \quad i = 1,\dots,m_l \tag{5.6}$$

for grids $\Gamma_h^{(l)}$, $l = 0,1,\dots,t$.

Using the relations

$$(A_l^{(i)} v^{(i)}, w^{(i)}) = \frac{h_t^2 a_i^{(t)}}{4} \int_{Z_i^{(t)}} \frac{dv_h}{ds} \cdot \frac{dw_h}{ds}\, ds \quad \forall v_h, w_h \in V_h^{(l)} \tag{5.7}$$

define 8×8 stiffness matrices $A_l^{(i)}$ for elements $Z_i^{(l)}$, where vectors

$v^{(i)}, w^{(i)} \in \mathbb{R}^8$ are restrictions of the functions $v_h, w_h \in V_h^{(l)}$, $l = 1,\ldots,t$, onto nodes of elements $Z_i^{(l)}$. Then by straightforward calculations we can readily establish the following fact similar to Lemma 4.1 for the two-dimensional problem.

Lemma 8.1. The matrix A of system (7.3) can be determined by using the relations

$$(Av, w) = \sum_{i=1}^{m_t} (A_t^{(i)} v^{(i)}, w^{(i)}) \quad \forall v, w \in \mathbb{R}^N . \tag{5.8}$$

Similarly to Section 4 we call the grid $Y = \Gamma_h^{(t)}$ a 'domain' and apply to constructing the preconditioner B for the matrix A from (5.5) the technique of multi-level domain decomposition methods with alternating Neumann-Dirichlet boundary conditions.

Set $Y^{(1)} = Y$ and define the 'subdomains'

$$Y_1^{(1)} = \bigcup_{i=1}^{m_{t-1}} (G_i^{(t-1)} \cap \Gamma_h^{(t)}) \tag{5.9}$$

and $Y_2^{(1)} = Y^{(1)} \backslash Y_1^{(1)}$. We thus prescribe the matrix $A_{11}^{(1)} \equiv A_{11}$ and define the matrix $B_{22}^{(1)}$ in the way similar to that of Section 4. Then partition the 'domain' $Y^{(2)} \equiv Y_2^{(1)}$ into 'subdomains' $Y_2^{(2)} = \Gamma_h^{(t-1)}$ and $Y_1^{(2)} = Y^{(2)} \backslash Y_2^{(2)}$ and define the corresponding matrices $A_{11}^{(2)}$ and $B_{22}^{(2)}$. It is not difficult to show that the latter is of the form

$$B_{22}^{(2)} = \begin{bmatrix} A_{11}^{(3)} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} . \tag{5.10}$$

Using the results obtained in Sections 3 and 4 define the $N \times N$ matrix

$$B = F[B_{11} \oplus B_{22} \oplus B_{33} \oplus B_{44}]F^T \tag{5.11}$$

where $B_{ii} = A_{11}^{(i)}$, $i = 1,2,3$, $B_{44} = A_{44} - A_{43}B_{33}^{-1}A_{34}$ and

$$F = \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ A_{21}B_{11}^{-1} & E_{22} & 0 & 0 \\ 0 & A_{32}B_{22}^{-1} & E_{33} & 0 \\ 0 & 0 & A_{43}B_{33}^{-1} & E_{44} \end{bmatrix} \tag{5.12}$$

and call it a DD-preconditioner for the matrix A from (5.5).

Remark. The first and second levels of decomposition of the 'domain' Y are shown in the case of two superelements $G_i^{(t-1)}$ in Figs.6 and 7. The arrow-heads at the end points of segments of length h mean that these end points do not belong to the given 'subdomain'. To illustrate $Y_2^{(2)}$ and $Y_2^{(1)}$, only visible sides of superelements $G_i^{(t-1)}$ are used.

On the one hand, the matrix B is written in the form of a three- level preconditioner, and, on the other hand, we have made use of only the two-level procedure of decomposition of the 'domain' Y, and at the last stage we have used the elimination procedure for defining the matrix B_{44}.

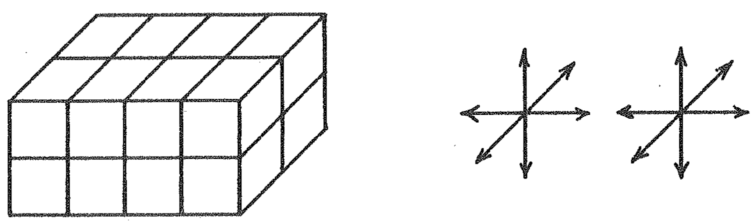

Figure 6. 'Subdomains' $Y_2^{(1)}$ and $Y_1^{(1)}$

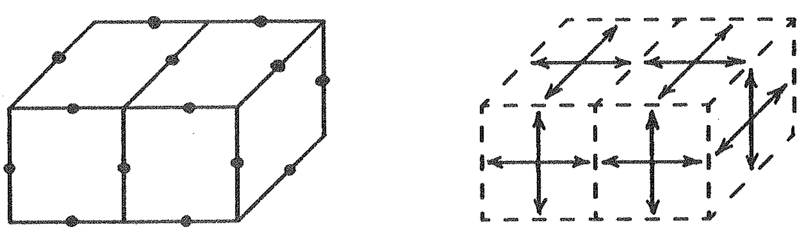

Figure 7. 'Subdomains' $Y_2^{(2)}$ and $Y_1^{(2)}$

By making straightforward calculations we can prove the following statements.

Lemma 5.1. The eigenvalues of the matrix $B^{-1}A$ belong to the segment $[1;b]$, where $b = (7 + \sqrt{19})/2$.

Corollary. The following estimates are valid:

$$\text{Cond } B^{-1}A \leqslant b < 5.68 \ . \tag{5.13}$$

Lemma 5.2. The following equality is valid:

$$B_{44} = \frac{1}{4} A_{t-1} \ . \tag{5.14}$$

Thus, as in Section 4, the matrix B from (5.11) can be called a two-grid DD-preconditioner for the matrix A from (5.5).

Using (5.7), (5.8), for the values $l = 1,\dots,t$ define $N_l \times N_l$ matrices

$$A_l = \begin{bmatrix} A_{11}^{(l)} & A_{12}^{(l)} & 0 & 0 \\ A_{21}^{(l)} & A_{22}^{(l)} & A_{23}^{(l)} & 0 \\ 0 & A_{32}^{(l)} & A_{33}^{(l)} & A_{34}^{(l)} \\ 0 & 0 & A_{43}^{(l)} & A_{44}^{(l)} \end{bmatrix} \tag{5.15}$$

and find the corresponding DD-preconditioners

$$B_l = F_l[B_{11}^{(l)} \oplus B_{22}^{(l)} \oplus B_{33}^{(l)} \oplus B_{44}^{(l)}]F_l^T \tag{5.16}$$

where $B_{11}^{(l)} = A_{11}^{(l)}$ and $B_{44}^{(l)} = A_{44}^{(l)} - A_{43}^{(l)}[B_{33}^{(l)}]^{-1}A_{34}^{(l)}$. Choose a positive integer $s \geqslant 1$, set $H_{44}^{(1)} = \hat{B}_1^{-1} \equiv B_1^{-1}$ and for the values $l = 2,\dots,t$ by formulae (4.18), (4.19) successively define $N_{l-1} \times N_{l-1}$ matrices $\hat{B}_{44}^{(l)}$ and $N_l \times N_l$ matrices

$$\hat{B}_l = F_l[B_{11}^{(l)} \oplus B_{22}^{(l)} \oplus B_{33}^{(l)} \oplus \hat{B}_{44}^{(l)}]F_l^T, \quad H_{44}^{(l+1)} = \hat{B}_l^{-1} . \tag{5.17}$$

Finally, set

$$B = \hat{B}_t . \tag{5.18}$$

The following statements can be proved [11].

Lemma 5.3. If $s = 3$, the following estimates are valid:

$$\text{Cond } B^{-1}A \leqslant \frac{3\sqrt{b} - 1}{3 - \sqrt{b}} < 9.97 . \tag{5.19}$$

Lemma 5.4. If $s = 4$, the following estimates are valid:

$$\text{Cond } B^{-1}A \leqslant \nu_{max} < 6.6 \tag{5.20}$$

where

$$\nu_{max} = \frac{2\sqrt{2b - 2\sqrt{b} + 1} - 3\sqrt{b} - 2}{4 - \sqrt{b}} . \tag{5.21}$$

6. NUMERICAL RESULTS

Let us apply to solving system (4.3) with the matrix A from (4.4) the generalized conjugate gradient method [7,13]:

$$u^{k+1} = u^k - \frac{1}{q_k}[B^{-1}\xi^k - e_{k-1}(u^k - u^{k-1})]$$

$$q_k = \frac{\|B^{-1}\xi^k\|_A^2}{\|\xi^k\|_{B^{-1}}^2} - e_{k-1}\ , \qquad e_k = q_k \frac{\|\xi^{k+1}\|_{B^{-1}}^2}{\|\xi^k\|_{B^{-1}}^2} \tag{6.1}$$

$$e_0 = 0\ , \quad k = 1,\dots,k_\varepsilon$$

with the matrix B from (4.21) for the value $s = 2$ or $s = 3$, where $\xi^k = Au^k - f$ and $\|\xi\|_A = (A\xi,\xi)^{1/2}$, $\xi \in \mathbb{R}^N$.

Choose the quantity k_ε in such a way that a given positive ε ($\varepsilon \ll 1$) will surely satisfy the inequality

$$\|u^{k_\varepsilon+1} - u^*\|_A \le \varepsilon\|u^0 - u^*\|_A \tag{6.2}$$

where $u^* = A^{-1}f$, for any initial guess $u^0 \in \mathbb{R}^N$.

Taking into account that method (6.1) obeys the estimate

$$\|u^k - u^*\|_A \le \frac{2q^k}{1 + q^{2k}}\|u^0 - u^*\|_A \tag{6.3}$$

where $q = (\sqrt{\nu} - 1)/(\sqrt{\nu} + 1)$ and ν is an arbitrary but fixed positive number such that $Cond\ B^{-1}A \le \nu$, we can choose for the required value of k_ε the maximal integer satisfying the inequality

$$k_\varepsilon < \frac{\ln \frac{\varepsilon}{2}}{\ln q}\ . \tag{6.4}$$

The following statements can be established.

Statement 6.1. To solve system (4.3) by method (6.1) with accuracy ε in the sense of inequality (6.2), it is sufficient to choose $k_\varepsilon = [1.21 \ln \frac{2}{\varepsilon}]$ in the case of $s = 2$ and $k_\varepsilon = [0.81 \ln \frac{2}{\varepsilon}]$ in the case of $s = 3$, where $[z]$ denotes the integral part of number z.

Statement 6.2. For the values $s = 2$ and $s = 3$ the number of arithmetic operations required for solving system (4.3) by method (6.1) with accuracy ε in the sense of inequality (6.2) can be estimated from above by the quantities $60N \ln \frac{2}{\varepsilon}$ and $55N \ln \frac{2}{\varepsilon}$, respectively.

Remark. Straightforward calculations indicate that for the model diffusion problem discussed the coefficient in Statement 6.2 for $s = 3$ is less than that for $s = 2$, i.e. theoretically the value $s = 3$ is more preferable. At the same time the numerical experiments have shown that in practice the choice of $s = 2$ is more preferable for obtaining (6.2) from the standpoint of the computational cost.

Statement 6.3. To solve system (5.1) with accuracy ε in the sense of inequality (6.2) by generalized conjugate gradient method (6.1) with the matrix B from (5.18), it is sufficient to choose $k_\varepsilon = [1.53 \ln \frac{2}{\varepsilon}]$ in the case of $s = 3$ and $k_\varepsilon = [1.22 \ln \frac{2}{\varepsilon}]$ in the case of $s = 4$.

Statement 6.4. For the values $s = 3$ and $s = 4$ the number of arithmetic operations required for solving system (5.1) by method (6.1) with accuracy ε in the sense of inequality (6.2) can be estimated from above by the quantities $75N \ln \frac{2}{\varepsilon}$ and $70N \ln \frac{2}{\varepsilon}$, respectively.

Remark. Straigtforward calculations indicate that for the three-dimensional model diffusion problem the coefficient in Statement 6.4 takes the least value for $s = 4$. At the same time the numerical experiments have shown that from the standpoint of costs the value of $s = 3$ is more preferable for obtaining estimate (6.2). The comparison of theoretical and experimental values of k_ε required for solving system (5.1) by method (6.1) in the sense of inequality (6.2) for $s = 3$ and $s = 7$ is given in Table 1.

	Theoretical value of k_ε		Experimental value of k_ε	
e	s = 3	s = 7	s = 3	s = 7
10^{-2}	8	6	6	6
10^{-4}	15	12	11	10
10^{-6}	22	17	17	15
10^{-8}	30	23	22	20

Table 1. Numerical results for the three-dimensional problem

7. CONCLUSION

All theoretical results obtained in Section 4 and 5 remain valid for the Neumann problem if in the formulae we use B^{+} instead of B^{-1}, i.e. we replace the inversion of the non-singular matrix by the generalized inversion. It is also interesting to note that the constructed estimates of the condition number for the two-grid method cannot be improved for the Neumann problem in the square domain Ω.

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