On the Coupling of Two Dimensional Hyperbolic and Elliptic Equations: Analytical and Numerical Approach

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Abstract. The coupling of linear hyperbolic and elliptic equations in a two dimensional domain is considered. Physical motivations for this investigation can be found in Fluid Dynamics (viscous/inviscid interactions for compressible flows, heat transfer in incompressible flows and other applications). A major point in this analysis consists in finding correct conditions at the interface separating the hyperbolic and the elliptic regions in a rigorous way. Then an effective iterative procedure is proposed, which alternates the solution of the hyperbolic equation and of the elliptic one within the respective regions. The strategy is quite important in view of the numerical computation since it permits to use different solvers in the two regions. The numerical approximation based on spectral collocation methods is detailed. The convergence analysis of the above iterative algorithm is provided both for the differential problem and its numerical approximation.

1. Introduction. In this paper we deal with the coupling of a linear elliptic equation with a hyperbolic one in a two dimensional domain Ω. The former is an Advection-Diffusion type equation (AD) set in a region $\Omega_2 \subset \Omega$, while the latter contains just the advection part (A) and is obtained from AD by dropping the diffusion term in the complementary region $\Omega_1 = \Omega \backslash \Omega_2$.

The one-dimensional version of this problem was studied in [1], where the coupling of parabolic and hyperbolic linear systems in one space variable is considered.

The problem at hand will be clearly stated in subsection 1.2 of this introduction along with the main results and a detailed outline of the paper.

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The main point of our investigation is concerned with the setting of correct transmission conditions across the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$: these conditions provide the matching between the solution to the AD equation and that to the A equation.

This type of problems arises from several physical applications which are modeled by a global AD equation in the whole Ω. However, in these problems the diffusive term is relevant only in the subregion Ω_2 (which clearly depends on the problem at hand), while it can be neglected in the rest of the domain Ω, without affecting the solution in a sensible way. We will give a few examples of concrete situations having such a feature in the subsection 1.1 below.

In the present paper we show that the coupled problem mentioned at the beginning (endowed with the correct transmission conditions) is consistent with the full AD problem in Ω. Furthermore, we propose a splitting method allowing the calculation of the solution to the coupled problem via separate calculations within Ω_1 and Ω_2, using the conditions at the interface in a convenient way. Eventually, we also provide a finite dimensional approximation of the problem at hand by the spectral collocation method.

Some of the results of this paper have been presented in [2] and [3]. We also refer to [4] and [5] for the coupling of Stokes-like problems.

1.1. Some physical examples. Fluid dynamics is among the fields that benefit largely from a coupling approach of the type studied here. As an example, consider viscous, compressible flows around rigid profiles (e.g. an aerofoil). Physical evidence suggests that viscosity effects are negligible apart from a small region close to the rigid body. This is one instance where the mathematical model of the problem may lead to the use of equations of different character (precisely, Euler and Navier-Stokes equations) in separate regions, just by dropping viscous terms when they are very small.

Another example is provided by a heat transfer problem such as a forced, incompressible flow over a heated plate. In such a case the thermal diffusivity is much more important in the boundary layer than elsewhere (here the reduced equation of conservation of energy can be assumed to describe the flow field). The velocity field can be evaluated independently from the temperature, while the latter is the solution to a linear AD equation in which the transport field is given precisely by the (known) velocity. As already noted, away from the boundary layer the diffusive term may be neglected. We refer to [6], where problems of this type are stated in large detail.

1.2. Statement of the problem and interface conditions. Hereafter we assume that Ω be a bounded, connected, open subset of $\mathbf{R}^2$, with boundary $\partial\Omega$; Ω_1 and Ω_2 are two open subsets of Ω, with $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega}$ (see Figure 1.1). Set

$$\Gamma_i = \partial\Omega \cap \partial\Omega_i, \quad i = 1, 2; \quad \Gamma = \partial\Omega_1 \backslash \Gamma_1 = \partial\Omega_2 \backslash \Gamma_2. \tag{1.1}$$

Let Γ_2^D, Γ_2^{Ne} be two (relatively) open subsets of Γ_2, with $\Gamma_2^D \cap \Gamma_2^{Ne} = \emptyset$, $\overline{\Gamma_2^D} \cup \overline{\Gamma_2^{Ne}} = \overline{\Gamma_2}$ (either may be empty). Denote by $\mathbf{n}$, $\mathbf{n}_1$, $\mathbf{n}_2$ the unit vector normal to $\partial\Omega$, $\partial\Omega_1$, $\partial\Omega_2$, respectively, oriented outwards. Let

- b_0 and f be scalar functions defined in Ω;
- ν be a positive function defined in Ω_2;
- $\mathbf{b}$ be a two dimensional vector valued function defined in Ω;
- ϕ be a scalar function defined on the boundary of Ω.

The following inequalities will be assumed throughout the paper, without any further explicit mention:

$$(i)\ \ \nu(x) \geq \nu_0 > 0,\ x \in \Omega_2;\quad (ii)\ \ \frac{1}{2}\operatorname{div} \mathrm{b}(x) + b_0(x) \geq \beta_0 > 0,\ x \in \Omega; \tag{1.2}$$

$$(\mathrm{b}\cdot \mathrm{n}_2)(x) \geq 0,\ x \in \Gamma_2^{Ne} \tag{1.3}$$

(the reason for these assumptions will be clear later on). Moreover, adopt the following notation, where Σ is any subset of $\partial\Omega_1$:

- $\Sigma^{in} = \{x \in \Sigma :\ (\mathrm{b}\cdot\mathrm{n}_1)(x) < 0\}$,
- $\Sigma^{out} = \{x \in \Sigma :\ (\mathrm{b}\cdot\mathrm{n}_1)(x) > 0\}$,
- $\Sigma^0 = \Sigma \backslash (\overline{\Sigma^{in}} \cup \overline{\Sigma^{out}})$;

we will use this notation either for $\Sigma = \partial\Omega_1$, or $\Sigma = \Gamma_1$, or $\Sigma = \Gamma$. The upper index *in* (respectively *out*) stands for "inflow" (respectively "outflow"), with respect to the domain Ω_1 (where a hyperbolic equation is going to be solved: the terminology is clearly related to fluid dynamical problems). Analogously, Σ^0 is the portion of Σ tangent to the vector field b.

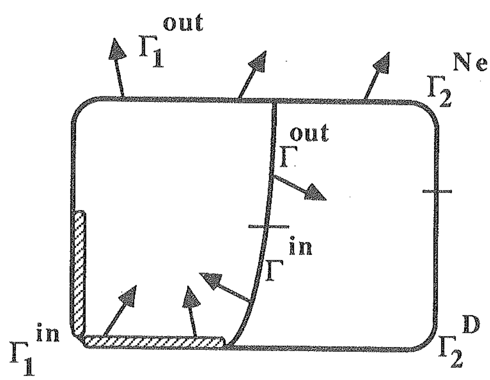

Figure 1.1. The geometry of the model problem: the arrows denote the local directions of the transport field b; shaded lines denote the inflow boundary Γ_1^{in}.

The problem we aim to solve is: *find a pair of real valued functions* u, w, *defined in* Ω_1 *and* Ω_2, *respectively, which satisfy the following boundary value problem:*

$$\operatorname{div}(\mathrm{b}u) + b_0 u = f \quad \text{in } \Omega_1, \tag{1.4}$$

$$\operatorname{div}(-\nu\nabla w + \mathrm{b}w) + b_0 w = f \quad \text{in } \Omega_2, \tag{1.5}$$

$$u = \phi \quad \text{on } \Gamma_1^{in}, \tag{1.6}$$

$$w = \phi \quad \text{on } \Gamma_2^{D}, \tag{1.7}$$

$$\nu\frac{\partial w}{\partial \mathrm{n}_2} = 0 \quad \text{on } \Gamma_2^{Ne}. \tag{1.8}$$

Obviously, suitable conditions at the interface Γ separating the two regions Ω_1 and Ω_2 are required. It can be conjectured in a natural way that *one* condition is needed along the whole Γ, in order to solve the elliptic problem in Ω_2 and that a *further* condition is required on Γ^{in} in order to solve the hyperbolic problem in Ω_1.

One of the scopes of our investigation is precisely to find these interface conditions. Of course, we pretend that the resulting coupled problem is well posed. Among all allowed choices, we make the most natural one, namely we take those interface conditions which are generated by a limit procedure on "globally viscous" problems, when "viscosity" vanishes within Ω_1. In this respect, we will find the following interface conditions:

$$-\nu\frac{\partial w}{\partial \mathrm{n}_2} + \mathrm{b}\cdot\mathrm{n}_2 w = -\mathrm{b}\cdot\mathrm{n}_1 u \quad \text{on } \Gamma, \tag{1.9}$$

$$u = w \qquad \text{on } \Gamma^{in}. \tag{1.10}$$

Actually, we will exploit an equivalent form of these two conditions, namely:

$$-\nu\frac{\partial w}{\partial \mathrm{n}_2} + \mathrm{b}\cdot\mathrm{n}_2 w = -\mathrm{b}\cdot\mathrm{n}_1 u \quad \text{on } \Gamma^{out}\cup\Gamma^0, \tag{1.11}$$

$$\nu\frac{\partial w}{\partial \mathrm{n}_2} = 0 \qquad \text{on } \Gamma^{in}, \tag{1.12}$$

$$u = w \qquad \text{on } \Gamma^{in}. \tag{1.13}$$

We point out that the two functions u and w exhibit a jump at $\Gamma^{out}\cup\Gamma^0$, in general. Actually, the interface conditions express the continuity of the flux along the whole interface, while the two solutions are required to join continuously only along Γ^{in}. However, the jump between u and w at $\Gamma^{out}\cup\Gamma^0$ turns out to be proportional to the value of ν at the interface. We notice that the decomposition we are dealing with is reasonable only if ν is extremely small in Ω_1, hence at the interface. Within Ω_2 the viscous term div $(\nu\nabla w)$ of equation (1.5) may be relevant either because the coefficient ν is "large" there or because of boundary layer effects. Our approach is well suited for the former situation, although also the latter might be recovered in some sense (with a scaling on the variables...).

Eventually, we state the complete problem we want to solve:

(P): *find a pair of real valued functions* u, w, *defined in* Ω_1 *and* Ω_2, *respectively, which satisfy* (1.4)-(1.8), (1.11)-(1.13).

To fix ideas, we regard problem (P) as the scalar version of a stationary linearized problem in fluid dynamics: in this framework, ν represents the kinematic viscosity, b the transport field, f the source term; the velocity field (u, w) is prescribed on the inflow boundary Γ_1^{in} of the "inviscid region" Ω_1 and along Γ_2^D, while along Γ_2^{Ne} a zero flux condition is expressed by (1.8). Problem (P) could also be viewed as a time discretization of an evolution advection-diffusion problem (namely, when a first order derivative with respect to time is added in the differential equations) by an implicit method. In such a case, b_0 is essentially the reciprocal of the time step. When dealing directly with the evolution problem, the interface conditions (1.11)-(1.13) are to be enforced at each time instant. Owing to the connection to the evolution problem, we term (1.4) a "hyperbolic equation" and consequently problem (P) is said to be a "coupled hyperbolic-elliptic problem", although actually it is a *degenerate elliptic problem.*

A thorough discussion is going to be carried out on problem (P). The main steps will concern:

(a) the theoretical justification of the interface conditions (1.11)-(1.13) and the analysis of the well posedness of problem (P);
(b) the numerical approximation of problem (P) by spectral collocation method;
(c) the setting of an effective iterative procedure yielding the solution of the problem as limit of a sequence of solutions of two independent sub-problems: a purely hyperbolic one within Ω_1 and a purely elliptic one within Ω_2;
(d) the adaptation of the above iterative procedure to the finite dimensional discretization of problem (P);
(e) the interpretation of the iterative procedure in terms of the Steklov-Poincaré interface operator.

For the sake of readability, we will state the main results concerning the previous steps in advance, postponing the detailed mathematical proofs until later. Therefore, the outline of the paper turns out to be the following.

In section 2 we deal with point (b); section 3 is devoted to the introduction of functional spaces and to face some of the questions raised in point (a): a precise statement of problem (P) is given, along with an existence and uniqueness theorem. Section 4 deals with points (c) and (d): in particular, the study of the iterative method is carried out by means of the investigation of the two separate sub-problems (hyperbolic in Ω_1 and elliptic in Ω_2) for both the differential and the numerical case. Point (e) is developed in section 4.1.4. Section 4.2.4 contains some numerical results and their discussion. The proofs of the results stated in sections 3 and 4 are detailed in sections 5 and 6, respectively.

We end this introduction with the index of the paper.

2. Numerical approximation of problem (P). In this section, we focus a very simple geometrical situation, see Figure 2.1. The loss of generality is motivated by our exigency of emphasizing the treatment of interface conditions. More general geometries can be reduced to this one, at least locally, by means of suitable mappings.

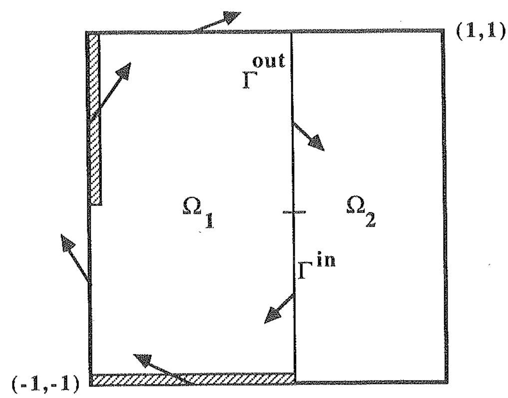

Figure 2.1. The computational domain: arrows denote the local directions of the transport field b; shaded lines denote the inflow boundary Γ_1^{in}.

We are given an integer $N > 0$ and denote by P_N the space of algebraic polynomials of degree at most N with respect to each variable x and y. We then denote by Ξ_N^i the set of $(N+1)^2$ collocation points of the Gauss-Lobatto formula within $\overline{\Omega_i}$, pertainig to either the Legendre or the Chebyshev weight function (see [7], Ch.2). We recall that a set $(\Xi_N^i)^0$ of $(N-1)^2$ of such points are internal to Ω_i, while the remaining $4N$ lie on the boundary of Ω_i (precisely on each side of Ω_i there are $N+1$ points, including the two extrema of the side).

If Σ is a subset of $\partial\Omega_i$, we denote by Σ_N the intersection $\Sigma \cap \Xi_N^i$ (this amounts to considering collocation points lying in Σ).

For each vector function $\mathbf{v} = (v_1, v_2)$ defined on $\overline{\Omega_i}$ we denote by $I_N^i\mathbf{v}$ the vector $(\tilde{v}_1, \tilde{v}_2)$ where $\tilde{v}_j$ $(j = 1, 2)$ is the polynomial of P_N interpolating v_j at the collocation points of Ξ_N^i $(i = 1, 2)$.

Moreover, for $P \in \Xi_N^i$ we denote by $\omega_P^{(i)}$ the corresponding weight in the Gauss-Lobatto integration formula for rectangular regions. For $P \in (\partial\Omega_i)_N$, we denote by $\theta_P^{(i)}$ the corresponding weight in the one-dimensional Gauss-Lobatto integration formula referred to $\partial\Omega_i$.

Finally, for v defined in Ω_i, $i = 1, 2$, set

$$L_N^i v = \frac{1}{2}[\,\mathrm{div}\, I_N^i(\mathbf{b}v) + \mathbf{b}\cdot\nabla v + v\ \mathrm{div}\,(I_N^i\mathbf{b})]\,: \tag{2.1}$$

L_N^i is the discrete skew-symmetric divergence operator (actually, $L_N^i v$ is the skew-symmetric decomposition of div $(\mathbf{b}v)$ relative to the discrete inner product associated with the Gauss-Lobatto integration formula (see section 4.2)).

Now we can state the numerical approximation to problem (P) as follows. We look for a pair of polynomials $u_N \in P_N$ and $w_N \in P_N$, satisfying

$$L_N^1 u_N + b_0 u_N = f \qquad \text{at } (\Xi_N^1)^0 \cup (\Gamma_1^{out} \cup \Gamma_1^0)_N \cup (\Gamma^{out} \cup \Gamma^0)_N; \tag{2.2}$$

$$- \operatorname{div}[I_N^2(\nu \nabla w_N)] + L_N^2 w_N + b_0 w_N = f \qquad \text{at } (\Xi_N^2)^0; \tag{2.3}$$

$$-[\mathbf{b}\cdot\mathbf{n}_1(u_N - \phi)](P) = [L_N^1 u_N + b_0 u_N - f](P)\frac{\omega_P^{(1)}}{\theta_P^{(1)}} \qquad \text{at } P \in (\Gamma_1^{in})_N; \tag{2.4}$$

$$w_N = \phi \qquad \text{at } (\Gamma_2^D)_N; \tag{2.5}$$

$$[-\nu\frac{\partial w_N}{\partial \mathbf{n}_2}](P) = R_2(P)\frac{\omega_P^{(2)}}{\theta_P^{(2)}} \qquad \text{at } P \in (\Gamma_2^{Ne})_N; \tag{2.6}$$

$$-[\mathbf{b}\cdot\mathbf{n}_1(u_N - w_N)](P) = [L_N^1 u_N + b_0 u_N - f](P)\frac{\omega_P^{(1)}}{\theta_P^{(1)}} \qquad \text{at } P \in (\Gamma^{in})_N; \tag{2.7}$$

$$[-\nu\frac{\partial w_N}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2(w_N - u_N)](P) = R_2(P)\frac{\omega_P^{(2)}}{\theta_P^{(2)}} \qquad \text{at } P \in (\Gamma^{out})_N; \tag{2.8}$$

$$[-\nu\frac{\partial w_N}{\partial \mathbf{n}_2}](P) = R_2(P)\frac{\omega_P^{(2)}}{\theta_P^{(2)}} \qquad \text{at } P \in (\Gamma^{in})_N, \tag{2.9}$$

where

$$R_2 = -\operatorname{div}[I_N^2(\nu\nabla w_N)] + L_N^2 w_N + b_0 w_N - f \tag{2.10}$$

is the residue coming from discrete integrations by parts.

REMARK 2.1 We just note that (2.2) corresponds to (1.4); (2.3) to (1.5); (2.4) to (1.6); (2.5) to (1.7); (2.6) to (1.8); (2.7) to (1.13); (2.8) to (1.11); (2.9) to (1.12). The boundary and interface conditions in (2.4), (2.6)-(2.9) are imposed in a weak form related to a variational formulation (see section 4.2). The strong form is obtained just by replacing the right hand sides of these formulas with zero. Since the quotient $\omega_P^{(i)}[\theta_P^{(i)}]^{-1}$ is proportional to N^{-2} (see, e.g. [7], Ch. 2), the weak form enforces the exact boundary and interface conditions up to the value of the residue of the equation times a constant that tends to zero as N tends to infinity. We chose to adopt the weak form because this suits better the analysis we will carry out, while it is equivalent to the strong formulation from the point of view of accuracy.□

The analysis of existence, uniqueness and stability estimates for the discrete problem (2.3)-(2.9) will be carried out under the following "coercivity assumption", which is the finite dimensional analogue of (1.2)(ii):

$$\frac{1}{2}\operatorname{div}[I_N^i \mathbf{b}(x)] + b_0(x) \geq \beta_N > 0, \; x \in \Omega_i, \; i = 1, 2. \tag{2.11}$$

Using standard interpolation error estimates, one can show that the constant β_N can be bounded from below independently of N, provided $\mathbf{b}$ is a smooth function, as a consequence of (1.2).

Under the assumptions (1.2)(i), (1.3) and (2.11), the hyperbolic-elliptic collocation problem (2.2)-(2.9) has one and only one solution. Uniqueness will be proved in section 6 (Proposition 6.10); existence follows from uniqueness as the problem has finite dimension.

3. Functional framework; existence and uniqueness for problem (P). In this section, we give a rigorous formulation of problem (P) and state the existence and uniqueness results for its solution. To this end, we must introduce some functional tools in advance.

Let A be an open, bounded subset of $\mathbf{R}^2$, with Lipschitz continuous boundary. If m is a positive integer, denote by $\mathbf{H}^m(A)$ the Sobolev space of real valued functions belonging to $\mathbf{L}^2(A)$ along with all derivatives up to the order m. $\mathbf{H}^m(A)$ is a Hilbert space with norm

$$\|v\|_{m,A} = \left[\sum_{i,j;i+j=0}^{i+j=m} \left\| \frac{\partial^i}{\partial x_1^i} \left(\frac{\partial^j v}{\partial x_2^j} \right) \right\|_{0,A}^2 \right]^{\frac{1}{2}}, \tag{3.1}$$

where $\|\cdot\|_{0,A}$ denotes the $\mathbf{L}^2(A)$ norm. Analogously, the $\mathbf{L}^\infty(A)$ norm is denoted by $\|\cdot\|_{\infty,A}$; $\mathbf{W}^{1,\infty}(A)$ represents the space of Lipschitz continuous functions on A, endowed with the usual norm

$$\|v\|_{1,\infty,A} = \|v\|_{\infty,A} + \|\frac{\partial v}{\partial x_1}\|_{\infty,A} + \|\frac{\partial v}{\partial x_2}\|_{\infty,A}, \qquad v \in \mathbf{W}^{1,\infty}.$$

It is well known (see, for instance, [8]) that the value at the boundary ∂A of all elements of $\mathbf{H}^m(A)$ can be given a meaning through a trace operator which maps linearly and continuously $\mathbf{H}^m(A)$ onto a subset of $\mathbf{L}^2(\partial A)$, denoted by $\mathbf{H}^{m-\frac{1}{2}}(\partial A)$ (a Hilbert space for the quotient norm $\|\cdot\|_{m-\frac{1}{2},\partial A}$).
In the following, the dual space of $\mathbf{H}^{\frac{1}{2}}(\partial A)$ will be called into play: this is denoted by $\mathbf{H}^{-\frac{1}{2}}(\partial A)$ and endowed with the dual norm $\|\cdot\|_{-\frac{1}{2},\partial A}$. Eventually, we recall some notations and properties for vector valued functions. The space

$$\mathbf{L}^2_{\text{div}}(A) = \{\mathbf{v} \in [\mathbf{L}^2(A)]^2 : \ \text{div}\,\mathbf{v} \in \mathbf{L}^2(A)\} \tag{3.2}$$

is a Hilbert space with the graph norm:

$$\|\mathbf{v}\|_{\mathbf{L}^2_{\text{div}}(A)} = \left[\|\mathbf{v}\|_{0,A}^2 + \|\,\text{div}\,\mathbf{v}\|_{0,A}^2\right]^{\frac{1}{2}} \tag{3.3}$$

(we adopt the same notation for norms as in the scalar case). Next, consider a (relatively) open subset Σ of ∂A and define

$$\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma) = \{v \in \mathbf{L}^2(\Sigma) : \ \tilde{v} \in \mathbf{H}^{\frac{1}{2}}(\partial A)\}, \tag{3.4}$$

where $\tilde{v} = \begin{cases} v \text{ in } \Sigma \\ 0 \text{ in } \partial A \backslash \Sigma \end{cases}$ is the trivial extension of v outside Σ. If $\Sigma = \partial A$, then $\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$ coincides with $\mathbf{H}^{\frac{1}{2}}(\partial A)$. For $v \in \mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$ we set

$$\|v\|_{\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)} = \|\tilde{v}\|_{\frac{1}{2},\partial A}$$

and denote by $\langle\cdot,\cdot\rangle$ the pairing between $\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$ and its topological dual $\left(\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)\right)'$. It is known (see, e.g., [8], [9]) that, if $\mathbf{v} \in \mathbf{L}^2_{\text{div}}(A)$, then $\mathbf{v}\cdot\mathbf{n} \in \left(\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)\right)'$, where $\mathbf{n}$ is the outward unit vector normal to ∂A. Moreover Green's formula holds for all $\mathbf{v} \in \mathbf{L}^2_{\text{div}}(A)$ and all $w \in \mathbf{H}^1(A)$, with $w_{|\partial A\setminus\Sigma} = 0$:

$$\int_A w \operatorname{div} \mathbf{v}\, dx + \int_A \nabla w\cdot\mathbf{v}\, dx = \langle \mathbf{v}\cdot\mathbf{n}, w\rangle. \tag{3.5}$$

In particular, if A is an open subset of Ω and $\mathbf{b} \in \left[\mathbf{W}^{1,\infty}(\Omega)\right]^2$ is the assigned transport field, then (3.5) becomes:

$$\int_A \nabla w\cdot\mathbf{b}z\, dx = -\int_A w \operatorname{div}(\mathbf{b}z)\, dx + \langle \mathbf{b}\cdot\mathbf{n}z, w\rangle, \tag{3.6}$$

for all $z \in \mathbf{L}^2(A)$ with $\operatorname{div}(\mathbf{b}z) \in \mathbf{L}^2(A)$ and all $w \in \mathbf{H}^1(A)$ with $w_{|\partial A\setminus\Sigma} = 0$. Moreover, for such a w the choice $z = w$ in (3.6) leads to

$$\int_A \nabla w\cdot\mathbf{b}w\, dx = -\int_A w^2 \operatorname{div}\mathbf{b}\, dx - \int_A \nabla w\cdot\mathbf{b}w\, dx + \int_{\partial A} \mathbf{b}\cdot\mathbf{n}w^2 ds,$$

whence

$$\int_A \nabla w\cdot\mathbf{b}w\, dx = -\frac{1}{2}\int_A w^2 \operatorname{div}\mathbf{b}\, dx + \frac{1}{2}\int_{\partial A} \mathbf{b}\cdot\mathbf{n}w^2 ds. \tag{3.7}$$

The previous formula suggests the following notation: if v is a scalar function defined on a (relatively) open subset Σ of $\partial\Omega \cup \Gamma$, we say that

$$|v|_\Sigma < +\infty, \text{ whenever } \sqrt{|\mathbf{b}\cdot\mathbf{n}|}v \in \mathbf{L}^2(\Sigma); \tag{3.8}$$

in such a case, we set

$$|v|^2_\Sigma = \int_\Sigma |\mathbf{b}\cdot\mathbf{n}|v^2 ds. \tag{3.9}$$

Notice that these formulas define a weighted norm in $\mathbf{L}^2(\Sigma)$, at least when the weight $|\mathbf{b}\cdot\mathbf{n}|$ does not vanish on Σ. In such a case, we set

$$\mathbf{L}^2_{\mathbf{b}}(\Sigma) = \{v : \Sigma \to \mathbf{R} : \sqrt{|\mathbf{b}\cdot\mathbf{n}|}v \in \mathbf{L}^2(\Sigma)\} : \tag{3.10}$$

this is a Banach space for the norm $|\cdot|_\Sigma$ (actually, a Hilbert space).

By the way of dealing with multipliers, we state a lemma which will be used later.

LEMMA 3.1 *Let A be an open, bounded subset of $\mathbf{R}^2$, with Lipschitz continuous boundary. Let Σ be a (relatively) open subset ∂A. Let ρ and z be such that*

$$\rho \in \mathbf{W}^{1,\infty}(\Sigma), \quad \rho > 0 \text{ on } \Sigma; \quad \sqrt{\rho}z \in \mathbf{L}^2(\Sigma). \tag{3.11}$$

(i) *If v satisfies $\rho v \in \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)\right]'$ and $\rho v = \rho z$ in the sense of $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)\right]'$, then*

$$\sqrt{\rho}v \in \mathbf{L}^2(\Sigma) \text{ and } v = z \text{ a.e. on } \Sigma. \tag{3.12}$$

(ii) *If* G *belongs to* $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)\right]'$ *and* z_n *is a sequence such that* $\sqrt{\rho} z_n \in \mathbf{L}^2(\Sigma)$, $\sqrt{\rho} z_n \rightharpoonup \sqrt{\rho} z$ *in* $\mathbf{L}^2(\Sigma)$, $\rho z_n \rightharpoonup \mathrm{G}$ *in* $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)\right]'$, *then* $\mathrm{G} = \rho z$.

Proof. Let ς be a smooth function with compact support in Σ. By (3.11), it is $\rho^{-\frac{1}{2}}\varsigma \in \mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)$. To prove part (i), we note that

$$\langle \rho v, \rho^{-\frac{1}{2}}\varsigma\rangle = \langle \rho z, \rho^{-\frac{1}{2}}\varsigma\rangle = \int_\Sigma \sqrt{\rho} z \varsigma \, ds. \tag{3.13}$$

Cauchy-Schwarz inequality in the last integral shows that $\sqrt{\rho} v \in \mathbf{L}^2(\Sigma)$ and then (3.13) gives the equality $\sqrt{\rho} v = \sqrt{\rho} z$ a.e on Σ. Because of (3.11), this gives $v = z$ a.e. on Σ. To prove part (ii), we note that in the equality

$$\langle \rho z_n, \rho^{-\frac{1}{2}}\varsigma\rangle = \int_\Sigma \sqrt{\rho} z_n \varsigma \, ds$$

the left hand side converges to $\langle \mathrm{G}, \rho^{-\frac{1}{2}}\varsigma\rangle$, while the right hand side converges to $\int_\Sigma \sqrt{\rho} z \varsigma \, ds$, whence $\rho^{-\frac{1}{2}}\mathrm{G} = \sqrt{\rho} z$ in $\mathbf{L}^2(\Sigma)$.□

With the preceding notations, problem (P) may be given a precise mathematical formulation and the following result may be proved.

THEOREM 3.2 *Assume the following regularity properties on the data:*

$$\partial\Omega_1 \text{ and } \partial\Omega_2 \text{ are Lipschitz continuous, piecewise } \mathrm{C}^{1,1};\ \Gamma \text{ is of class } \mathrm{C}^{1,1}; \tag{3.14}$$

$$\nu \in \mathbf{L}^\infty(\Omega_2),\ \mathbf{b} \in \left[\mathbf{W}^{1,\infty}(\Omega)\right]^2,\ b_0 \in \mathbf{L}^\infty(\Omega),\ f \in \mathbf{L}^2(\Omega); \tag{3.15}$$

$$\phi \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega),\ \text{ with } \phi_{|\Gamma_2^D} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_2^D),\ \ \phi_{|\Gamma_1^{in}} \in \mathbf{L}_\mathbf{b}^2(\Gamma_1^{in}). \tag{3.16}$$

Finally, assume (1.2) *and* (1.3) *(which are meaningful, due to* (3.14) *and* (3.15)*). Then there is a unique pair* (u, w) *which solves problem* (P) *in the following sense:*

(a) $u \in \mathbf{L}^2(\Omega_1)$, $w \in \mathbf{H}^1(\Omega_2)$;
(b) equation (1.4) *holds in the sense of distributions in* Ω_1;
(c) equation (1.5) *holds in the sense of distributions in* Ω_2;
(d) boundary condition (1.6) *holds a.e. on* Γ_1^{in};
(e) boundary condition (1.7) *holds in* $\mathbf{H}^{\frac{1}{2}}(\Gamma_2^D)$;
(f) boundary condition (1.8) *holds in the sense of* $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_2^{Ne})\right]'$;
(g) interface condition (1.11) *holds in the sense of* $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma^{out} \cup \Gamma^0)\right]'$;
(h) interface condition (1.12) *holds in the sense of* $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma^{in})\right]'$;
(i) interface condition (1.13) *holds a.e. on* Γ^{in}.

Finally, problem (P) *is limit of a family of globally elliptic variational problems.*□

We will give two different proofs of existence: one in section 5, by building a family of globally elliptic approaching problems (in this way, we will also derive the interface conditions); the other proof of existence will be based on the iterative procedure introduced in section 4. Uniqueness will also be obtained in the framework of this iterative procedure (see section 6.1.3). Here we point out that boundary and interface conditions defining problem (P) cannot be given a classical meaning, because of the irregularity of the solution in the general case.

REMARK 3.3 The regularity assumption made on ϕ in (3.16) is natural, in some sense. Indeed, we will show in Theorem 4.1 that the solution u to a purely hyperbolic problem with transport field $\mathbf{b}$ is such that $\mathbf{b}\cdot\mathbf{n}_1 u^2$ is integrable along the outflow boundary. If we have in mind that the hyperbolic domain is an element of a decomposition of a larger hyperbolic region, then we realize that the inflow boundary of our domain is but the outflow boundary for another adjoining subdomain. Therefore, our assumption on the inflow data matches the expected regularity for the outflow coming from the adjoining domain.□

4. Solution of the coupled problem (P) via an iterative procedure.

Our goal in this section is to exhibit the solution of problem (P) (see (1.4)-(1.8), (1.11)-(1.13)) as a limit of solutions of two subproblems within Ω_1 and Ω_2, respectively. This is done by attributing condition (1.13) to the hyperbolic problem in Ω_1 and conditions (1.11), (1.12) to the elliptic problem in Ω_2. Therefore, we consider the following iterative procedure.

Let u^0, w^0 be given on Γ^{in}. We define a sequence (u^n, w^n), $n \geq 1$ by solving for each n the following hyperbolic problem within Ω_1

$$\begin{cases} (i) & \operatorname{div}(\mathbf{b}u^n) + b_0 u^n = f & \text{in } \Omega_1 \\ (ii) & u^n = \phi & \text{on } \Gamma_1^{in} \\ (iii) & u^n = \psi^n & \text{on } \Gamma^{in} \end{cases} \tag{4.1}$$

and then the following elliptic problem within Ω_2

$$\begin{cases} (i) & \operatorname{div}(-\nu\nabla w^n + \mathbf{b}w^n) + b_0 w^n = f & \text{in } \Omega_2 \\ (ii) & w^n = \phi & \text{on } \Gamma_2^D \\ (iii) & \nu\dfrac{\partial w^n}{\partial \mathbf{n}_2} = 0 & \text{on } \Gamma_2^{Ne} \\ (iv) & -\nu\dfrac{\partial w^n}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2 w^n = -\mathbf{b}\cdot\mathbf{n}_1 u^n & \text{on } \Gamma^{out}\cup\Gamma^0 \\ (v) & \nu\dfrac{\partial w^n}{\partial \mathbf{n}_2} = 0 & \text{on } \Gamma^{in}, \end{cases} \tag{4.2}$$

where

$$\psi^n = \theta w^{n-1} + (1-\theta)u^{n-1} \quad \text{on } \Gamma^{in}, \qquad \theta > 0 \tag{4.3}$$

Formally, the limit of the sequence (u^n, w^n), if existing, solves the coupled problem (P).

The transformation of the original problem (P) into (4.1), (4.2) provides:

(a) a viable algorithm for finding the solution to the problem, which can be conveyed easily to finite dimensions (see section 4.2);
(b) an alternative way of proving the existence of a solution (see section 4.1);
(c) a proof of uniqueness (see section 4.1);

We point out that the splitting we have made carries several advantages from the computational point of view. First of all, the two subproblems can be faced by standard numerical methods for hyperbolic and elliptic problems, respectively. Furthermore, the two parts of the interface boundary play a separate role in the interaction mechanism between Ω_1 and Ω_2. Actually, the hyperbolic solution u^n is influenced by the elliptic one w^{n-1} through Γ^{in} only, while, symmetrically, u^n influences w^n through Γ^{out} solely. This makes the iterative procedure very effective, as documented by some numerical tests (see section 4.2.4).

The iterative procedure will be discussed both for the differential problem (4.1), (4.2) (see section 4.1) and for its finite dimensional approximation (see section 4.2). In both cases, we will face the question of solvability of the subproblems and prove some a priori estimates that will be used in the discussion of convergence of the iterative procedure. Finally, we will introduce the *influence operator* S associated with the coupled problem (P): this is precisely the pseudodifferential operator acting solely on the values of the solution at the interface. This operator is known as the *Steklov-Poincaré operator* in the differential problem and the *capacitance matrix* in the discrete approximation. The iterative algorithm can be interpreted by means of the operator S as an iterative procedure acting solely on the interface variables.

4.1. The differential case. This subsection is divided into four parts: in the first one, we give the precise formulation of a hyperbolic problem of type (4.1) and state an a priori estimate for the solution; the second one contains the analogous arguments for an elliptic problem of type (4.2); the third one is devoted to the combination of the results of the two previous cases, in order to get a global estimate for the iterative scheme (4.1), (4.2). This will allow us to get a proof of convergence for the scheme, along with a new proof of existence of solutions to the coupled problem (P); uniqueness will also follow in this framework. Finally, we will introduce the Steklov-Poincaré operator associated with the coupled problem (P).

4.1.1. The hyperbolic problem in Ω_1. With the usual notations, we consider the following problem:
(P_H): $u \in \mathbb{L}^2(\Omega_1)$ satisfies

$$\operatorname{div}(\mathbf{b}u) + b_0 u = f \quad \text{in } \mathcal{D}'(\Omega_1), \tag{4.4}$$
$$u = \lambda \quad \text{a.e. on } \partial\Omega_1^{in}, \tag{4.5}$$

where $\lambda \in \mathbb{H}^{-\frac{1}{2}}(\partial\Omega_1)$ is given, satisfying

$$\lambda_{|\partial\Omega_1^{in}} \in \mathbb{L}^2_{\mathbf{b}}(\partial\Omega_1^{in}). \tag{4.6}$$

Our aim is to find how the inflow data λ propagates to $\partial\Omega_1^{out}$ through the solution of (P_H). This influence is shown in the following theorem through an a priori estimates that will play an important role in what follows.

THEOREM 4.1 *Under the same assumptions on the data Ω_1, $\mathbf{b}$, b_0 and f as in Theorem 3.2, if (4.6) holds, then problem (P_H) has a unique solution* u. *This satisfies $u_{|\partial\Omega_1^{out}} \in \mathbb{L}^2_{\mathbf{b}}(\partial\Omega_1^{out})$ and*

$$\left(\frac{\beta_0}{2} - \delta\right)\|u\|^2_{0,\Omega_1} + \frac{1}{2}|u|^2_{\partial\Omega_1^{out}} \leq \frac{1}{4\delta}\|f\|^2_{0,\Omega_1} + \frac{1}{2}|\lambda|^2_{\partial\Omega_1^{in}}, \tag{4.7}$$

for all $\delta > 0$. If $f = 0$, then (4.7) holds with $\delta = 0$, namely

$$\frac{\beta_0}{2}\|u\|^2_{0,\Omega_1} + \frac{1}{2}|u|^2_{\partial\Omega_1^{out}} \leq \frac{1}{2}|\lambda|^2_{\partial\Omega_1^{in}}. \tag{4.8}$$

Finally, problem (P_H) is limit of a family of elliptic problems.□

The proof of this theorem will be given in section 6.1.1, still using a regularization argument. Here we state the estimate (4.7) for problem (4.1) as a corollary.

COROLLARY 4.2 *Under the same assumptions on the data as in Theorem 3.2, if*

$$\psi^n \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in}), \tag{4.9}$$

then problem (4.1) has a unique solution u^n. This satisfies $u^n_{|\Gamma^{out}} \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{out})$ and

$$(\frac{\beta_0}{2} - \delta)\|u^n\|^2_{0,\Omega_1} + \frac{1}{2}|u^n|^2_{\partial\Omega_1^{out}} \leq \frac{1}{4\delta}\|f\|^2_{0,\Omega_1} + \frac{1}{2}|\phi|^2_{\Gamma_1^{in}} + |\psi^n|^2_{\Gamma^{in}}, \tag{4.10}$$

for all $\delta > 0$. Moreover, if $f = \phi = 0$, then (4.10) holds with $\delta = 0$, namely

$$\frac{\beta_0}{2}\|u^n\|^2_{0,\Omega_1} + \frac{1}{2}|u^n|^2_{\partial\Omega_1^{out}} \leq \frac{1}{2}|\psi^n|^2_{\Gamma^{in}}. \tag{4.11}$$

□

Of course, u^n solves problem (4.1) in a sense analogous to that of problem (P_H). The assumption (4.9) holds, provided the initial guesses u^0, w^0 are regular enough (which we may certainly assume).

4.1.2. The elliptic problem in Ω_2. Keeping in mind the usual notations and geometry, we suppose that ϕ and μ are given, with

$$\phi \in \mathbf{H}^{\frac{1}{2}}(\Gamma_2^D), \quad \mu \in \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma)\right]', \; \mu_{|\Gamma^{out}} \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{out}). \tag{4.12}$$

Let V be the following affine subspace of $\mathbf{H}^1(\Omega_2)$:

$$V = \{v \in \mathbf{H}^1(\Omega_2) : \; v = \phi \text{ on } \Gamma_2^D\}.$$

Consider the following variational problem in Ω_2:

(P_E) : find $w \in V$ such that, for all $v \in V$,

$$\int_{\Omega_2}(\nu\nabla w - \mathbf{b}w)\cdot\nabla(w-v)\,dx + \int_{\Omega_2} b_0 w(w-v)\,dx + \int_{\Gamma^{in}\cup\Gamma_2^{Ne}} \mathbf{b}\cdot\mathbf{n}_2 w(w-v)\,ds =$$
$$= \int_{\Omega_2} f(w-v)\,dx - \int_{\Gamma^{out}} \mathbf{b}\cdot\mathbf{n}_2\mu(w-v)\,ds. \tag{4.13}$$

Problem (P_E) is the variational formulation of an elliptic boundary value problem in Ω_2, analogous to (4.2). More precisely, (P_E) is equivalent to finding $w \in \mathrm{H}^1(\Omega_2)$ such that

$$\operatorname{div}(-\nu\nabla w + \mathbf{b}w) + b_0 w = f \qquad \text{in } \Omega_2, \tag{4.14}$$

$$w = \phi \qquad \text{on } \Gamma_2^D, \tag{4.15}$$

$$\nu\frac{\partial w}{\partial \mathbf{n}_2} = 0 \qquad \text{in } \left[\mathrm{H}_{00}^{\frac{1}{2}}(\Gamma_2^{Ne})\right]', \tag{4.16}$$

$$-\nu\frac{\partial w}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2 w = \mathbf{b}\cdot\mathbf{n}_2\mu \qquad \text{in } \left[\mathrm{H}_{00}^{\frac{1}{2}}(\Gamma^{out}\cup\Gamma^0)\right]', \tag{4.17}$$

$$\nu\frac{\partial w}{\partial \mathbf{n}_2} = 0 \qquad \text{in } \left[\mathrm{H}_{00}^{\frac{1}{2}}(\Gamma^{in})\right]'. \tag{4.18}$$

The following result holds.

THEOREM 4.3 *Under the assumptions of Theorem 3.2, if (4.12) holds, then problem (P_E) has one and only one solution* w. *This satisfies the estimate*

$$(1-\delta)\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2 + \{\beta_0 - \delta(\|\mathbf{b}\|_{1,\infty,\Omega_2} + \|b_0\|_{\infty,\Omega_2})\}\|w\|_{0,\Omega_2}^2 + (\frac{1}{2}-\delta)|w|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2$$
$$\leq (1+\frac{1}{4\delta})\|f\|_{0,\Omega_2}^2 + \frac{1}{2}|\phi|_{\Gamma_2^D}^2 + C(\Omega_2,\mathbf{b},b_0;\delta)\|\phi\|_{\frac{1}{2},\Gamma_2^D}^2 + (\frac{1}{2}+\delta)|\mu|_{\Gamma^{out}}^2, \tag{4.19}$$

where $\delta > 0$ is arbitrary and $C(\Omega_2,\mathbf{b},b_0;\delta)$ is a positive quantity depending only on its argument. Moreover, if $f = \phi = 0$, then (4.19) holds with $\delta = 0$, namely

$$\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2 + \beta_0\|w\|_{0,\Omega_2}^2 + \frac{1}{2}|w|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2 \leq \frac{1}{2}|\mu|_{\Gamma^{out}}^2. \tag{4.20}$$

□

Again, we state the estimate (4.19) for problem (4.2) as a corollary.

COROLLARY 4.4 *Under the same assumptions on the data as in Theorem 3.2, if (4.9) holds, then the solution to (4.2) satisfies*

$$(1-\delta)\|\sqrt{\nu}|\nabla w^n|\|_{0,\Omega_2}^2 + \{\beta_0 - \delta(\|\mathbf{b}\|_{1,\infty,\Omega_2} + \|b_0\|_{\infty,\Omega_2})\}\|w^n\|_{0,\Omega_2}^2 + (\frac{1}{2}-\delta)|w^n|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2$$
$$\leq (1+\frac{1}{4\delta})\|f\|_{0,\Omega_2}^2 + \frac{1}{2}|\phi|_{\Gamma_2^D}^2 + C(\Omega_2,\mathbf{b},b_0;\delta)\|\phi\|_{\frac{1}{2},\Gamma_2^D}^2 + (\frac{1}{2}+\delta)|u^n|_{\Gamma^{out}}^2, \tag{4.21}$$

for all $\delta > 0$. Moreover, if $f = \phi = 0$, then (4.21) holds with $\delta = 0$, namely

$$\|\sqrt{\nu}|\nabla w^n|\|_{0,\Omega_2}^2 + \beta_0\|w^n\|_{0,\Omega_2}^2 + \frac{1}{2}|w^n|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2 \leq \frac{1}{2}|u^n|_{\Gamma^{out}}^2. \tag{4.22}$$

□

The comments on the validity of (4.2) and of (4.9) are analogous to those made for Corollary 4.2. The proof of Theorem 4.3 will be given in section 6.1.2.

4.1.3. Convergence of the iterative procedure; existence and uniqueness results for the coupled problem (P). We begin by giving an equivalent formulation of the convergence problem for the iterative scheme (4.1), (4.2).

Let $\theta \in \mathbf{R}$ and $\psi \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$ be given. Let u^ψ, w^ψ be respectively the solutions to the two boundary value problems

$$\begin{cases} (i) & \operatorname{div}(\mathbf{b}u^\psi) + b_0 u^\psi = f & \text{in } \mathcal{D}'(\Omega_1) \\ (ii) & u^\psi = \phi & \text{a.e. on } \Gamma_1^{in} \\ (iii) & u^\psi = \psi & \text{a.e. on } \Gamma^{in}; \end{cases} \tag{4.23}$$

$$\begin{cases} (i) & \operatorname{div}(-\nu\nabla w^\psi + \mathbf{b}w^\psi) + b_0 w^\psi = f & \text{in } \mathcal{D}'(\Omega_2) \\ (ii) & w^\psi = \phi & \text{on } \Gamma_2^D \\ (iii) & \nu\dfrac{\partial w^\psi}{\partial \mathbf{n}_2} = 0 & \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_2^{Ne})\right]' \\ (iv) & -\nu\dfrac{\partial w^\psi}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2 w^\psi = -\mathbf{b}\cdot\mathbf{n}_1 u^\psi & \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma^{out} \cup \Gamma^0)\right]' \\ (v) & \nu\dfrac{\partial w^\psi}{\partial \mathbf{n}_2} = 0 & \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma^{in})\right]'. \end{cases} \tag{4.24}$$

Define the application

$$T_\theta : \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in}) \to \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in}); \quad T_\theta : \psi \mapsto T_\theta(\psi) = \theta w^\psi{}_{|\Gamma^{in}} + (1-\theta)\psi. \tag{4.25}$$

PROPOSITION 4.5 *The following two conditions are equivalent, for a given $\psi \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$.*

1. *The pair (u^ψ, w^ψ) solves the coupled problem* (P).
2. *There exists $\theta \in \mathbf{R}$ such that ψ is a fixed point for T_θ.*

Moreover, each fixed point for T_θ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma^{in})$. Finally, if ψ is a fixed point for T_1, then it is a fixed point for T_θ, for all $\theta \in \mathbf{R}$.□

LEMMA 4.6 *Under the assumptions of Theorem 3.2, the solutions u_0^ψ, w_0^ψ to (4.23), (4.24), respectively, with $f = \phi = 0$, satisfy:*

$$\||\sqrt{\nu}|\nabla w_0^\psi|\|^2_{0,\Omega_2} + \frac{\beta_0}{2}\|u_0^\psi\|^2_{0,\Omega_1} + \beta_0\|w_0^\psi\|^2_{0,\Omega_2} + \frac{1}{2}|w_0^\psi|^2_{\Gamma^{in}\cup\Gamma_2^{Ne}} \le \frac{1}{2}|\psi|^2_{\Gamma^{in}}. \tag{4.26}$$

□

In view of Proposition 4.5, we will search for fixed points of the application T_1.

LEMMA 4.7 *The application T_1 has a unique fixed point ψ.*□

The following estimate holds for the iterative scheme (4.1), (4.2) with homogeneous data.

LEMMA 4.8 *Under the assumptions of Theorem 3.2, if the initial guesses u^0, w^0 are such that $\psi^0 \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$, then the solutions u_0^n, w_0^n to* (4.1), (4.2) *with $f = \phi = 0$ satisfy*

$$\||\sqrt{\nu}|\nabla w_0^n|\|^2_{0,\Omega_2} + \frac{\beta_0}{2}\|u_0^n\|^2_{0,\Omega_1} + \beta_0\|w_0^n\|^2_{0,\Omega_2} + \frac{1}{2}|w_0^n|^2_{\Gamma^{in}\cup\Gamma_2^{Ne}} \le \frac{1}{2}|\psi^n|^2_{\Gamma^{in}}. \tag{4.27}$$

□

This estimate allows us to prove the following convergence result.

LEMMA 4.9 *There exists* $\delta > 0$ *such that if* $\psi^0 \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$ *and* $\theta \in]0, 1+\delta[$, *then the sequence* ψ^n *defined in* (4.3) *converges in* $\mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$ *to the (unique) fixed point of* T_1.□

Finally, we can state the convergence theorem for the iterative scheme, containing also the existence and uniqueness results for the coupled problem (P). The proof will be given in section 6.1.3.

THEOREM 4.10 *Under the assumptions of Theorem* 3.2, *if the initial guesses* u^0, w^0 *are such that* $\psi^0 \in \mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$, *then the sequence* (u^n, w^n) *converges to a limit pair* (u, w), *in the following sense:*

$$u^n \to u \quad \text{in } \mathbf{L}^2(\Omega_1); \qquad w^n \to w \quad \text{in } \mathbf{H}^1(\Omega_2).$$

The limit pair provides the unique solution to the coupled problem (P).□

REMARK 4.11 It will be clear from the proof of the theorem that the rate of convergence depends on ν and b, in principle.□

4.1.4. <u>The Steklov-Poincaré operator associated with problem (P).</u> In this section we introduce the Steklov-Poincaré pseudodifferential operator associated with problem (P) (see (1.4)-(1.8), (1.11)-(1.13)). This can be defined as *an operator which permits to reduce the solution of the coupled problem to the solution of an equation involving only the interface values.* For simplicity, we will detail only the homogeneous case $f = \phi = 0$.

Let $\psi \in \mathbf{H}^{\frac{1}{2}}_{00}(\Gamma)$ be a given function. Correspondingly, denote by $U = U(\psi)$ the solution to the hyperbolic problem within Ω_1

$$\begin{cases} (i) & \operatorname{div}(\mathbf{b}U) + b_0 U = 0 & \text{in } \Omega_1 \\ (ii) & U = 0 & \text{on } \Gamma_1^{in} \\ (iii) & U = \psi & \text{on } \Gamma^{in}. \end{cases} \tag{4.28}$$

Next, denote by $W = W(\psi)$ the solution to the elliptic problem within Ω_2

$$\begin{cases} (i) & \operatorname{div}(-\nu\nabla W + \mathbf{b}W) + b_0 W = 0 & \text{in } \Omega_2 \\ (ii) & W = 0 & \text{on } \Gamma_2^D \\ (iii) & \nu\dfrac{\partial W}{\partial \mathbf{n}_2} = 0 & \text{on } \Gamma_2^{Ne} \\ (iv) & W = \psi & \text{on } \Gamma. \end{cases} \tag{4.29}$$

Finally, define

$$S^1\psi = \begin{cases} \mathbf{b}\cdot\mathbf{n}_1 U & \text{on } \Gamma^{out}, \\ 0 & \text{on } \Gamma^{in}; \end{cases} \tag{4.30}$$

$$S^2\psi = \begin{cases} -\nu\dfrac{\partial W}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2 W & \text{on } \Gamma^{out}, \\ \dfrac{\partial W}{\partial \mathbf{n}_2} & \text{on } \Gamma^{in} \end{cases} \tag{4.31}$$

(actually, $S^1\psi$ depends only on the values of ψ on Γ^{in}).

The interface operator

$$S = S^1 + S^2 \,:\, \mathbf{H}^{\frac{1}{2}}_{00}(\Gamma) \to [\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma)]' \tag{4.32}$$

is the Steklov-Poincaré operator associated to problem (P). As a matter of fact, solving problem (P) (with homogeneous data $f = \phi = 0$) is equivalent to the *simultaneous solution* of the two independent problems (4.28) and (4.29) along with the interface equation

$$S\psi = 0 \tag{4.33}$$

(this equation is homogeneous because we are dealing with a homogeneous coupled problem). We can also give a *sequential version* of this procedure, which is strictly related to the iterative argument introduced at the beginning of this section 4. Indeed, the interface equation (4.33) may be equivalently reformulated by means of the operator $(S^2)^{-1}$, the inverse of S^2 (this exists, because (4.29) is well posed). In this case, we end up with the equation

$$-(S^2)^{-1}S^1\psi = \psi, \tag{4.34}$$

that is, we look for fixed points of the application $-(S^2)^{-1}S^1$. Now, it is readily seen that the restriction of this application to $\mathbf{L}^2_{\mathrm{b}}(\Gamma^{in})$ coincides with the map T_θ defined in (4.25), when $\theta = 1$ and $f = \phi = 0$: dropping the restriction, we simply write

$$-(S^2)^{-1}S^1 = T_1. \tag{4.35}$$

More generally, for any θ (4.25) gives

$$T_\theta\psi = \theta T_1\psi + (1-\theta)\psi, \tag{4.36}$$

whence (4.34) is equivalent also to the equation

$$T_\theta\psi = \psi. \tag{4.37}$$

Thus, the interface equation is equivalent to the search for fixed points of the map T_θ in the homogeneous case. The natural iterative procedure applied to (4.37) gives

$$\psi^{n+1} = T_\theta\psi^n. \tag{4.38}$$

On one hand, (4.35) and (4.36) allow us to read this iterative procedure as

$$\psi^{n+1} = -\theta(S^2)^{-1}S^1\psi^n + (1-\theta)\psi^n :$$

the application of the map S^2 and the definition (4.32) then give

$$S^2(\psi^{n+1} - \psi^n) = -\theta S\psi^n,$$

which is nothing else than the Richardson iterative method applied to equation (4.33) with "preconditioner" $(S^2)^{-1}$ (the terminology is borrowed from numerical analysis). On the other hand, (4.3) and (4.25) show that the iterative procedure (4.38) is equivalent to the iterative algorithm (4.1), (4.2). Therefore, we conclude that *the iterative algorithm* (4.1), (4.2) *is equivalent to the Richardson iterative method applied to equation* (4.33) *with "preconditioner"* $(S^2)^{-1}$.

4.2. The discrete case. The iterative procedure (4.1), (4.2) can be realized numerically by the spectral collocation method as follows.

For each n, we look for two polynomials $u_N^n \in P_N$, $w_N^n \in P_N$ satisfying the following hyperbolic problem in Ω_1:

$$\begin{cases} (i) & L_N^1 u_N^n + b_0 u_N^n = f \quad \text{at } (\Xi_N^1)^0 \cup (\Gamma_1^{out} \cup \Gamma_1^0)_N \cup (\Gamma^{out} \cup \Gamma^0)_N, \\ (ii) & -[\mathbf{b}\cdot\mathbf{n}_1(u_N^n - \phi)](P) = [L_N^1 u_N^n + b_0 u_N^n - f](P)\dfrac{\omega_P^{(1)}}{\theta_P^{(1)}} \quad \text{at } P \in (\Gamma_1^{in})_N, \\ (iii) & -[\mathbf{b}\cdot\mathbf{n}_1(u_N^n - \psi^n)](P) = [L_N^1 u_N^n + b_0 u_N^n - f](P)\dfrac{\omega_P^{(1)}}{\theta_P^{(1)}} \quad \text{at } P \in (\Gamma^{in})_N; \end{cases} \tag{4.39}$$

and the following elliptic problem in Ω_2:

$$\begin{cases} (i) & -\operatorname{div}[I_N^2(\nu \nabla w_N^n)] + L_N^2 w_N^n + b_0 w_N^n = f \quad \text{at } (\Xi_N^2)^0, \\ (ii) & w_N^n = \phi \quad \text{at } (\Gamma_2^D)_N, \\ (iii) & [-\nu \dfrac{\partial w_N^n}{\partial \mathbf{n}_2}](P) = R_2(P)\dfrac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma_2^{Ne})_N, \\ (iv) & [-\nu \dfrac{\partial w_N^n}{\partial \mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2(w_N^n - u_N^n)](P) = R_2(P)\dfrac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma^{out})_N, \\ (v) & [-\nu \dfrac{\partial w_N^n}{\partial \mathbf{n}_2}](P) = R_2(P)\dfrac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma^{in})_N. \end{cases} \tag{4.40}$$

The residue R_2 was defined in (2.10); moreover, the initial values u_N^0 and w_N^0 have to be prescribed at the collocation points of Γ. For $n \geq 1$, ψ^n is defined as follows, according to (4.3):

$$\psi^n = \theta w_N^{n-1} + (1-\theta) u_N^{n-1}, \qquad \theta > 0 \tag{4.41}$$

Clearly, (4.39)-(4.40) is an iterative procedure that yields at the limit the solution to the finite dimensional coupled problem (2.2)-(2.9).

We define the following discrete inner products, associated with the Gauss-Lobatto collocation points:

$$\begin{cases} (i) & (u,v)_{N,\Omega_i} = \sum_{P \in \Xi_N^i} u(P)v(P)\omega_P^{(i)}, \\ (ii) & (u,v)_{N,\Sigma} = \sum_{P \in \Sigma_N} u(P)v(P)\theta_P^{(i)}, \end{cases} \tag{4.42}$$

where Σ is any subset of $\partial\Omega_i$ (Σ_N was defined at the beginning of section 2) and u, v are defined in Ω_i, $i = 1,2$. Correspondingly, we introduce the norm associated to (4.42)(i):

$$\|v\|_{N,\Omega_i} = (v,v)_{N,\Omega_i}^{\frac{1}{2}}, \quad i = 1,2. \tag{4.43}$$

Upon the space of algebraic polynomials of degree less than or equal to N, this norm is equivalent to the continuous $\mathbb{L}^2$ norm with constants independent of N (see, e.g., [7], Ch. 2). Precisely, there is a constant C_0 independent of N such that

$$\|v\|_{0,\Omega_i} \leq \|v\|_{N,\Omega_i} \leq C_0\|v\|_{0,\Omega_i} \quad i = 1,2, \tag{4.44}$$

for all $v \in P_N$. Finally, we introduce the following notation, in agreement with (3.9): for all $v \in P_N$ and all subset Σ of $\partial\Omega_1$, we set

$$|v|_{N,\Sigma} = (|\mathbf{b}\cdot\mathbf{n}_1|, v^2)^{\frac{1}{2}}_{N,\Sigma}. \tag{4.45}$$

We recall that $|v|_{N,\Sigma}$ is uniformly equivalent to the norm $|v|_\Sigma$ for all functions $v \in P_N$. In particular, there is a constant C_1 independent of N such that

$$|v|_{N,\Sigma} \le C_1 |v|_\Sigma . \tag{4.46}$$

The theoretical results we are going to give in the following three subsections will be derived in section 6.2 for the collocation method using Legendre Gaussian points only (the case of Chebyshev points is not covered by our analysis).

4.2.1. The hyperbolic collocation problem in Ω_1. The following stability result (which is the discrete analogue of (4.10)) holds for problem (4.39) (see section 6.2.1 for the proof).

LEMMA 4.12 *Under the assumption* (2.11), *the solution* u_N^n *to* (4.39) *satisfies*

$$\beta_N \|u_N^n\|^2_{N,\Omega_1} + |u_N^n|^2_{N,\partial\Omega_1^{out}} \le \frac{1}{\beta_N}\|f\|^2_{N,\Omega_1} + |\phi|^2_{N,\Gamma_1^{in}} + |\psi^n|^2_{N,\Gamma^{in}}. \tag{4.47}$$

□

COROLLARY 4.13 *If in particular it is* $f = 0$ *and* $\phi = 0$ *on* Γ_1^{in}, *then the inequality* (4.47) *reduces to the following "outflow-inflow" estimate:*

$$\beta_N \|u_N^n\|^2_{N,\Omega_1} + |u_N^n|^2_{N,\partial\Omega_1^{out}} \le |\psi^n|^2_{N,\Gamma^{in}}. \tag{4.48}$$

□

REMARK 4.14 (Boundary points in which $\mathbf{b}\cdot\mathbf{n}_1 = 0$). On the boundary subset $(\Gamma_1^0)_N$ we have collocated the differential equation, both in (2.2) and in (4.39). In other words, the collocation scheme here proposed treats $(\Gamma_1^0)_N$ as a part of the outflow boundary. In this way, the stability inequality (4.47) states that the discrete solution u_N^n depends solely on the values of the "inflow data" on $\Gamma_1^{in} \cup \Gamma^{in}$. As a by-product, one deduces that $u_N^n = 0$, whenever $f = 0$, $\phi = 0$ and $\psi^n = 0$ (homogeneous case), i.e. the discrete hyperbolic problem has a unique solution.□

4.2.2. The elliptic collocation problem in Ω_2. The following stability result holds for problem (4.40) (see section 6.2.2 for the proof).

LEMMA 4.15 *Under the assumption* (2.11), *if* $\phi = 0$ *then the solution* w_N^n *to problem* (4.40) *satisfies*

$$2\|\sqrt{\nu}|\nabla w_N^n|\|^2_{N,\Omega_2} + \beta_N \|w_N^n\|^2_{N,\Omega_2} + (|\mathbf{b}\cdot\mathbf{n}_2|, (w_N^n)^2)_{N,\Gamma_2^{Ne}} \le$$
$$\le \frac{1}{\beta_N}\|f\|^2_{N,\Omega_2} + |u_N^n|^2_{N,\Gamma^{out}}. \tag{4.49}$$

□

The inequality (4.49) is the discrete counterpart of (4.19) for the homogeneous case.

4.2.3. Convergence of the discrete iterative procedure; existence and uniqueness results for the coupled collocation problem. To begin with, from the previous hyperbolic and elliptic stability estimates we deduce a global inequality (analogous to (4.27)) for the homogeneous case, which will be useful in proving convergence of the iterative scheme.

Under the same assumptions of Corollary 4.13 and Lemma 4.15, by a simple combination of (4.48) and (4.49) we get the following hyperbolic-elliptic inequality:

$$2\|\sqrt{\nu}|\nabla w_N^n|\|_{N,\Omega_2}^2 + \beta_N\{\|u_N^n\|_{N,\Omega_1}^2 + \|w_N^n\|_{N,\Omega_2}^2\} + |w_N^n|_{N,\Gamma^{in}}^2 \le |\psi^n|_{N,\Gamma^{in}}^2. \quad (4.50)$$

As a consequence of this estimate, we will prove the existence and uniqueness of the solution to the coupled collocation problem (2.2)-(2.9). Let us define the following *error functions*:

$$e_1^n = u_N - u_N^n, \quad e_2^n = w_N - w_N^n, \quad (4.51)$$

where (u_N, w_N) is the solution to problem (2.2)-(2.9), while u_N^n, w_N^n are the solutions to (4.39) and (4.40), respectively. The following theorem states the convergence of (u_N^n, w_N^n) to (u_N, w_N). The proof will be given in section 6.2.3.

THEOREM 4.16 *There exists $\delta > 0$ such that for all $\theta \in (1-\delta, 1+\delta)$ we have*

$$e_1^n \to 0, \quad e_2^n \to 0, \quad \text{as } n \to +\infty.$$

Therefore, the sequence (u_N^n, w_N^n) converges to the solution (u_N, w_N) of the coupled collocation problem (2.2)-(2.9). *Moreover, the rate of convergence is independent of* N.□

REMARK 4.17 An argument analogous to that of Remark 4.11 can be repeated in the present case as well.□

4.2.4. Some numerical results. Several numerical experiments based on the spectral collocation method (2.2)-(2.9) are reported in [2]. They show that the numerical solution is in a very good agreement with the differential solution, for the commonly used case of collocation points of Chebyshev type, even without using the (expensive) skew-symmetric decomposition (2.1). Here we make some additional remarks about the convergence properties of our iterative procedure (4.39), (4.40).

There are situations in which the differential problem (P) decouples in a natural way into two problems, one of them being independent of the other. This is the case, for instance, when either $\Gamma^{out} = \emptyset$, or $\Gamma^{in} = \emptyset$ (see Figure 4.1).

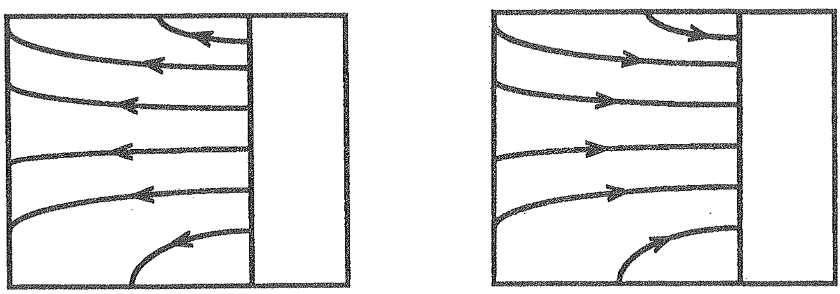

Figure 4.1. Examples of convergence in one iteration: the orientation of the characteristic lines is shown in Ω_1.

In these situations, our iterative procedure converges just in one iteration (in the former case, the elliptic problem is solved first, while in the latter the hyperbolic problem must be solved in advance). We note that this feature occurs in the first case because we used (1.12) instead of (1.9). This choice is helpful also in a more general context, namely when no characteristic lines exist entering Ω_1 across Γ^{in} and leaving Ω_1 across Γ^{out} (see Figure 4.2). In this case, the hyperbolic problem must be solved first and a successive resolution in Ω_2 and Ω_1 provides the exact solution.

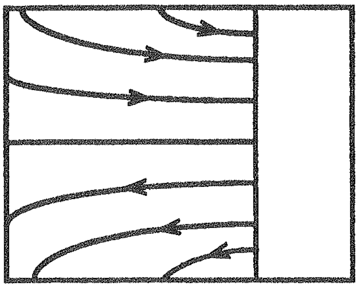

Figure 4.2. Examples of convergence in two iterations: the orientation of the characteristic lines is shown in Ω_1.

Here we report some results obtained for situations in which the coupling of the two problems is more severe. In all cases, we choose the following data:

$$\Omega = (-1,1)\times(-1,1), \quad \Omega_1 = (-1,0)\times(-1,1), \quad \Omega_2 = (0,1)\times(-1,1);$$
$$f = 1, \quad b_0 = 1, \quad \mathrm{b} = \alpha \mathrm{B}.$$

$(N+1)^2$ Chebyshev collocation points are considered within each subdomain and the skew-symmetric decomposition (2.1) is not used. We denote by n_{it} the minimum value of the integer n such that

$$\max \frac{|u^n - u^{n-1}|}{|u^n|} < 10^{-8},$$

the maximum being taken on all grid points lying on Γ.

Table 4.1 shows the values of n_{it} for various choices of ν, N and α in the case $\mathrm{B} = (y,1)^T$.

Tables 4.2 and 4.3 deal with the case $\mathrm{B} = (y,-x)^T$. Table 4.2 shows the values of n_{it} for various choices of ν and α, with $N = 8$. Table 4.3 shows the dependence of n_{it} on N and α, with $\nu = 10^{-3}$.

In all of these cases, the relaxation parameter θ appearing in (4.41) was chosen dynamically so as to minimize the interface error at each step.

For the case $\mathrm{B} = (y,1)^T$, Table 4.1 shows that the rate of convergence is independent of the number of grid points, while it has a mild dependence on the size of ν and $|\mathrm{b}|$, in agreement with the results of our investigation.

For the other case $\mathrm{B} = (y,-x)^T$, Table 4.2 shows that the rate of convergence is still mildly depending on the size of ν and b, while Table 4.3 shows a dependence also on the number of grid points, when $\alpha = 100$. We conjecture that this pathology is due to the ill-conditioning of the problem (the rate $|\mathrm{b}|\nu^{-1}$ ranges from 0 to 10^5 on the computational domain).

ν	α	N	n_{it}
0.1	0.1	8,16,20	4
0.1	1	8,16,20	5
0.1	10	8,16,20	9
0.1	100	16	13
0.01	0.1	8,16,20	4
0.01	1	8,16,20	8
0.01	10	8,16,20	14
0.01	100	16	15
0.001	0.1	8,16,20	5
0.001	1	8,16,20	11
0.001	10	8,16,20	16
0.001	100	16	16

Table 4.1.

ν	α	n_{it}
0.1	1	5
0.1	10	16
0.1	100	41
0.01	1	6
0.01	10	18
0.01	100	55
0.001	1	6
0.001	10	19
0.001	100	50

Table 4.2.

α	N	n_{it}
1	8	6
1	10	6
1	16	6
10	8	19
10	10	20
10	16	20
100	8	50
100	10	57
100	16	64

Table 4.3.

5. Vanishing viscosity approximation to problem (P). This section is devoted to the proof of existence of a solution to problem (P), stated in Theorem 3.2, via a "vanishing viscosity" argument. Throughout the section we shall assume (1.2), (1.3), (3.14), (3.15) and (3.16) without any further explicit mention.

Let $\epsilon > 0$ be a small parameter and choose sequences of $\mathbf{C}^\infty$ functions $\{\phi_\epsilon\}$, $\{b_{0\epsilon}\}$, $\{\nu_\epsilon\}$ such that

$$\phi_\epsilon \in \mathbf{C}^\infty(\partial\Omega), \quad \phi_{\epsilon|\Gamma_2^D} \to \phi_{|\Gamma_2^D} \text{ in } \mathbf{H}^{\frac{1}{2}}(\Gamma_2^D), \quad \phi_{\epsilon|\Gamma_1^{in}} \to \phi_{|\Gamma_1^{in}} \text{ in } \mathbf{L}_b^2(\Gamma_1^{in}); \tag{5.1}$$

$$b_{0\epsilon} \in \mathbf{C}^\infty(\Omega), \quad \nu_\epsilon \in \mathbf{C}^\infty(\Omega_2); \ b_{0\epsilon} \overset{*}{\rightharpoonup} b_0 \text{ in } \mathbf{L}^\infty(\Omega), \quad \nu_\epsilon \overset{*}{\rightharpoonup} \nu \text{ in } \mathbf{L}^\infty(\Omega_2); \tag{5.2}$$

as $\epsilon \to 0$ (in (5.2) $\overset{*}{\rightharpoonup}$ denotes the weak* convergence): the reason for such a regularization on the data will be clear later on. As a consequence of (1.2) and (5.2), we see that

$$\nu_\epsilon(x) \geq \frac{1}{2}\nu_0, \ x \in \Omega_2, \quad \frac{1}{2}\operatorname{div} \mathbf{b}(x) + b_{0\epsilon}(x) \geq \frac{1}{2}\beta_0, \ x \in \Omega, \tag{5.3}$$

for ϵ small enough: form now on we shall consider only values of ϵ such that (5.3) holds. Set

$$a_\epsilon(x) = \begin{cases} \epsilon & \text{if } x \in \Omega_1, \\ \nu_\epsilon(x) & \text{if } x \in \Omega_2. \end{cases} \tag{5.4}$$

Consider the closed, linear affine subspace of $\mathbf{H}^1(\Omega)$:

$$V_\epsilon = \{v \in \mathbf{H}^1(\Omega) : \ v_{|\Gamma_2^D} = \phi_\epsilon\} \tag{5.5}$$

and the following variational problem:

(P_ϵ) : to find $z_\epsilon \in V_\epsilon$ such that, for all $v \in V_\epsilon$,

$$\int_\Omega (a_\epsilon \nabla z_\epsilon - \mathbf{b} z_\epsilon)\cdot\nabla(z_\epsilon - v)dx + \int_\Omega b_{0\epsilon} z_\epsilon (z_\epsilon - v)dx + \int_{\Gamma_1^{out}\cup\Gamma_2^{Ne}} |\mathbf{b}\cdot\mathbf{n}| z_\epsilon(z_\epsilon - v)ds =$$
$$= \int_\Omega f(z_\epsilon - v)dx - \int_{\Gamma_1^{in}} \mathbf{b}\cdot\mathbf{n}_1 \phi_\epsilon (z_\epsilon - v)ds. \tag{5.6}$$

Note that in the last integral on the left hand side it is $|\mathbf{b}\cdot\mathbf{n}| = \mathbf{b}\cdot\mathbf{n}_1$ on Γ_1^{out} and $|\mathbf{b}\cdot\mathbf{n}| = \mathbf{b}\cdot\mathbf{n}_2$ on Γ_2^{Ne} (by (1.3)).

As we shall see in a moment, (P_ϵ) is equivalent to a boundary value problem quite similar to (P), the difference lying in the regularization on the data and in the presence of an elliptic singular perturbation (which affects also the boundary and interface conditions). To state this equivalence, we introduce the following notation which will be adopted from now on: if z_ϵ is a solution to (P_ϵ), we denote by

$$u_\epsilon = z_{\epsilon|\Omega_1} \quad \text{and} \quad w_\epsilon = z_{\epsilon|\Omega_2} \tag{5.7}$$

the restrictions of z_ϵ to Ω_1 and Ω_2, respectively.

LEMMA 5.1 *Let z_ϵ be a solution to (P_ϵ). Then*

$$-\epsilon\Delta u_\epsilon + \operatorname{div}(\mathbf{b}u_\epsilon) + b_{0\epsilon}u_\epsilon = f \qquad \text{in } \Omega_1, \tag{5.8}$$

$$\operatorname{div}(-\nu_\epsilon\nabla w_\epsilon + \mathbf{b}w_\epsilon) + b_{0\epsilon}w_\epsilon = f \qquad \text{in } \Omega_2, \tag{5.9}$$

$$-\epsilon\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} + \mathbf{b}\cdot\mathbf{n}_1 u_\epsilon = \mathbf{b}\cdot\mathbf{n}_1\phi_\epsilon \qquad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_1^{in})\right]', \tag{5.10}$$

$$\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} = 0 \qquad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_1^{out}\cup\Gamma_1^0)\right]', \tag{5.11}$$

$$w_\epsilon = \phi_\epsilon \qquad \text{on } \Gamma_2^D, \tag{5.12}$$

$$\nu_\epsilon\frac{\partial w_\epsilon}{\partial \mathbf{n}_2} = 0 \qquad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma_2^{Ne})\right]', \tag{5.13}$$

$$-\epsilon\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} + \mathbf{b}\cdot\mathbf{n}_1 u_\epsilon = \nu_\epsilon\frac{\partial w_\epsilon}{\partial \mathbf{n}_2} - \mathbf{b}\cdot\mathbf{n}_2 w_\epsilon \qquad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)\right]', \tag{5.14}$$

$$u_\epsilon = w_\epsilon \qquad \text{on } \Gamma. \tag{5.15}$$

Proof. Let z_ϵ solve (P_ϵ) and write (5.6) with $v = z_\epsilon + \psi$, where $\psi \in \mathcal{D}(\Omega)$ (that is, a smooth function with compact support). It follows

$$\operatorname{div}(-a_\epsilon\nabla z_\epsilon + \mathbf{b}z_\epsilon) + b_{0\epsilon}z_\epsilon = f \qquad \text{in } \mathcal{D}'(\Omega), \tag{5.16}$$

whence (5.8) and (5.9) are obtained by localization.
Setting

$$\mathbf{F}_\epsilon = -a_\epsilon \nabla z_\epsilon + \mathbf{b} z_\epsilon \tag{5.17}$$

(the *flux* corresponding to z_ϵ), by (5.16) we see that $\mathbf{F}_\epsilon \in \mathbf{L}^2_{\text{div}}(\Omega)$. Hence, Green's formula (3.5) (with $A = \Omega$ and $\Sigma = \Gamma_1$) gives, for all $\psi \in \mathbf{H}^1(\Omega)$ with $\psi = 0$ on Γ_2,

$$\int_\Omega \psi \operatorname{div} \mathbf{F}_\epsilon dx + \int_\Omega \mathbf{F}_\epsilon \cdot \nabla \psi dx = \langle \mathbf{F}_\epsilon \cdot \mathbf{n}_1, \psi \rangle.$$

Now, by the regularity assumption on b, it is

$$\langle \mathbf{F}_\epsilon \cdot \mathbf{n}_1, \psi \rangle = \langle -\epsilon \frac{\partial u_\epsilon}{\partial \mathbf{n}_1}, \psi \rangle + \int_{\Gamma_1} \mathbf{b} \cdot \mathbf{n}_1 u_\epsilon \psi ds.$$

Thus, (5.10) and (5.11) follow easily by taking $v = z_\epsilon + \psi$ in (5.6), with $\psi \in \mathbf{C}^\infty(\bar{\Omega})$, $\psi = 0$ in Ω_2. An analogous argument shows (5.13). Since $z_\epsilon \in V_\epsilon$, (5.12) and (5.15) are obvious. Finally, (5.14) may be proved by a standard argument which consists in applying Green's formula separately to Ω_1 and Ω_2, then using (5.8) and (5.9).□

REMARK 5.2 The way we have chosen for regularizing problem (P) has proven effective in a low regularity framework such as ours: remind that the original inflow data satisfies (3.16), hence a pure Dirichlet condition on Γ_1^{in} for the approaching problems is not stable as $\epsilon \to 0$. Moreover, the a priori estimates we are going to obtain are strongly based on the type of regularization we adopted.□

Existence and uniqueness of the solution to problem (P_ϵ) is a consequence of our assumptions (in particular, of (5.3) and (1.3)).

THEOREM 5.3 *For all $\epsilon > 0$, problem (P_ϵ) has one and only one solution z_ϵ.*

Proof. The bilinear, continuous form appearing in (5.6)

$$a(u, v) \underset{def}{=} \int_\Omega (a_\epsilon \nabla u - \mathbf{b} u) \cdot \nabla v \, dx + \int_{\Gamma_1^{out} \cup \Gamma_2^{Ne}} |\mathbf{b} \cdot \mathbf{n}| uv \, ds + \int_\Omega b_{0\epsilon} uv \, dx$$

is coercive in $V_\epsilon - V_\epsilon$ (algebraic difference). In fact, Green's formula (3.7) gives, for all $u,\ v \in V_\epsilon$,

$$\begin{aligned} a(u-v, u-v) &= \int_\Omega a_\epsilon |\nabla(u-v)|^2 dx + \int_\Omega (\frac{1}{2} \operatorname{div} \mathbf{b} + b_{0\epsilon})(u-v)^2 dx + \\ &+ \frac{1}{2} \int_{\Gamma_1^{out} \cup \Gamma_2^{Ne}} |\mathbf{b} \cdot \mathbf{n}| (u-v)^2 ds - \frac{1}{2} \int_{\Gamma_1^{in}} \mathbf{b} \cdot \mathbf{n}_1 (u-v)^2 ds \overset{(5.3)}{\geq} \\ &\geq \min(\epsilon, \frac{1}{2}\nu_0) \int_\Omega |\nabla(u-v)|^2 dx + \frac{1}{2}\beta_0 \int_\Omega (u-v)^2 dx + \\ &+ \frac{1}{2} \int_{\Gamma_1^{out} \cup \Gamma_2^{Ne}} |\mathbf{b} \cdot \mathbf{n}| (u-v)^2 ds \geq \min(\epsilon, \frac{1}{2}\nu_0, \frac{1}{2}\beta_0) \|u-v\|^2_{1,\Omega}. \end{aligned}$$

Therefore, Lax-Milgram Lemma gives existence and uniqueness of the solution. These properties are not uniform in ϵ, since the coerciveness constant vanishes as $\epsilon \to 0$.□

REMARK 5.4 Assumption (1.3) has proven convenient for coerciveness, although it is not necessary. For instance, one can do without it in the coerciveness computation, whenever the integral on Γ_2^{Ne} can be absorbed by the $\mathbf{H}^1$ norm near Γ_2^{Ne}. This requires a balance between ν_ϵ, $\mathbf{b}$ and $b_{0\epsilon}$ we do not detail here.□

Now we are in a position to begin our asymptotic analysis, as $\epsilon \to 0$.

LEMMA 5.5 *There is a constant $C > 0$ such that, for all $\epsilon > 0$,*

$$\|z_\epsilon\|_{0,\Omega} \leq C, \tag{5.18}$$

$$\sqrt{\epsilon}\|\nabla u_\epsilon\|_{0,\Omega_1} \leq C, \tag{5.19}$$

$$\|\nabla w_\epsilon\|_{0,\Omega_2} \leq C. \tag{5.20}$$

Proof. For all $\epsilon > 0$, let $v_\epsilon \in \mathbf{H}^1(\Omega)$ be a function such that

$$v_{\epsilon|\Gamma_2^D} = \phi_\epsilon, \quad \|v_\epsilon\|_{1,\Omega} \leq C\|\phi_\epsilon\|_{\frac{1}{2},\Gamma_2^D}$$

(there are such v's). By (5.1), it follows that

$$\|v_\epsilon\|_{1,\Omega} \leq M, \tag{5.21}$$

for some $M > 0$. Write (5.6) for $v = v_\epsilon$ and apply Green's formula (3.7):

$$\int_\Omega a_\epsilon|\nabla z_\epsilon|^2 dx + \int_\Omega (\frac{1}{2}\operatorname{div}\mathbf{b} + b_{0\epsilon}) {z_\epsilon}^2 dx + \frac{1}{2}\int_{\Gamma_1^{out}\cup\Gamma_2^{Ne}\cup\Gamma_1^{in}} |\mathbf{b}\cdot\mathbf{n}| {z_\epsilon}^2 ds =$$

$$= \int_\Omega a_\epsilon \nabla z_\epsilon \cdot \nabla v_\epsilon \, dx - \int_\Omega (\mathbf{b} z_\epsilon)\cdot\nabla v_\epsilon \, dx + \int_\Omega b_{0\epsilon} z_\epsilon v_\epsilon \, dx + \frac{1}{2}\int_{\Gamma_2^D} \mathbf{b}\cdot\mathbf{n}_2 {\phi_\epsilon}^2 ds +$$

$$+ \int_{\Gamma_1^{out}\cup\Gamma_2^{Ne}} |\mathbf{b}\cdot\mathbf{n}| z_\epsilon v_\epsilon \, ds - \int_{\Gamma_1^{in}} \mathbf{b}\cdot\mathbf{n}_1 \phi_\epsilon (u_\epsilon - v_\epsilon) ds + \int_\Omega f(z_\epsilon - v_\epsilon)\, dx.$$

This inequality, together with (5.3), (1.3) and the well known algebraic inequality $ab \leq \delta a^2 + \frac{1}{4\delta}b^2$, valid for all $a, b \geq 0$ and all $\delta > 0$, gives

$$\epsilon\|\nabla u_\epsilon\|_{0,\Omega_1}^2 + \frac{1}{2}\nu_0\|\nabla w_\epsilon\|_{0,\Omega_2}^2 + \frac{1}{2}\beta_0\|z_\epsilon\|_{0,\Omega}^2 + \frac{1}{2}|u_\epsilon|_{\Gamma_1^{in}}^2 + \frac{1}{2}|u_\epsilon|_{\Gamma_1^{out}}^2 + \frac{1}{2}|w_\epsilon|_{\Gamma_2^{Ne}}^2 \leq$$

$$\leq \delta\epsilon\||\nabla u_\epsilon|\|_{0,\Omega_1}^2 + \delta\|\sqrt{\nu_\epsilon}|\nabla w_\epsilon|\|_{0,\Omega_2}^2 + \frac{1}{4\delta}\|v_\epsilon\|_{1,\Omega}^2 +$$

$$+ \{\delta\|z_\epsilon\|_{0,\Omega}^2 + \frac{1}{4\delta}\|v_\epsilon\|_{1,\Omega}^2\}\{\|\mathbf{b}\|_{1,\infty,\Omega} + \|b_{0\epsilon}\|_{\infty,\Omega}\} + \delta\|z_\epsilon\|_{0,\Omega}^2 +$$

$$\|v_\epsilon\|_{0,\Omega}^2 + (1 + \frac{1}{4\delta})\|f\|_{0,\Omega}^2 + \delta|u_\epsilon|_{\Gamma_1^{in}}^2 + \frac{1}{2\delta}|\phi_\epsilon|_{\Gamma_1^{in}}^2 + \delta|v_\epsilon|_{\Gamma_1^{in}}^2 +$$

$$+ \delta|w_\epsilon|_{\Gamma_2^{Ne}}^2 + \frac{1}{4\delta}|v_\epsilon|_{\Gamma_2^{Ne}}^2 + \frac{1}{2}|\phi_\epsilon|_{\Gamma_2^D}^2 + \frac{1}{4\delta}|v_\epsilon|_{\Gamma_1^{out}}^2 + \delta|u_\epsilon|_{\Gamma_1^{out}}^2. \tag{5.22}$$

Hence, (5.17)-(5.20) follow by (5.1), (5.2) and (5.21).□

We note here that (5.22) gives an additional information on some boundary integrals that will be exploited in section 6.

Lemma 5.5 implies the following Corollary immediately.

COROLLARY 5.6 *There exist*

$$u \in \mathbf{L}^2(\Omega_1), \quad w \in \mathbf{H}^1(\Omega_2) \tag{5.23}$$

such that, possibly taking subsequences,

$$u_\epsilon \rightharpoonup u \text{ weakly in } \mathbf{L}^2(\Omega_1); \quad w_\epsilon \rightharpoonup w \text{ weakly in } \mathbf{H}^1(\Omega_2). \tag{5.24}$$

Moreover,

$$\epsilon \nabla u_\epsilon \to 0 \text{ strongly in } \mathbf{L}^2(\Omega_1). \tag{5.25}$$

□

Analogous results hold for the flux $\mathbf{F}_\epsilon$ defined in (5.17).

PROPOSITION 5.7 *There is a constant $C > 0$ such that, for all $\epsilon > 0$,*

$$\|\mathbf{F}_\epsilon\|_{\mathbf{L}^2_{\mathrm{div}}(\Omega)} \leq C. \tag{5.26}$$

Consequently, on a subsequence, we find that

$$\mathbf{F}_\epsilon \rightharpoonup \mathbf{F} \text{ weakly in } \mathbf{L}^2_{\mathrm{div}}(\Omega), \tag{5.27}$$

where

$$\mathbf{F} = \begin{cases} \mathrm{b}u & \text{in } \Omega_1, \\ -\nu \nabla w + \mathrm{b}w & \text{in } \Omega_2, \end{cases} \tag{5.28}$$

u, w *being the functions provided by Corollary* 5.6.

Proof. The estimate (5.26) follows easily by Lemma 5.5. Consequently, there exists some $\mathbf{F} \in \mathbf{L}^2_{\mathrm{div}}(\Omega)$ such that (5.27) holds. By uniqueness of the $\mathbf{L}^2(\Omega)$ weak limit, (5.2), (5.24) and (5.25) give formula (5.28) for $\mathbf{F}$.□

The limit functions u and w satisfy differential equations and boundary and interface conditions. A first result is provided by next theorem.

THEOREM 5.8 *The functions* u *and* w *provided by Corollary* 5.6 *satisfy:*

$$\mathrm{div}\,(\mathrm{b}u) + b_0 u = f \qquad \text{in } \mathcal{D}'(\Omega_1), \tag{5.29}$$

$$\mathrm{div}\,(-\nu\nabla w + \mathrm{b}w) + b_0 w = f \qquad \text{in } \mathcal{D}'(\Omega_2), \tag{5.30}$$

$$\mathrm{b}\cdot\mathrm{n}_1 u = \mathrm{b}\cdot\mathrm{n}_1 \phi \qquad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_1^{in})\right]', \tag{5.31}$$

$$w = \phi \qquad \text{on } \Gamma_2^D, \tag{5.32}$$

$$\nu \frac{\partial w}{\partial \mathrm{n}_2} = 0 \qquad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_2^{Ne})\right]', \tag{5.33}$$

$$-\mathrm{b}\cdot\mathrm{n}_1 u = -\nu\frac{\partial w}{\partial \mathrm{n}_2} + \mathrm{b}\cdot\mathrm{n}_2 w \qquad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma)\right]'. \tag{5.34}$$

Proof. Let $\psi_i \in \mathcal{D}(\Omega_i)$, $i = 1, 2$, and choose $v = z_\epsilon + \psi_i$ in (5.6). Thanks to (5.2), (5.24) and (5.25), we may take the limit in (5.6), obtaining the two differential equations (5.29) and (5.30). The boundary condition (5.32) follows by (5.1), (5.12) and (5.24). Moreover, (5.27) implies the weak convergence of $\mathbf{F}_\epsilon \cdot \mathbf{n}$ to $\mathbf{F}\cdot\mathbf{n}$ in $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_1^{in})\right]'$, $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma)\right]'$ and $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma_2^{Ne})\right]'$, so that (5.28), (5.10), (5.1) and Lemma 3.1, part (ii), give (5.31). Again (5.28), (5.24) and (5.13) imply (5.33), while (5.34) follows by (5.28) and (5.14). Finally, assumption (3.16) on ϕ and Lemma 3.1, part (i), (with $A = \Omega$, $\Sigma = \Gamma^{in}$, $\rho = |\mathbf{b}\cdot\mathbf{n}_1|$, $v = u$ and $z = \phi$) show that $u_{|\Gamma_1^{in}} \in \mathbf{L}^2_{\mathbf{b}}(\Gamma_1^{in})$ and that the boundary condition (5.31) gives $u = \phi$ a.e. on Γ_1^{in}, which is the sense stated in Theorem 3.2.□

To complete the proof of Theorem 3.2, we must show that u and w satisfy a further condition on Γ^{in}. This needs a preliminary study of higher regularity for u_ϵ and w_ϵ.

PROPOSITION 5.9 *The solution z_ϵ to (P_ϵ) is such that $a_\epsilon \nabla z_\epsilon \in \mathbf{H}^1(\Omega\backslash\Lambda)$, where Λ is an arbitrarily small neighborhood of $\partial\Omega$.*

Proof. Since the local regularity result is well known in the interior of Ω_i, here we check the local regularity near a point $P \in \Gamma$, where the transmission condition is enforced.

After a localization and flattening of Γ, we are left with the problem of finding $\tilde{z}_\epsilon \in \mathbf{H}^1(\mathbf{R}^2)$ such that, for all $v \in \mathbf{H}^1(\mathbf{R}^2)$,

$$\int_{\mathbf{R}^2} (\tilde{a}_\epsilon \nabla \tilde{z}_\epsilon - \tilde{\mathbf{b}}\tilde{z}_\epsilon)\cdot\nabla v\, dx + \int_{\mathbf{R}^2} \tilde{b}_{0\epsilon}\tilde{z}_\epsilon v\, dx = \int_{\mathbf{R}^2} \tilde{f} v\, dx \tag{5.35}$$

(tildas take into account that the original solution has been multiplied by a localizing factor and that a change of variables has been made in order to flatten the interface; note that $\tilde{a}_\epsilon$ shows a jump across $\{x_2 = 0\}$ and that it behaves like ϵ on $\{x_2 < 0\}$). We note that (5.35) is equivalent to the differential equation

$$\mathrm{div}\,(-\tilde{a}_\epsilon \nabla \tilde{z}_\epsilon + \tilde{\mathbf{b}}\tilde{z}_\epsilon) + \tilde{b}_{0,\epsilon}\tilde{z}_\epsilon = \tilde{f} \quad \text{in } \mathcal{D}'(\mathbf{R}^2). \tag{5.36}$$

To shorten notations, we will drop the tildas, the subscript ϵ and the domain of integration $\mathbf{R}^2$ in the rest of this computation. Moreover, we will denote by C, $C(\epsilon)$,... possibly different constants.

With the aim of applying a difference quotients technique, we fix $h > 0$ and set, for a given v defined in $\mathbf{R}^2$,

$$v_+(x_1, x_2) = v(x_1 + h, x_2), \quad (x_1, x_2) \in \mathbf{R}^2; \qquad \delta_h v = v_+ - v.$$

Plug $v = \delta_h z$ in (5.35): it follows

$$\int (a\nabla z - \mathbf{b}z)\cdot\nabla(\delta_h z)\, dx + \int b_0 z(\delta_h z)\, dx = \int f(\delta_h z)\, dx. \tag{5.37}$$

Next, make the change of variables $(x_1, x_2) \mapsto (x_1 + h, x_2)$ in (5.35), then choose $v_+ = \delta_h z$ in the transformed equation: it follows

$$\int (a_+ \nabla z_+ - \mathbf{b}_+ z_+)\cdot\nabla(\delta_h z)\, dx + \int b_{0+} z_+(\delta_h z)\, dx = \int f_+(\delta_h z)\, dx. \tag{5.38}$$

Taking the difference between (5.37) and (5.38), observing that ∇ and δ_h commute and recalling the identity $\delta_h(uv) = u\delta_h v + v_+\delta_h u$, we find

$$-\int a_+|\nabla(\delta_h z)|^2 dx - \int \delta_h a \nabla z\cdot\nabla(\delta_h z)\, dx + \int \mathbf{b}_+\delta_h z\cdot\nabla(\delta_h z)\, dx +$$
$$+\int z\delta_h \mathbf{b}\cdot\nabla(\delta_h z)\, dx - \int b_{0+}|\delta_h z|^2 dx - \int \delta_h b_0 z \delta_h z\, dx = -\int \delta_h f \delta_h z\, dx.$$

Therefore,

$$\int a_+|\nabla(\delta_h z)|^2 dx \le \|\delta_h a\|_\infty \|\nabla z\|_0 \|\nabla(\delta_h z)\|_0 + \|\mathbf{b}\|_\infty \|\delta_h z\|_0 \|\nabla(\delta_h z)\|_0 + \\ + \|\delta_h \mathbf{b}\|_\infty \|z\|_0 \|\nabla(\delta_h z)\|_0 + \|b_0\|_\infty \|\delta_h z\|_0^2 + \\ + \|\delta_h b_0\|_\infty \|z \delta_h z\|_0 + \|\delta_h f\|_{-1} \|\delta_h z\|_1, \tag{5.39}$$

where $\|\cdot\|_{-1}$ denotes the norm of the dual space of $\mathbf{H}^1(\mathbf{R}^2) = \mathbf{H}_0^1(\mathbf{R}^2)$. Now,

$$\|z\delta_h z\|_0 \le \|z\|_0 \|\delta_h z\|_0, \quad \|\delta_h z\|_1^2 = \|\delta_h z\|_0^2 + \|\nabla(\delta_h z)\|_0^2$$

and, for h small,

$$\|\delta_h a\|_\infty \le Ch\|\frac{\partial a}{\partial x_1}\|_\infty, \quad \|\delta_h \mathbf{b}\|_\infty \le Ch\|\mathbf{b}\|_{1,\infty}, \quad \|\delta_h b_0\|_0 \le Ch\|b_0\|_1,$$
$$\|\delta_h f\|_{-1} \le Ch\|f\|_0, \quad \|\delta_h z\|_0 \le Ch\|z\|_1.$$

Thus, (5.39) gives in a standard way

$$\|\nabla(\delta_h z)\|_0^2 \le C(\epsilon) h^2 \{\|a\|_{1,\infty}^2 + \|\mathbf{b}\|_{1,\infty}^2 + \|b_0\|_1^2 + \|f\|_0^2 + \|z\|_1^2\}.$$

Taking the limit as $h \to 0$, we find

$$\frac{\partial}{\partial x_1}(a\nabla z) \in \mathbf{L}^2(\mathbf{R}^2) \quad \text{and} \quad \|\frac{\partial}{\partial x_1}(a\nabla z)\|_0 \le C(\epsilon), \tag{5.40}$$

because a is regular in the x_1 direction. By (5.36), we know that

$$\frac{\partial}{\partial x_2}(a\nabla z) = -\frac{\partial}{\partial x_1}(a\nabla z) + \operatorname{div}(\mathbf{b}z) + b_0 z - f,$$

so that (5.40) implies

$$\|\frac{\partial}{\partial x_2}(a\nabla z)\|_0 \le C(\epsilon),$$

that is

$$\|(a\nabla z)\|_1 \le C(\epsilon). \tag{5.41}$$

Note that the constant $C(\epsilon)$ is unbounded as $\epsilon \to 0$. Going back to the original variables, (5.41) gives the local $\mathbf{H}^1$ regularity of $a_\epsilon \nabla z_\epsilon$ near the point P, whence the proof is complete.□

REMARK 5.10 We did not care about the regularity near the fixed boundary $\partial\Omega$, which will be of no use in the sequel. We just warn the reader that $\mathbf{H}^2$ regularity of the solution near a point of $\partial\Omega$ may not hold, even for very smooth data, unless some compatibility condition depending also on the geometry of Ω is satisfied.□

The local regularity provided by Proposition 5.9 enables us to get an a priori estimate for z_ϵ.

LEMMA 5.11 *Let ψ be a nonnegative, smooth function defined in $\mathbf{R}^2$, vanishing in a neighborhood of $\partial\Omega \cup \Gamma^{out}$. There is a constant $C_\psi > 0$ such that, for all ϵ,*

$$\int_\Omega \psi|\operatorname{div}(\mathbf{b}z_\epsilon)|^2 dx \le C_\psi. \tag{5.42}$$

Proof. Let ψ be such a function and take the $\mathbf{L}^2(\Omega)$ scalar product of (5.16) by $\psi \operatorname{div}(\mathbf{b}z_\epsilon)$. Recalling (5.7), it follows:

$$-\epsilon\int_{\Omega_1}\Delta u_\epsilon\psi\mathbf{b}\cdot\nabla u_\epsilon\,dx-\epsilon\int_{\Omega_1}\Delta u_\epsilon\psi\operatorname{div}\mathbf{b}\,u_\epsilon\,dx-$$
$$-\int_{\Omega_2}\operatorname{div}(\nu_\epsilon\nabla w_\epsilon)\psi\mathbf{b}\cdot\nabla w_\epsilon\,dx-\int_{\Omega_2}\operatorname{div}(\nu_\epsilon\nabla w_\epsilon)\psi\operatorname{div}\mathbf{b}\,w_\epsilon\,dx+$$
$$+\int_\Omega\psi|\operatorname{div}(\mathbf{b}z_\epsilon)|^2dx+\int_\Omega\psi b_{0\epsilon}z_\epsilon\operatorname{div}(\mathbf{b}z_\epsilon)\,dx=\int_\Omega\psi f\operatorname{div}(\mathbf{b}z_\epsilon)\,dx. \tag{5.43}$$

Proposition 5.9 allows us to apply Green's formula separately to Ω_1 and Ω_2, since the restriction of $\psi\operatorname{div}(\mathbf{b}z_\epsilon)$ to Ω_i belongs to $\mathbf{H}^1(\Omega_i)$, $i=1,2$. Therefore, we find

$$-\epsilon\int_{\Omega_1}\Delta u_\epsilon\psi\mathbf{b}\cdot\nabla u_\epsilon\,dx=\epsilon\int_{\Omega_1}\nabla u_\epsilon\cdot\nabla\psi\,\mathbf{b}\cdot\nabla u_\epsilon\,dx+$$
$$\epsilon\int_{\Omega_1}\psi\nabla u_\epsilon\cdot\nabla(\mathbf{b}\cdot\nabla u_\epsilon)\,dx-\epsilon\int_{\partial\Omega_1}\psi\frac{\partial u_\epsilon}{\partial\mathbf{n}_1}\mathbf{b}\cdot\nabla u_\epsilon\,ds. \tag{5.44}$$

The second integral on the right hand side of (5.44) may be worked out as follows:

$$\epsilon\int_{\Omega_1}\psi\nabla u_\epsilon\cdot\nabla(\mathbf{b}\cdot\nabla u_\epsilon)\,dx=\frac{\epsilon}{2}\int_{\Omega_1}\psi\mathbf{b}\cdot\nabla(|\nabla u_\epsilon|^2)\,dx+\epsilon\int_{\Omega_1}\psi\nabla u_\epsilon\cdot(\nabla u_\epsilon\cdot\nabla\mathbf{b})\,dx=$$
$$=-\frac{\epsilon}{2}\int_{\Omega_1}\operatorname{div}(\psi\mathbf{b})|\nabla u_\epsilon|^2dx+\frac{\epsilon}{2}\int_{\partial\Omega_1}\psi\mathbf{b}\cdot\mathbf{n}_1|\nabla u_\epsilon|^2ds+\epsilon\int_{\Omega_1}\psi\nabla u_\epsilon\cdot(\nabla u_\epsilon\cdot\nabla\mathbf{b})\,dx.$$

Therefore, the first integral in (5.43) becomes:

$$-\epsilon\int_{\Omega_1}\Delta u_\epsilon\psi\mathbf{b}\cdot\nabla u_\epsilon\,dx=\epsilon\int_{\Omega_1}\nabla u_\epsilon\cdot\nabla\psi\,\mathbf{b}\cdot\nabla u_\epsilon\,dx+$$
$$+\epsilon\int_{\Omega_1}\psi\nabla u_\epsilon\cdot(\nabla u_\epsilon\cdot\nabla\mathbf{b})\,dx-\frac{\epsilon}{2}\int_{\Omega_1}\operatorname{div}(\psi\mathbf{b})|\nabla u_\epsilon|^2dx-$$
$$-\epsilon\int_{\partial\Omega_1}\psi\frac{\partial u_\epsilon}{\partial\mathbf{n}_1}\mathbf{b}\cdot\nabla u_\epsilon\,ds+\frac{\epsilon}{2}\int_{\partial\Omega_1}\psi\mathbf{b}\cdot\mathbf{n}_1|\nabla u_\epsilon|^2ds. \tag{5.45}$$

Analogously, the third integral in (5.43) becomes:

$$-\int_{\Omega_2}\operatorname{div}(\nu_\epsilon\nabla w_\epsilon)\psi\mathbf{b}\cdot\nabla w_\epsilon\,dx=\int_{\Omega_2}\nu_\epsilon\nabla w_\epsilon\cdot\nabla\psi\,\mathbf{b}\cdot\nabla w_\epsilon\,dx+$$
$$+\int_{\Omega_2}\nu_\epsilon\psi\nabla w_\epsilon\cdot(\nabla w_\epsilon\cdot\nabla\mathbf{b})\,dx-\frac{1}{2}\int_{\Omega_2}\operatorname{div}(\psi\mathbf{b})\nu_\epsilon|\nabla w_\epsilon|^2dx-$$
$$-\int_{\partial\Omega_2}\psi\nu_\epsilon\frac{\partial w_\epsilon}{\partial\mathbf{n}_2}\mathbf{b}\cdot\nabla w_\epsilon\,ds+\frac{1}{2}\int_{\partial\Omega_2}\psi\nu_\epsilon\mathbf{b}\cdot\mathbf{n}_2|\nabla w_\epsilon|^2ds. \tag{5.46}$$

Using (5.16), second and fourth integrals in (5.43) become respectively:

$$-\epsilon\int_{\Omega_1}\Delta u_\epsilon\psi\operatorname{div}\mathbf{b}\,u_\epsilon\,dx=\int_{\Omega_1}(f-\operatorname{div}(\mathbf{b}u_\epsilon)-b_{0\epsilon}u_\epsilon)\,\psi u_\epsilon\operatorname{div}\mathbf{b}\,dx, \tag{5.47}$$

$$-\int_{\Omega_2} \operatorname{div}(\nu_\epsilon \nabla w_\epsilon)\psi \operatorname{div}\mathbf{b}\, w_\epsilon\, dx = \int_{\Omega_2} (f - \operatorname{div}(\mathbf{b}w_\epsilon) - b_{0\epsilon}w_\epsilon)\, \psi w_\epsilon \operatorname{div}\mathbf{b}\, dx. \quad (5.48)$$

Taking (5.45)-(5.48) into account, (5.43) becomes

$$\int_\Omega \psi|\operatorname{div}(\mathbf{b}z_\epsilon)|^2 dx - \epsilon\int_{\partial\Omega_1}\psi\frac{\partial u_\epsilon}{\partial \mathbf{n}_1}\mathbf{b}\cdot\nabla u_\epsilon\, ds + \frac{\epsilon}{2}\int_{\partial\Omega_1}\psi\mathbf{b}\cdot\mathbf{n}_1|\nabla u_\epsilon|^2 ds -$$
$$-\int_{\partial\Omega_2}\psi\nu_\epsilon\frac{\partial w_\epsilon}{\partial \mathbf{n}_2}\mathbf{b}\cdot\nabla w_\epsilon\, ds + \frac{1}{2}\int_{\partial\Omega_2}\psi\nu_\epsilon\mathbf{b}\cdot\mathbf{n}_2|\nabla w_\epsilon|^2 ds \le$$
$$\le C(\psi)\left\{\|\mathbf{b}\|_{1,\infty,\Omega}\left[\|\sqrt{\epsilon}|\nabla u_\epsilon|\|^2_{0,\Omega_1} + \|\sqrt{\nu_\epsilon}|\nabla w_\epsilon|\|^2_{0,\Omega_2}\right] + \right.$$
$$\left. + \left[\|\sqrt{\psi}\operatorname{div}(\mathbf{b}z_\epsilon)\|_{0,\Omega} + \|\mathbf{b}\|_{1,\infty,\Omega}\|z_\epsilon\|_{0,\Omega}\right]\left[\|f\|_{0,\Omega} + \|b_{0\epsilon}\|_{\infty,\Omega}\|z_\epsilon\|_{0,\Omega}\right]\right\}, \quad (5.49)$$

for a constant $C(\psi)$ independent of ϵ. Recalling (5.2), (5.24) and (5.25), we have that

$$\{\text{right hand side of } (5.49)\} \le C(\psi)\{1 + \|\sqrt{\psi}\operatorname{div}(\mathbf{b}z_\epsilon)\|_{0,\Omega}\}, \quad (5.50)$$

where $C(\psi)$ is a new constant, depending on Ω, $\mathbf{b}$, b_0, ν, f, ϕ and ψ, but not on ϵ. Now we deal with the boundary integrals on the left hand side of (5.49). To begin with, we recall that the only nontrivial contribution is along Γ^{in}, because of the assumption on the support of ψ. Next, denote by $\mathbf{t}_1$ (respectively, $\mathbf{t}_2$) the unit vector tangent to Γ^{in}, oriented in the usual way with respect to $\mathbf{n}_1$ (respectively, $\mathbf{n}_2$): note that $\mathbf{t}_1 = -\mathbf{t}_2$). Thanks to Proposition 5.9, the gradient of z_ϵ on Γ^{in} may be decomposed along the directions $\mathbf{n}_i$ and $\mathbf{t}_i$, that is

$$\nabla w_\epsilon = \frac{\partial w_\epsilon}{\partial \mathbf{n}_2}\mathbf{n}_2 + \frac{\partial w_\epsilon}{\partial \mathbf{t}_2}\mathbf{t}_2, \quad \nabla u_\epsilon = \frac{\partial u_\epsilon}{\partial \mathbf{n}_1}\mathbf{n}_1 + \frac{\partial u_\epsilon}{\partial \mathbf{t}_1}\mathbf{t}_1 \quad \text{on } \Gamma^{in}.$$

Moreover, the transmission condition (5.14) gives

$$\frac{\partial w_\epsilon}{\partial \mathbf{n}_2} = -\frac{\epsilon}{\nu_\epsilon}\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} \qquad \text{on } \Gamma^{in}.$$

Therefore, the boundary integrals on the left hand side of (5.49) become

$$-\epsilon\int_{\partial\Omega_1}\psi\frac{\partial u_\epsilon}{\partial \mathbf{n}_1}\mathbf{b}\cdot\nabla u_\epsilon\, ds + \frac{\epsilon}{2}\int_{\partial\Omega_1}\psi\mathbf{b}\cdot\mathbf{n}_1|\nabla u_\epsilon|^2 ds -$$
$$-\int_{\partial\Omega_2}\psi\nu_\epsilon\frac{\partial w_\epsilon}{\partial \mathbf{n}_2}\mathbf{b}\cdot\nabla w_\epsilon\, ds + \frac{1}{2}\int_{\partial\Omega_2}\psi\nu_\epsilon\mathbf{b}\cdot\mathbf{n}_2|\nabla w_\epsilon|^2 ds =$$
$$= -\frac{1}{2}\left[\int_{\Gamma^{in}}(\nu_\epsilon - \epsilon)\psi\mathbf{b}\cdot\mathbf{n}_1\left|\frac{\partial u_\epsilon}{\partial \mathbf{t}_1}\right|^2 ds + \int_{\Gamma^{in}}\epsilon\left(1 - \frac{\epsilon}{\nu_\epsilon}\right)\psi\mathbf{b}\cdot\mathbf{n}_1\left|\frac{\partial u_\epsilon}{\partial \mathbf{n}_1}\right|^2 ds\right].$$

Each of the two integrals between square brackets is nonpositive, for ϵ small, because $\mathbf{b}\cdot\mathbf{n}_1 < 0$ on $\Gamma_1^{in} \cup \Gamma^{in}$. Therefore, the contribution of the boundary integrals on the left hand side of (5.49) is nonnegative and (5.42) follows easily by (5.50).□

REMARK 5.12 We note that the local regularity result of Proposition 5.9 has been used *only* to apply Green's formula, to give the boundary integrals in (5.49) a meaning and to decompose the gradient along the normal and the tangent directions. In particular, we need not the $\mathbf{H}^2$ local estimates be uniform in ϵ (which cannot be expected).□

As a consequence of Lemma 5.11, we get a convergence result.

PROPOSITION 5.13 *Let Ω' be any open subset of Ω_1, such that $\overline{\Omega'} \cap [\overline{\Gamma_1} \cup \overline{\Gamma^{out}}] = \emptyset$. Possibly taking a subsequence,* $\mathbf{b}u_\epsilon$ *converges to* $\mathbf{b}u$ *weakly in $\mathbf{L}^2_{\text{div}}(\Omega')$, as $\epsilon \to 0$ (u is the function found in Corollary 5.6).*

Proof. By (5.17) and (5.42), $\mathbf{b}u_\epsilon$ is bounded in $\mathbf{L}^2_{\text{div}}(\Omega')$, uniformly in ϵ. Therefore, a subsequence must converge weakly in $\mathbf{L}^2_{\text{div}}(\Omega')$. Possibly taking a further subsequence, by (5.24), the limit must be $\mathbf{b}u$.□

We are in a position to find the remaining interface condition fulfilled by the limit functions u and w.

PROPOSITION 5.14 *The functions* u, w *defined in Corollary* 5.6 *satisfy*

$$-\mathbf{b}\cdot\mathbf{n}_1 u = \mathbf{b}\cdot\mathbf{n}_2 w \quad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma^{in})\right]'. \tag{5.51}$$

Proof. Let Γ' be any (relatively) open subset of Γ^{in}, with $\overline{\Gamma'} \subset \Gamma^{in}$. Let Ω' be the intersection of Ω_1 with a small neighborhood of Γ'. By Proposition 5.13, $\mathbf{b}u_\epsilon$ converges to $\mathbf{b}u$ weakly in $\mathbf{L}^2_{\text{div}}(\Omega')$, whence $\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon$ converges to $\mathbf{b}\cdot\mathbf{n}_1 u$ weakly in $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega')$. Choosing test functions in $\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma')$, it follows that the convergence is weak in $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma')\right]'$. Recalling (3.15) and (5.15), we get that

$$-\mathbf{b}\cdot\mathbf{n}_1 u = \mathbf{b}\cdot\mathbf{n}_2 w \quad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\Gamma')\right]'.$$

Since Γ' is arbitrary, this gives (5.51). Again, Lemma 3.1, part (i), allows us to read condition (5.51) as $u = w$ a.e. on Γ^{in}, which is the sense stated in Theorem 3.2.□

The proof of the existence part of Theorem 3.2 is complete. Clearly, the very procedure we have followed gives (P) as a limit of globally elliptic problems, as stated in the existence theorem. The uniqueness part will be proved by means of a different argument in section 6.1.3.

6. The iterative procedure of section 4: the proofs. In this section we prove the results stated in section 4.

6.1. The differential case. We begin by detailing the differential problem first, distinguishing between the hyperbolic and the elliptic subproblems.

6.1.1. Proofs of the results of section 4.1.1. Our goal is to prove Theorem 4.1. The strategy of approaching (P_H) by regularized problems has proven effective also in this case, yielding the a priori estimate (4.7). As a by-product, we will also get a direct proof of existence and uniqueness of a solution to (P_H).

Let $\epsilon > 0$ be a small parameter and choose sequences of functions $\{\lambda_\epsilon\}$, $\{b_{0\epsilon}\}$ satisfying

$$\lambda_\epsilon \in \mathbf{C}^\infty(\partial\Omega_1), \quad \lambda_{\epsilon|\partial\Omega_1^{in}} \to \lambda_{|\partial\Omega_1^{in}} \text{ in } \mathbf{L}^2_{\mathbf{b}}(\partial\Omega_1^{in}), \tag{6.1}$$

$$b_{0\epsilon} \in \mathbf{C}^\infty(\Omega_1), \quad b_{0\epsilon} \overset{*}{\rightharpoonup} b_0 \text{ in } \mathbf{L}^\infty(\Omega_1). \tag{6.2}$$

As in the preceding section, we may assume that (5.3) holds for all values of ϵ which will appear in the sequel.

Consider the variational problem

$(P_{H\epsilon})$: to find $u_\epsilon \in \mathbf{H}^1(\Omega_1)$ such that, for all $v \in \mathbf{H}^1(\Omega_1)$,

$$\int_{\Omega_1} (\epsilon\nabla u_\epsilon - \mathbf{b}u_\epsilon)\cdot\nabla(u_\epsilon - v)\, dx + \int_{\Omega_1} b_{0\epsilon}u_\epsilon(u_\epsilon - v)\, dx + \int_{\partial\Omega_1^{out}} \mathbf{b}\cdot\mathbf{n}_1 u_\epsilon(u_\epsilon - v)\, ds =$$
$$= \int_{\Omega_1} f(u_\epsilon - v)\, dx - \int_{\partial\Omega_1^{in}} \mathbf{b}\cdot\mathbf{n}_1\lambda_\epsilon(u_\epsilon - v)\, ds. \quad (6.3)$$

Problem $(P_{H\epsilon})$ is an elliptic regularization to the hyperbolic problem (P_H), analogous to the one introduced in the previous section for the coupled problem (P). Therefore, the techniques developed in section 5 still apply to the present case and give a number of properties, which we summarize as follows.

PROPOSITION 6.1 *For all $\epsilon > 0$, problem $(P_{H\epsilon})$ has a unique solution u_ϵ, which satisfies:*

$$-\epsilon\Delta u_\epsilon + \mathrm{div}\,(\mathbf{b}u_\epsilon) + b_{0\epsilon}u_\epsilon = f \quad \text{in } \mathcal{D}'(\Omega_1), \quad (6.4)$$
$$-\epsilon\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} + \mathbf{b}\cdot\mathbf{n}_1 u_\epsilon = \mathbf{b}\cdot\mathbf{n}_1\lambda_\epsilon \quad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\partial\Omega_1^{in})\right]', \quad (6.5)$$
$$\frac{\partial u_\epsilon}{\partial \mathbf{n}_1} = 0 \quad \text{in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\partial\Omega_1^{out})\right]'; \quad (6.6)$$
$$\epsilon\nabla u_\epsilon \to 0 \text{ strongly in } \mathbf{L}^2(\Omega_1), \quad \text{as } \epsilon \to 0. \quad (6.7)$$

Moreover, there is a function $u \in \mathbf{L}^2(\Omega_1)$ such that (possibly taking a subsequence)

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } \mathbf{L}^2(\Omega_1), \quad (6.8)$$
$$\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon \rightharpoonup \mathbf{b}\cdot\mathbf{n}_1 u \quad \text{weakly in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\partial\Omega_1^{in})\right]', \quad (6.9)$$
$$\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon \rightharpoonup \mathbf{b}\cdot\mathbf{n}_1 u \quad \text{weakly in } \left[\mathbf{H}_{00}^{\frac{1}{2}}(\partial\Omega_1^{out})\right]'. \quad (6.10)$$

Finally, the limit function satisfies (4.4) *and* (4.5).

Proof. The proof of (6.4)-(6.9) is obtained by repeating the corresponding arguments of section 5. We just note that the boundary integrals of (5.49) (in the proof of Lemma 5.11) do not contain any interface contribution: now the whole boundary $\partial\Omega_1$ behaves like Γ_1 did in the preceding case. To show (6.10), let us repeat the proof of Lemma 5.11, this time choosing a nonnegative, smooth ψ vanishing in a neighborhood of $\partial\Omega_1^{in}$. All steps in the proof of the lemma can be followed again, up to the treatment of the boundary integrals in (5.49). Now, we may exploit (6.6) to see that the two integrals on $\partial\Omega_1$ (presently, there is no contribution from $\partial\Omega_2$) reduce to

$$\frac{\epsilon}{2}\int_{\partial\Omega_1^{out}} \psi\mathbf{b}\cdot\mathbf{n}_1(\frac{\partial u_\epsilon}{\partial \mathbf{t}_1})^2 ds.$$

This is a nonnegative quantity added to the left hand side of (5.49), hence it can be dropped in the subsequent computation. As a consequence, we get that Lemma 5.11 still holds with this ψ, hence $\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon \rightharpoonup \mathbf{b}\cdot\mathbf{n}_1 u$ in $\left[\mathbf{H}_{00}^{\frac{1}{2}}(\partial\Omega_1^{out})\right]'$. This proves (6.10).□

REMARK 6.2 More precisely, the strategy adopted in section 5 permits to obtain (6.10) along $\partial\Omega_1^{out}$, except at singular points. We do not stick to this matter, because we will exploit this convergence along Γ^{out}, which is regular.□

In addition to the previous properties, the following estimate is fulfilled.

PROPOSITION 6.3 *For all $\epsilon > 0$ and all $\delta > 0$,*

$$\epsilon|||\nabla u_\epsilon|||_{0,\Omega_1}^2 + (\frac{\beta_0}{2} - \delta)||u_\epsilon||_{0,\Omega_1}^2 + \frac{1}{2}|u_\epsilon|_{\partial\Omega_1^{out}}^2 \leq \frac{1}{4\delta}||f||_{0,\Omega_1}^2 + \frac{1}{2}|\lambda_\epsilon|_{\partial\Omega_1^{in}}^2. \quad (6.11)$$

Moreover, if $f = 0$ then (6.11) holds with $\delta = 0$.

Proof. Choose $v = 0$ in (6.3): Green's formula (3.7) gives

$$\epsilon\int_{\Omega_1}|\nabla u_\epsilon|^2 dx + \int_{\Omega_1}(\frac{1}{2}\operatorname{div}\mathbf{b} + b_{0\epsilon}){u_\epsilon}^2 dx + \frac{1}{2}\int_{\partial\Omega_1^{out}}\mathbf{b}\cdot\mathbf{n}_1{u_\epsilon}^2 ds -$$
$$-\frac{1}{2}\int_{\partial\Omega_1^{in}}\mathbf{b}\cdot\mathbf{n}_1{u_\epsilon}^2 ds = \int_{\Omega_1} f u_\epsilon\, dx - \int_{\partial\Omega_1^{in}}\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon\lambda_\epsilon\, ds.$$

Owing to (5.3), we get (6.11) easily. Note that the small parameter δ comes into play just in order to treat the first term on the right hand side of the previous equality. Therefore, one can forget it whenever $f = 0$.□

Proof of Theorem 4.1. For all $\sigma > 0$, set

$$(\partial\Omega_1)_\sigma = \{x \in \partial\Omega_1^{out} : (\mathbf{b}\cdot\mathbf{n}_1)(x) > \sigma\}. \quad (6.12)$$

Obviously, $\partial\Omega_1^{out} = \cup_{\sigma>0}(\partial\Omega_1)_\sigma$; moreover, (6.11) and (6.1) imply that

$$\sigma\int_{(\partial\Omega_1)_\sigma}{u_\epsilon}^2 ds \leq C,$$

for all $\sigma > 0$, with a constant C independent of ϵ and σ. Therefore, for all $\sigma > 0$ there is $g_\sigma \in \mathbf{L}^2((\partial\Omega_1)_\sigma)$ such that $u_\epsilon \rightharpoonup g_\sigma$ weakly in $\mathbf{L}^2((\partial\Omega_1)_\sigma)$, as $\epsilon \to 0$ (possibly taking a subsequence). Thus,

$$\mathbf{b}\cdot\mathbf{n}_1 u_\epsilon \rightharpoonup \mathbf{b}\cdot\mathbf{n}_1 g_\sigma \quad \text{weakly in } \mathbf{L}^2((\partial\Omega_1)_\sigma) : \quad (6.13)$$

by (6.10), g_σ does not depend on the above subsequence and

$$g_\sigma = u \text{ on } (\partial\Omega_1)_\sigma, \quad \text{whence } u \in \mathbf{L}^2((\partial\Omega_1)_\sigma), \quad (6.14)$$

for all $\sigma > 0$. By (6.1), (6.7), (6.13), (6.14) and by the lower semicontinuity of $|\cdot|$, the limit of (6.11) as $\epsilon \to 0$ gives

$$(\frac{\beta_0}{2} - \delta)||u||_{0,\Omega_1}^2 + \frac{1}{2}|u|_{(\partial\Omega_1)_\sigma}^2 \leq \frac{1}{4\delta}||f||_{0,\Omega_1}^2 + \frac{1}{2}|\lambda|_{\partial\Omega_1^{in}}^2. \quad (6.15)$$

The family of functions $\{G_\sigma\}$ defined by

$$G_\sigma = \begin{cases} \mathbf{b}\cdot\mathbf{n}_1 u^2, & \text{on } (\partial\Omega_1)_\sigma, \\ 0, & \text{on } \partial\Omega_1^{out}\setminus(\partial\Omega_1)_\sigma \end{cases}$$

is such that: $G_\sigma \geq 0$ on $\partial\Omega_1^{out}$, for all $\sigma > 0$; $G_{\sigma_1} \geq G_{\sigma_2}$, if $\sigma_1 \leq \sigma_2$; G_σ is integrable on $\partial\Omega_1^{out}$, this integral equals $|u|^2_{(\partial\Omega_1)_\sigma}$, hence it is uniformly bounded with σ, due to (6.15). Thus, Beppo Levi's theorem on monotone convergence applies and gives the a.e. pointwise convergence of G_σ to some $G_0 \in \mathbf{L}^1(\partial\Omega_1^{out})$, along with the convergence of the corresponding integrals. It follows that $\mathbf{b}\cdot\mathbf{n}_1 u^2 \in \mathbf{L}^1(\partial\Omega_1^{out})$ and (6.15) yields (4.7).

It remains to discuss uniqueness: by linearity, it is enough to do this in the homogeneous case $f = \lambda = 0$. With this assumption, (4.7) gives immediately that $u = 0$ a.e. in Ω_1 and a.e. on $\partial\Omega_1^{out}$ (whence uniqueness), *but only for the solution obtained as limit of solutions to the regularized problems* $(P_{H\epsilon})$. Next Proposition shows that uniqueness holds independently of the regularization. Introduce the space

$$\mathbf{W} = \{v \in \mathbf{L}^2(\Omega_1) : \mathbf{b}\cdot\nabla v \in \mathbf{L}^2(\Omega_1),\ v = 0 \text{ on } \partial\Omega_1^{in}\}, \tag{6.16}$$

endowed with the norm

$$\left[\|v\|^2_{0,\Omega_1} + \|\mathbf{b}\cdot\nabla v\|^2_{0,\Omega_1}\right]^{\frac{1}{2}}, \qquad v \in \mathbf{W}.$$

The vanishing of the elements of $\mathbf{W}$ on $\partial\Omega_1^{in}$ is intended in the following sense: $\mathbf{b}\cdot\mathbf{n}_1 v = 0$ in $\left[\mathbf{H}^{\frac{1}{2}}_{00}(\partial\Omega_1^{in})\right]'$ (which is meaningful, since $\mathbf{b}v$ belongs to $\mathbf{L}^2_{\mathrm{div}}(\Omega_1)$), or even a.e. on $\partial\Omega_1^{in}$, due to Lemma 3.1, part (i).

PROPOSITION 6.4 *Let* $u \in \mathbf{W}$ *be such that*

$$\mathbf{b}\cdot\nabla u + (b_0 + \mathrm{div}\,\mathbf{b})u = 0 \qquad \text{a.e. in } \Omega_1. \tag{6.17}$$

Then, $u = 0$ *a.e. in* Ω_1 *and a.e. on* $\partial\Omega_1^{out}$. *In particular, problem* (P_H) *with homogeneous data* $f = \lambda = 0$ *has only the trivial solution, whence the nonhomogeneous problem has a unique solution.*

Proof. It is easy to check that the space $\mathbf{W}$ is complete. Moreover, the family $Z = \{v \in \mathbf{C}^1(\overline{\Omega_1}) : v_{|\partial\Omega_1^{in}} = 0\}$ is dense in $\mathbf{W}$. This can be shown by means of the same technique used to prove the density of the smooth functions in Sobolev spaces (see [8]). As a consequence, we claim that Green's formula (3.7) holds for all $v \in \mathbf{W}$, namely

$$\frac{1}{2}\int_{\partial\Omega_1^{out}} \mathbf{b}\cdot\mathbf{n}v^2 ds = \int_{\Omega_1} v\mathbf{b}\cdot\nabla v\, dx + \frac{1}{2}\int_{\Omega_1} v^2\, \mathrm{div}\,\mathbf{b}\, dx. \tag{6.18}$$

Indeed, let $\{v_n\}$ be a sequence in Z, converging to v in $\mathbf{W}$: in particular, $\mathbf{b}v_n \to \mathbf{b}v$ in $\mathbf{L}^2_{\mathrm{div}}(\Omega_1)$, whence

$$\mathbf{b}\cdot\mathbf{n}v_n \to \mathbf{b}\cdot\mathbf{n}v \quad \text{in } \left[\mathbf{H}^{\frac{1}{2}}_{00}(\partial\Omega_1^{out})\right]'. \tag{6.19}$$

Now, (3.7) gives

$$\frac{1}{2}\int_{\partial\Omega_1^{out}} \mathbf{b}\cdot\mathbf{n}v_n^2 ds = \int_{\Omega_1} v_n\mathbf{b}\cdot\nabla v_n\, dx + \frac{1}{2}\int_{\Omega_1} v_n^2\, \mathrm{div}\,\mathbf{b}\, dx, \tag{6.20}$$

for all n. The right hand side of this equality converges to the right hand side of (6.18), because $v_n \to v$ in $\mathbf{W}$. Therefore, the left hand side of (6.20) has a limit, which turns out to be equal to the left hand side of (6.18) (this can be shown by the same argument used at the beginning of the proof of Theorem 4.1, recalling also (6.19)). Thus, (6.18) holds. Now, take the $\mathbf{L}^2(\Omega_1)$ scalar product of (6.17) by u and apply (6.18): it follows

$$\frac{1}{2}\int_{\partial\Omega_1^{out}} \mathbf{b}\cdot\mathbf{n}u^2 ds + \int_{\Omega_1} u^2(\frac{1}{2}\operatorname{div}\mathbf{b} + b_0)\, dx = 0.$$

Hence, (1.2) gives $u = 0$ a.e. in Ω_1 and a.e. on $\partial\Omega_1^{out}$. The consequent application to problem (P_H) is trivial, because (6.17) is nothing else than (4.4) when $f = 0$.□

REMARK 6.5 Proposition 6.4 does not provide any information on the behavior of the solution to the homogeneous problem along $\partial\Omega_1^0$, where $\mathbf{b}\cdot\mathbf{n}_1 = 0$. Clearly, this lack of information is not relevant, for "regular" solutions.□

By now, the proof of Theorem 4.1 is complete.

6.1.2. Proofs of the results of section 4.1.2. Proof of Theorem 4.3. By Green's formula (3.7), the variational equality (4.13) gives

$$\begin{aligned}\int_{\Omega_2} \nu|\nabla w|^2 dx + \int_{\Omega_2} (\frac{1}{2}\operatorname{div}\mathbf{b} + b_0)w^2 dx - \frac{1}{2}\int_{\Gamma^{out}\cup\Gamma_2^D} \mathbf{b}\cdot\mathbf{n}_2 w^2 ds+ \\ + \frac{1}{2}\int_{\Gamma^{in}\cup\Gamma_2^{Ne}} \mathbf{b}\cdot\mathbf{n}_2 w^2 ds = \int_{\Omega_2} \nu\nabla w\cdot\nabla v\, dx - \int_{\Omega_2} (\mathbf{b}w)\cdot\nabla v\, dx+ \\ + \int_{\Omega_2} b_0 wv\, dx + \int_{\Gamma^{in}\cup\Gamma_2^{Ne}} \mathbf{b}\cdot\mathbf{n}_2 wv\, ds + \int_{\Omega_2} f(w-v)\, dx - \int_{\Gamma^{out}} \mathbf{b}\cdot\mathbf{n}_2\mu(w-v) ds.\end{aligned}$$

Recalling (1.2), (1.3) and noticing that $\mathbf{b}\cdot\mathbf{n}_2$ is negative on Γ^{out} and positive on Γ^{in}, for all $\delta > 0$ we get

$$\begin{aligned}\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2 + \beta_0\|w\|_{0,\Omega_2}^2 + (\frac{1}{2}-\delta)|w|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2 \le \delta\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2 + \frac{1}{4\delta}\|v\|_{1,\Omega_2}^2+ \\ + \{\delta\|w\|_{0,\Omega_2}^2 + \frac{1}{4\delta}\|v\|_{1,\Omega_2}^2\}\{\|\mathbf{b}\|_{1,\infty,\Omega_2} + \|b_0\|_{\infty,\Omega_2}\}+ \\ + \delta\|w\|_{0,\Omega_2}^2 + \|v\|_{0,\Omega_2}^2 + (1+\frac{1}{4\delta})\|f\|_{0,\Omega_2}^2 + \frac{1}{2}|\phi|_{\Gamma_2^D}^2+ \\ + (\frac{1}{2}+\delta)|\mu|_{\Gamma^{out}}^2 + \frac{1}{4\delta}|v|_{\partial\Omega_2\setminus\Gamma_2^D}^2. \qquad (6.21)\end{aligned}$$

Now, we may choose a test function v such that $\|v\|_{1,\Omega_2} \le C\|\phi\|_{\frac{1}{2},\Gamma_2^D}$, hence (6.21) gives

$$\begin{aligned}\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2 + \beta_0\|w\|_{0,\Omega_2}^2 + (\frac{1}{2}-\delta)|w|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2 \le \delta\|\sqrt{\nu}|\nabla w|\|_{0,\Omega_2}^2+ \\ + \delta\|w\|_{0,\Omega_2}^2\{\|\mathbf{b}\|_{1,\infty,\Omega_2} + \|b_0\|_{\infty,\Omega_2}\} + \delta\|w\|_{0,\Omega_2}^2 + (1+\frac{1}{4\delta})\|f\|_{0,\Omega_2}^2+ \\ + \frac{1}{2}|\phi|_{\Gamma_2^D}^2 + (\frac{1}{2}+\delta)|\mu|_{\Gamma^{out}}^2 + C(\Omega_2,\mathbf{b},b_0;\delta)\|\phi\|_{\frac{1}{2},\Gamma_2^D}^2,\end{aligned}$$

for a suitable positive constant $C(\Omega_2, \mathbf{b}, b_0; \delta)$ depending only on its argument. This proves (4.19). Finally, if $f = \phi = 0$, then we may choose $v = 0$ in (4.13): $\delta = 0$ is then allowed in (6.21) and (4.20) follows.□

6.1.3. Proofs of the results of section 4.1.3. Proof of Proposition 4.5. Let ψ be given in $\mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$.

(1) $\Rightarrow$ (2). If (u^ψ, w^ψ) solves problem (P), then it is $u^\psi = w^\psi$ a.e. on Γ^{in}. By (4.23), it is $u^\psi = \psi$ a.e. in Γ^{in}, whence $w^\psi = \psi$ a.e. on Γ^{in}. Therefore, for all $\theta \in \mathbb{R}$, $T_\theta\psi = \theta(w^\psi)_{|\Gamma^{in}} + (1-\theta)\psi = \psi$, that is, ψ is a fixed point for T_θ. In particular, it follows that the fixed point ψ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma^{in})$ and that it does not depend on θ.

(2) $\Rightarrow$ (1). Let ψ be a fixed point for T_θ, for some $\theta \in \mathbb{R}$. Then, $\psi = T_\theta\psi = \theta(w^\psi)_{|\Gamma^{in}} + (1-\theta)\psi$, that is, $w^\psi = \psi$ a.e. on Γ^{in}. Thus, $u^\psi = \psi = w^\psi$ a.e. in Γ^{in} and the pair (u^ψ, w^ψ) solves the coupled problem (P).□

Proof of Lemma 4.6. The proof is immediate, by a simple combination of the estimates (4.8) and (4.20) applied to (4.23), (4.24).□

Proof of Lemma 4.7. Our aim is to show that the map T_1 is a contraction. To this end, let ψ_1, ψ_2 be given in $\mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$. By linearity, $T_1\psi_1 - T_1\psi_2 = T_1(\psi_1 - \psi_2) = (w_0^{\psi_1-\psi_2})_{|\Gamma^{in}}$ (notation of Lemma 4.6), since we have to solve the two problems (4.23), (4.24) with $f = \phi = 0$ and $\psi = \psi_1 - \psi_2$. Therefore, Lemma 4.6 applies and (4.26) holds. Now, if γ denotes the trace constant in Ω_2 (that is, $\|v\|_{\frac{1}{2},\partial\Omega_2} \leq \gamma\|v\|_{1,\Omega_2}$, for all $v \in \mathbf{H}^1(\Omega_2)$), we have that

$$\|\sqrt{\nu}|\nabla w_0^\psi|\|^2_{0,\Omega_2} + \beta_0\|w_0^\psi\|^2_{0,\Omega_2} \geq \min(\nu_0,\beta_0)\gamma^{-2}\|w_0^\psi\|^2_{\frac{1}{2},\partial\Omega_2} \geq C\frac{1}{2}|w_0^\psi|^2_{\Gamma^{in}}, \quad (6.22)$$

where

$$C = \frac{2\gamma^{-2}\min(\nu_0,\beta_0)}{\max_{\Gamma^{in}}|\mathbf{b}\cdot\mathbf{n}_1|} > 0.$$

Thus, a combination of (4.26) and (6.22) gives

$$|w_0^\psi|^2_{\Gamma^{in}} \leq \frac{1}{C+1}|\psi|^2_{\Gamma^{in}},$$

namely

$$|T_1\psi_1 - T_1\psi_2|^2_{\Gamma^{in}} \leq \frac{1}{C+1}|\psi_1 - \psi_2|^2_{\Gamma^{in}}. \quad (6.23)$$

Therefore, T_1 is a contraction in $\mathbf{L}^2_{\mathbf{b}}(\Gamma^{in})$, hence it admits a unique fixed point.□

REMARK 6.6 Since fixed points for T_θ do not depend on θ, Lemma 4.7 provides the existence and uniqueness of the fixed point for T_θ, for all $\theta \in \mathbb{R}$. Actually, a convenient choice of θ might diminish the value of the contraction constant appearing in (6.23), thus improving the rate of convergence in numerical approximations.□

Proof of Lemma 4.8. By the assumption on ψ^0, Corollary 4.2 implies that (4.9) holds for all n. After this remark, the proof is immediate, by a simple combination of the estimates (4.11) and (4.22).□

Proof of Lemma 4.9. Let ψ be the fixed point of T_1. For all n, set

$$d^n = \psi^n - \psi, \quad U^n = u^n - u^\psi, \quad W^n = w^n - w^\psi, \quad (6.24)$$

where u^n, w^n solve (4.1), (4.2), while u^ψ, w^ψ solve (4.23), (4.24). It is immediate to see that U^n, W^n solve the two boundary value problems (4.1), (4.2) with $f = \phi = 0$

and ψ^n replaced by d^n. Therefore, we can read (4.27) with u_0^n, w_0^n and ψ^n replaced by U^n, W^n and d^n, respectively: it follows

$$\|\sqrt{\nu}|\nabla W^n|\|_{0,\Omega_2}^2 + \frac{\beta_0}{2}\|U^n\|_{0,\Omega_1}^2 + \beta_0\|W^n\|_{0,\Omega_2}^2 + \frac{1}{2}|W^n|_{\Gamma^{in}\cup\Gamma_2^{Ne}}^2 \leq \frac{1}{2}|d^n|_{\Gamma^{in}}^2. \quad (6.25)$$

This inequality and a trick analogous to (6.22) give

$$|W^n|_{\Gamma^{in}}^2 \leq K^2 |d^n|_{\Gamma^{in}}^2, \quad (6.26)$$

for a suitable constant $K < 1$. Since $(w^\psi)_{|\Gamma^{in}} = T_1\psi = \psi$, by (6.24) and (4.3) it follows that $d^{n+1} = \theta W^n + (1-\theta)d^n$, whence

$$|d^{n+1}|_{\Gamma^{in}} \leq |\theta||W^n|_{\Gamma^{in}} + |1-\theta||d^n|_{\Gamma^{in}} \overset{(6.26)}{\leq} (|\theta|K + |1-\theta|)|d^n|_{\Gamma^{in}}. \quad (6.27)$$

Therefore, if θ belongs to the interval $]0, \frac{2}{1+K}[$ (which includes the value 1), then the constant $|\theta|K + |1-\theta|$ is smaller than 1 and (6.27) entails that $|d^n|_{\Gamma^{in}} \to 0$, as $n \to +\infty$. It follows that $\psi^n \to \psi$ in $\mathbf{L}_\mathrm{b}^2(\Gamma^{in})$.□

Proof of Theorem 4.10. Again, let ψ be the fixed point of T_1 and denote by u^ψ, w^ψ the corresponding solutions to (4.23), (4.24). By Lemma 4.9, the sequence d^n (defined in (6.24)) converges to 0 in $\mathbf{L}_\mathrm{b}^2(\Gamma^{in})$. By (6.25), the sequences of functions U^n, W^n (still defined in (6.24)) satisfy $U^n \to 0$ in $\mathbf{L}^2(\Omega_1)$ and $W^n \to 0$ in $\mathbf{H}^1(\Omega_2)$. Therefore, $u^n \to u^\psi$ in $\mathbf{L}^2(\Omega_1)$ and $w^n \to w^\psi$ in $\mathbf{H}^1(\Omega_2)$. By Proposition 4.5, we know that (u^ψ, w^ψ) is a solution to the coupled problem (P).

Finally, we discuss uniqueness. If ψ is the unique fixed point of T_1, the solutions u^ψ, w^ψ to (4.23), (4.24) are such that u^ψ is uniquely determined, in the sense of Proposition 6.4, and consequently w^ψ is unique as a solution to (4.24). By Proposition 4.5, (u^ψ, w^ψ) provides the unique solution (in the above sense) to the coupled problem (P).□

6.2. The discrete case. The analysis we are going to carry out in this section is concerned solely with the collocation method using the Legendre Gaussian points. In particular, it does not apply to the case in which the collocation points pertain to the Chebyshev-Lobatto formula.

LEMMA 6.7 *Within the domain Ω_i, $i = 1, 2$, the operator L_N^i defined by (2.1) satisfies the following property:*

$$(L_N^i v, v)_{N,\Omega_i} = \frac{1}{2}(\operatorname{div}(I_N^i \mathrm{b}), v^2)_{N,\Omega_i} + \frac{1}{2}(\mathrm{b}\cdot\mathbf{n}_i, v^2)_{N,\partial\Omega_i} \quad (6.28)$$

for all polynomial v *of degree* N *(the discrete inner products are defined in (4.42)).*

Proof. Since the one-dimensional Gauss-Lobatto quadrature formula is exact when the integrand is a one-dimensional algebraic polynomial of degree less than or equal to $2N-1$ (see, e.g., [10]), it is easy to show that the following discrete rule of integration by parts holds:

$$(\operatorname{div} \mathrm{g}, v)_{N,\Omega_i} = -(\mathrm{g}, \nabla v)_{N,\Omega_i} + (\mathrm{g}\cdot\mathbf{n}_i, v)_{N,\partial\Omega_i} \quad (6.29)$$

for all polynomials g, v such that $g_i v \in P_{2N}$, $i = 1, 2$. From (6.29) it follows:

$$(\operatorname{div}(I_N^i \mathbf{b}v), v)_{N,\Omega_i} = -(\mathbf{b}v, \nabla v)_{N,\Omega_i} + (\mathbf{b}\cdot\mathbf{n}_i, v^2)_{N,\partial\Omega_i}. \tag{6.30}$$

The inequality (6.28) follows from (6.30) and from the definition (2.1).□

Let us now define the bilinear form $c_N^i : P_N \times P_N \to \mathbf{R}$,

$$c_N^i(u,v) \underset{def}{=} (L_N^i u, v)_{N,\Omega_i} + ((\mathbf{b}\cdot\mathbf{n}_i)^- u, v)_{N,\partial\Omega_i}, \quad i = 1, 2: \tag{6.31}$$

here and in the following we denote by $u^{\pm}$ the positive and negative parts of the function u, respectively. From (6.28) we easily deduce that

$$c_N^i(v,v) = \frac{1}{2}(\operatorname{div}(I_N^i \mathbf{b}), v^2)_{N,\Omega_i} + \frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_i)^+, v^2)_{N,\partial\Omega_i} + \frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_i)^-, v^2)_{N,\partial\Omega_i}, \quad i = 1, 2 \tag{6.32}$$

for all polynomial $v \in P_N$.

6.2.1. Proofs of the results of section 4.2.1.

LEMMA 6.8 *The hyperbolic problem* (4.39) *can be written equivalently as follows:*

$$\begin{aligned} &\text{find } u_N^n \in P_N \text{ such that for all } v \in P_N \\ &c_N^1(u_N^n, v) + (b_0 u_N^n, v)_{N,\Omega_1} = (f, v)_{N,\Omega_1} + ((\mathbf{b}\cdot\mathbf{n}_1)^- \phi, v)_{N,\Gamma_1^{in}} + \\ &\qquad + ((\mathbf{b}\cdot\mathbf{n}_1)^- \psi^n, v)_{N,\Gamma^{in}}. \end{aligned} \tag{6.33}$$

Proof. The finite dimensional problem (6.33) can be equivalently reformulated by restricting test functions to the $(N+1)^2$ Lagrangean functions $v^{(k)} \in P_N$ such that $v^{(k)}(P_j) = \delta_{kj}$ for all $P_j \in \Xi_N^1$. If P_k is an internal point and $v^{(k)}$ is the associated Lagrangean function, then it is $v^{(k)} \equiv 0$ on $\partial\Omega_1$. Hence (6.33) with $v = v^{(k)}$ gives (4.39)(i) immediately, owing to (6.31).
Next, write (6.33) with $v = v^{(j)}$, where $P_j \in \partial\Omega_1$. Using (4.39)(i), (4.42) as well as (6.31), we find

$$[L_N^1 u_N^n + b_0 u_N^n - f](P_j)\omega_{P_j}^{(1)} = [(\mathbf{b}\cdot\mathbf{n}_1)^-(\chi - u_N^n)](P_j)\theta_{P_j}^{(1)}, \tag{6.34}$$

where

$$\chi = \begin{cases} \phi & \text{at } (\Gamma_1^{in})_N, \\ \psi^n & \text{at } (\Gamma^{in})_N. \end{cases} \tag{6.35}$$

Therefore, (4.39)(ii) and (4.39)(iii) follow easily by (6.34).□

Proof of Lemma 4.12. We use the equivalent variational formulation (6.33). Taking $v = u_N^n$ and using (6.32) along with definitions (4.43) and (4.45), by (2.11) we obtain

$$\begin{aligned} &\beta_N \|u_N^n\|_{N,\Omega_1}^2 + \frac{1}{2}|u_N^n|_{N,\partial\Omega_1^{out}}^2 + \frac{1}{2}|u_N^n|_{N,\partial\Omega_1^{in}}^2 = \\ &\qquad (f, u_N^n)_{N,\Omega_1} + ((\mathbf{b}\cdot\mathbf{n}_1)^- \chi, u_N^n)_{N,\partial\Omega_1}. \end{aligned} \tag{6.36}$$

By Cauchy-Schwarz inequality we have

$$(f, u_N^n)_{N,\Omega_1} \leq \frac{1}{2\beta_N} \|f\|^2_{N,\Omega_1} + \frac{\beta_N}{2} \|u_N^n\|^2_{N,\Omega_1}, \tag{6.37}$$

$$((\mathbf{b}\cdot\mathbf{n}_1)^- \chi, u_N^n)_{N,\partial\Omega_1} \leq \frac{1}{2} |\phi|^2_{N,\Gamma_1^{in}} + \frac{1}{2} |\psi^n|^2_{N,\Gamma^{in}} + \frac{1}{2} |u_N^n|^2_{N,\partial\Omega_1^{in}}. \tag{6.38}$$

The inequality (4.47) follows by (6.36)-(6.38).□

6.2.2. Proofs of the results of section 4.2.2. We turn now to the elliptic problem (4.40). Let us define the following polynomial subset:

$$P_N^D \underset{def}{=} \{v \in P_N : \; v = 0 \text{ on } \Gamma_2^D\}. \tag{6.39}$$

This is precisely the set of algebraic polynomials of degree less than or equal to N that vanish on the portion of $\partial\Omega_2$ where a Dirichlet condition is prescribed. We start proving the following equivalence statement.

LEMMA 6.9 *Assume $\phi = 0$ on Γ_2^D. The collocation problem* (4.40) *is equivalent to the following discrete variational problem:*

$$\begin{aligned} &\text{find } w_N^n \in P_N^D \text{ such that for all } v \in P_N^D \\ &(\nu\nabla w_N^n, \nabla v)_{N,\Omega_2} + c_N^2(w_N^n, v) + (b_0 w_N^n, v)_{N,\Omega_2} = \\ &\qquad = (f, v)_{N,\Omega_2} + ((\mathbf{b}\cdot\mathbf{n}_1)^+ u_N^n, v)_{N,\Gamma}. \end{aligned} \tag{6.40}$$

Proof. We first note that the same argument used to get (6.29) gives the discrete integration by parts formula:

$$(\nu\nabla w_N^n, \nabla v)_{N,\Omega_2} = -(\operatorname{div}[I_N^2(\nu\nabla w_N^n)], v)_{N,\Omega_i} + (\nu\frac{\partial w_N^n}{\partial \mathbf{n}_2}, v)_{N,\partial\Omega_2}. \tag{6.41}$$

As we noticed in the proof of Lemma 6.8, problem (6.40) can be equivalently reformulated by letting the test functions v range within the space of Lagrangean functions $v^{(k)} \in P_N^D$ associated with the collocation points of $\Xi_N^2 \cap (\overline{\Omega_2}\backslash\Gamma_2^D)$.

(i) If $v^{(k)}$ is associated with a collocation point P_k of $(\Xi_N^2)^0$, then $v^{(k)}$ vanishes on $\partial\Omega_2$, hence all boundary termes involving $v^{(k)}$ can be dropped. Therefore, from (6.40), (6.41) and (6.31) it follows that the set of equations (4.40)(i) holds.

(ii) Let $v^{(k)}$ be now the Lagrangean function associated with a collocation point P_k of $(\Gamma_2^{Ne})_N$. Taking $v = v^{(k)}$ in (6.40) and using (6.41) and (6.31) we obtain

$$R_2(P_k)\omega_{P_k}^{(2)} + [(\mathbf{b}\cdot\mathbf{n}_2)^- w_N^n](P_k)\theta_{P_k}^{(2)} = ((\mathbf{b}\cdot\mathbf{n}_1)^+ u_N^n, v^{(k)})_{N,\Gamma} - [\nu\frac{\partial w_N^n}{\partial \mathbf{n}_2}](P_k)\theta_{P_k}^{(2)}. \tag{6.42}$$

The second term in the left hand side vanishes because of (1.3). Morover, the first term in the right hand side vanishes, since $v^{(k)}$ is zero on Γ. Therefore, we obtain (4.40)(iii).

(iii) Now, take $v = v^{(k)}$ in (6.40), with $v^{(k)}$ associated to a point P_k of $(\Gamma^{in})_N$. Relation (6.42) holds also in the present case, but now we have that

$$((\mathbf{b}\cdot\mathbf{n}_1)^+ u_N^n, v^{(k)})_{N,\Gamma} = [(\mathbf{b}\cdot\mathbf{n}_1)^+ u_N^n](P_k)\theta_{P_k}^{(2)}.$$

This term vanishes along with the first term in the right hand side, because $(\mathbf{b}\cdot\mathbf{n}_2)^- = (\mathbf{b}\cdot\mathbf{n}_1)^+ = 0$ on Γ^{in}. Thus, the set of relations (4.40)(v) holds.

(iv) Finally, let $v^{(k)}$ be the Lagrangean function associated with a collocation point P_k of $(\Gamma^{out})_N$. Taking $v = v^{(k)}$ in (6.40) and using (6.42) we obtain

$$R_2(P_k)\omega^{(2)}_{P_k} = [-\nu\frac{\partial w^n_N}{\partial \mathbf{n}_2} - (\mathbf{b}\cdot\mathbf{n}_2)^- w^n_N + (\mathbf{b}\cdot\mathbf{n}_1)^+ u^n_N](P_k)\theta^{(2)}_{P_k}.$$

Noticing that $(\mathbf{b}\cdot\mathbf{n}_2)^- = (\mathbf{b}\cdot\mathbf{n}_1)^+ = -\mathbf{b}\cdot\mathbf{n}_2$ on Γ^{out}, the set of relations (4.40)(iv) follows.□

We are in a position to prove the stability estimate (4.49).

Proof of Lemma 4.15. Take $v = w^n_N$ in (6.40) and use (2.11) and (6.32). It follows

$$(\nu\nabla w^n_N, \nabla w^n_N|||^2_{N,\Omega_2} + \beta_N\|w^n_N\|^2_{N,\Omega_2} + \frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_2)^+, (w^n_N)^2)_{N,\partial\Omega_2} +$$
$$+\frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_2)^-, (w^n_N)^2)_{N,\partial\Omega_2} = (f, w^n_N)^2_{N,\Omega_2} + ((\mathbf{b}\cdot\mathbf{n}_1)^+u^n_N, w^n_N)_{N,\Gamma}. \tag{6.43}$$

We notice that

$$((\mathbf{b}\cdot\mathbf{n}_2)^-, (w^n_N)^2)_{N,\partial\Omega_2} = ((\mathbf{b}\cdot\mathbf{n}_2)^-, (w^n_N)^2)_{N,\Gamma}, \tag{6.44}$$

since $w^n_N = 0$ on Γ^D_2 and $(\mathbf{b}\cdot\mathbf{n}_2)^- = 0$ on Γ^{Ne}_2. Moreover, using once again the identity $(\mathbf{b}\cdot\mathbf{n}_1)^+ = (\mathbf{b}\cdot\mathbf{n}_2)^-$ on Γ and the Cauchy-Schwarz inequality, it follows

$$((\mathbf{b}\cdot\mathbf{n}_1)^+u^n_N, w^n_N)_{N,\Gamma} \le \frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_2)^-, (u^n_N)^2)_{N,\Gamma} + \frac{1}{2}((\mathbf{b}\cdot\mathbf{n}_2)^-, (w^n_N)^2)_{N,\Gamma},$$
$$(f, w^n_N)_{N,\Omega_2} \le \frac{1}{2\beta_N}\|f\|^2_{N,\Omega_2} + \frac{\beta_N}{2}\|w^n_N\|^2_{N,\Omega_2}. \tag{6.45}$$

Owing to (6.44) and (6.45), we get (4.49) easily from (6.43).□

6.2.3. Proofs of the results of section 4.2.3. Owing to the previous stability inequalities we can deduce the uniqueness result for the collocation coupled problem (2.2)-(2.9), stated in Theorem 4.16.

PROPOSITION 6.10 *The collocation coupled problem* (2.2)-(2.9) *has a unique solution.*

Proof. It is enough to prove that the homogeneous problem (namely the one with $f = \phi = 0$) has the only trivial solution $u_N = w_N = 0$. Let us introduce the unknown function

$$\psi_N \underset{def}{=} (w_N)_{|\Gamma^{in}}. \tag{6.46}$$

As soon as ψ_N is available on Γ^{in}, we can solve the hyperbolic problem (2.2), (2.4), (2.7) first. Next we can solve the elliptic problem (2.3), (2.5), (2.6), (2.8), (2.9). We can therefore read the stability estimate (4.50) with ψ^n replaced by ψ_N, u^n_N by u_N and w^n_N by w_N, obtaining

$$2\|\sqrt{\nu}|\nabla w_N|\|^2_{N,\Omega_2} + \beta_N[\|u_N\|^2_{N,\Omega_1} + \|w_N\|^2_{N,\Omega_2}] + |w_N|^2_{N,\Gamma^{in}} \le |\psi_N|^2_{N,\Gamma^{in}}.$$

By (6.46), we may drop the two boundary terms in the previous inequality and conclude that $u_N = w_N = 0$.□

Now we turn to the proof of the convergence results of Theorem 4.16.

By simply taking the difference between equations (2.2)-(2.9) and the corresponding equations (4.39), (4.40), we see immediately that the following statement holds.

PROPOSITION 6.11 *The error functions defined in* (4.51) *satisfy the same equations* (4.39), (4.40), *with* u_N^n *and* w_N^n *replaced respectively by* e_1^n *and* e_2^n, $f = 0$, $\phi = 0$ *and*

$$\psi^n = \theta e_2^{n-1} + (1-\theta) e_1^{n-1}, \qquad \theta > 0. \tag{6.47}$$

□

In view of the above statement, we can write the inequality (4.50) for e_1^n and e_2^n, obtaining:

$$2\|\sqrt{\nu}|\nabla e_2^n|\|_{N,\Omega_2}^2 + \beta_N[\|e_1^n\|_{N,\Omega_1}^2 + \|e_2^n\|_{N,\Omega_2}^2] + |e_2^n|_{N,\Gamma^{in}}^2 \leq |\psi^n|_{N,\Gamma^{in}}^2. \tag{6.48}$$

Now, (1.2), (4.44), (4.46) and (6.22) give

$$2\|\sqrt{\nu}|\nabla e_2^n|\|_{N,\Omega_2}^2 + \beta_N\|e_2^n\|_{N,\Omega_2}^2 \geq K|e_2^n|_{N,\Gamma^{in}}^2, \tag{6.49}$$

where

$$K = \frac{\gamma^{-2}\min(\nu_0, \beta_N)}{C_1^2 \max_{\Gamma^{in}} |\mathbf{b}\cdot\mathbf{n}_1|} > 0.$$

Thus, a combination of (6.48) and (6.49) gives

$$|e_2^n|_{N,\Gamma^{in}}^2 \leq \frac{1}{K+1}|\psi^n|_{N,\Gamma^{in}}^2 : \tag{6.50}$$

this inequality expresses a contraction of the input error ψ^n on Γ^{in}. More precisely, let us define a discrete interface operator as follows (see (4.25))

$$T_N : P_N(\Gamma^{in}) \to P_N(\Gamma^{in}), \qquad \psi^n \mapsto T_N\psi^n \underset{def}{=} (e_2^n)_{|\Gamma^{in}}. \tag{6.51}$$

The inequality (6.50) reads equivalently in terms of this operator:

$$|T_N\psi^n|_{N,\Gamma^{in}}^2 \leq \frac{1}{K+1}|\psi^n|_{N,\Gamma^{in}}^2, \qquad \forall\psi^n \in P_N(\Gamma^{in}). \tag{6.52}$$

If $\theta = 1$ in (6.47), then it is $(e_2^n)_{|\Gamma^{in}} = \psi^{n+1}$, $\forall n$. Thus, (6.52) yields

$$|\psi^{n+1}|_{N,\Gamma^{in}}^2 \leq \frac{1}{K+1}|\psi^n|_{N,\Gamma^{in}}^2, \qquad \forall n \geq 2, \tag{6.53}$$

hence the operator T_N is a contraction and $\psi^n \to 0$ as $n \to +\infty$. As shown in the proof of Proposition 6.10, this (trivial) limit value at Γ^{in} allows us to reconstruct the whole solution to the coupled homogeneous collocation problem (2.2)-(2.9), which turns out to be identically zero. Therefore, we get e_1^n, $e_2^n \to 0$, as $n \to +\infty$. In view of the definition (4.51), we infer that the sequence (u_N^n, w_N^n) converges to the (unique) solution of the collocation problem (2.2)-(2.9), in the case $\theta = 1$.

If $0 < \theta \neq 1$, then we define

$$T_{N,\theta} \underset{def}{=} \theta T_N + (1-\theta)\mathbb{I}, \tag{6.54}$$

where $\mathbb{I}$ is the identity operator. From (6.47) we deduce that

$$\psi^{n+1} = T_{N,\theta}\psi^n, \qquad \forall n \geq 2.$$

Since $T_{N,1} = T_N$, for θ sufficiently close to 1 the operator $T_{N,\theta}$ is still a contraction (with a contraction constant possibly smaller than $(K+1)^{-1}$).□

REFERENCES

[1] Gastaldi, F. and Quarteroni, A. *On the Coupling of Hyperbolic and Parabolic Systems, Analytical and Numerical Approach*, to appear in Applied Numerical Math..

[2] Gastaldi, F., Quarteroni, A. and Sacchi Landriani, G. *Effective methods for the treatment of interfaces separating equations of different character*, in *Computers and Experiments in Fluid Flow*, G.M.Carlomagno and C.A.Brebbia ed.s, Computational Mechanics Publications, Springer-Verlag, Berlin, Heidelberg, New York (1989), 65-74.

[3] Gastaldi, F., Quarteroni, A. and Sacchi Landriani, G. *Coupling of two-dimensional hyperbolic and elliptic equations*, submitted to Proc. ICOSAHOM 1989, C. Canuto and A. Quarteroni ed.s, North Holland.

[4] Quarteroni, A. and Valli, A. *Domain decomposition for a generalized Stokes problem*, Proc. 2nd ECMI Conference, Glasgow, 1988, to appear.

[5] Quarteroni, A., Sacchi Landriani, G. and Valli, A. *Coupling of viscous and inviscid Stokes equations via a domain decomposition method for finite elements*, Preprint n. 287, Department of Mathematics, University of Trento (1989).

[6] Schlichting, H. *Boundary-layer theory*, McGraw-Hill, New York (1979).

[7] Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A. *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, Heidelberg, Berlin (1988).

[8] Adams, R. *Sobolev spaces*, Academic press, New York (1975).

[9] Lions, J.L. and Magenes, E. *Non-homogeneous boundary value problems and applications*, vol. I, Springer-Verlag, Berlin, Heidelberg, New York (1972).

[10] Davis, P.J. and Rabinowitz, P. *Methods of numerical integration*, 2^{nd} ed., Academic Press, London, New York (1984).