CHAPTER 2

On the Coupling of Two Dimensional Hyperbolic and Elliptic Equations: Analytical and Numerical Approach
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Abstract. The coupling of linear hyperbolic and elliptic equations in a two dimensional domain is considered. Physical motivations for this investigation can be found in Fluid Dynamics (viscous/inviscid interactions for compressible flows, heat transfer in incompressible flows and other applications). A major point in this analysis consists in finding correct conditions at the interface separating the hyperbolic and the elliptic regions in a rigorous way. Then an effective iterative procedure is proposed, which alternates the solution of the hyperbolic equation and of the elliptic one within the respective regions. The strategy is quite important in view of the numerical computation since it permits to use different solvers in the two regions. The numerical approximation based on spectral collocation methods is detailed. The convergence analysis of the above iterative algorithm is provided both for the differential problem and its numerical approximation.

1. Introduction. In this paper we deal with the coupling of a linear elliptic equation with a hyperbolic one in a two dimensional domain $\Omega$. The former is an Advection-Diffusion type equation (AD) set in a region $\Omega_2 \subset \Omega$, while the latter contains just the advection part (A) and is obtained from AD by dropping the diffusion term in the complementary region $\Omega_1 = \Omega \setminus \Omega_2$.

The one-dimensional version of this problem was studied in [1], where the coupling of parabolic and hyperbolic linear systems in one space variable is considered.

The problem at hand will be clearly stated in subsection 1.2 of this introduction along with the main results and a detailed outline of the paper.

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The main point of our investigation is concerned with the setting of correct transmission conditions across the interface $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$: these conditions provide the matching between the solution to the AD equation and that to the $A$ equation.

This type of problems arises from several physical applications which are modeled by a global AD equation in the whole $\Omega$. However, in these problems the diffusive term is relevant only in the subregion $\Omega_2$ (which clearly depends on the problem at hand), while it can be neglected in the rest of the domain $\Omega$, without affecting the solution in a sensible way. We will give a few examples of concrete situations having such a feature in the subsection 1.1 below.

In the present paper we show that the coupled problem mentioned at the beginning (endowed with the correct transmission conditions) is consistent with the full AD problem in $\Omega$. Furthermore, we propose a splitting method allowing the calculation of the solution to the coupled problem via separate calculations within $\Omega_1$ and $\Omega_2$, using the conditions at the interface in a convenient way. Eventually, we also provide a finite dimensional approximation of the problem at hand by the spectral collocation method.

Some of the results of this paper have been presented in [2] and [3]. We also refer to [4] and [5] for the coupling of Stokes-like problems.

1.1. Some physical examples. Fluid dynamics is among the fields that benefit largely from a coupling approach of the type studied here. As an example, consider viscous, compressible flows around rigid profiles (e.g. an aerofoil). Physical evidence suggests that viscosity effects are negligible apart from a small region close to the rigid body. This is one instance where the mathematical model of the problem may lead to the use of equations of different character (precisely, Euler and Navier-Stokes equations) in separate regions, just by dropping viscous terms when they are very small.

Another example is provided by a heat transfer problem such as a forced, incompressible flow over a heated plate. In such a case the thermal diffusivity is much more important in the boundary layer than elsewhere (here the reduced equation of conservation of energy can be assumed to describe the flow field). The velocity field can be evaluated independently from the temperature, while the latter is the solution to a linear AD equation in which the transport field is given precisely by the (known) velocity. As already noted, away from the boundary layer the diffusive term may be neglected. We refer to [6], where problems of this type are stated in large detail.

1.2. Statement of the problem and interface conditions. Hereafter we assume that $\Omega$ be a bounded, connected, open subset of $\mathbb{R}^2$, with boundary $\partial \Omega$; $\Omega_1$ and $\Omega_2$ are two open subsets of $\Omega$, with $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega}$ (see Figure 1.1). Set

$$\Gamma_i = \partial \Omega \cap \partial \Omega_i, \quad i = 1, 2; \quad \Gamma = \partial \Omega_1 \setminus \Gamma_1 = \partial \Omega_2 \setminus \Gamma_2. \quad (1.1)$$

Let $\Gamma_2^D$, $\Gamma_2^{Ne}$ be two (relatively) open subsets of $\Gamma_2$, with $\Gamma_2^D \cap \Gamma_2^{Ne} = \emptyset$, $\overline{\Gamma_2^D} \cup \overline{\Gamma_2^{Ne}} = \overline{\Gamma_2}$ (either may be empty). Denote by $n$, $n_1$, $n_2$ the unit normal vector to $\partial \Omega$, $\partial \Omega_1$, $\partial \Omega_2$, respectively, oriented outwards. Let

- $b_0$ and $f$ be scalar functions defined in $\Omega$;
- $\nu$ be a positive function defined in $\Omega_2$;
- $b$ be a two dimensional vector valued function defined in $\Omega$;
- $\phi$ be a scalar function defined on the boundary of $\Omega$. 


The following inequalities will be assumed throughout the paper, without any further explicit mention:

\[(i) \quad \nu(x) \geq \nu_0 > 0, \quad x \in \Omega_2; \quad (ii) \quad \frac{1}{2} \text{div} \ b(x) + b_0(x) \geq \beta_0 > 0, \quad x \in \Omega; \quad (1.2)\]
\[(b \cdot n_2)(x) \geq 0, \quad x \in \Gamma_2^{Ne} \quad (1.3)\]

(The reason for these assumptions will be clearer later on.) Moreover, adopt the following notation, where \(\Sigma\) is any subset of \(\partial \Omega_1\):

- \(\Sigma^{in} = \{ x \in \Sigma : (b \cdot n_1)(x) < 0 \}\),
- \(\Sigma^{out} = \{ x \in \Sigma : (b \cdot n_1)(x) > 0 \}\),
- \(\Sigma^0 = \Sigma \setminus (\Sigma^{in} \cup \Sigma^{out})\).

We will use this notation either for \(\Sigma = \partial \Omega_1\), or \(\Sigma = \Gamma_1\), or \(\Sigma = \Gamma\). The upper index \(\text{in}\) (respectively \(\text{out}\)) stands for "inflow" (respectively "outflow"), with respect to the domain \(\Omega_1\) (where a hyperbolic equation is going to be solved: the terminology is clearly related to fluid dynamical problems). Analogously, \(\Sigma^0\) is the portion of \(\Sigma\) tangent to the vector field \(b\).

![Figure 1.1. The geometry of the model problem: the arrows denote the local directions of the transport field \(b\); shaded lines denote the inflow boundary \(\Gamma_1^{in}\).](image)

The problem we aim to solve is: find a pair of real valued functions \(u, w\), defined in \(\Omega_1\) and \(\Omega_2\), respectively, which satisfy the following boundary value problem:

\[
\begin{align*}
\text{div} (bu) + b_0u &= f & \text{in } \Omega_1, \quad (1.4) \\
\text{div} (-\nu \nabla w + bw) + b_0w &= f & \text{in } \Omega_2, \quad (1.5) \\
u_0^2 w &= \phi & \text{on } \Gamma, \quad (1.6) \\
u \frac{\partial u}{\partial n_1} &= 0 & \text{on } \Gamma_1^{in}, \quad (1.7) \\
\nu \frac{\partial w}{\partial n_2} &= 0 & \text{on } \Gamma_2^{Ne}. \quad (1.8)
\end{align*}
\]

Obviously, suitable conditions at the interface \(\Gamma\) separating the two regions \(\Omega_1\) and \(\Omega_2\) are required. It can be conjectured in a natural way that one condition is needed along the whole \(\Gamma\), in order to solve the elliptic problem in \(\Omega_2\) and that a further condition is required on \(\Gamma_1^{in}\) in order to solve the hyperbolic problem in \(\Omega_1\).
One of the scopes of our investigation is precisely to find these interface conditions. Of course, we pretend that the resulting coupled problem is well posed. Among all allowed choices, we make the most natural one, namely we take those interface conditions which are generated by a limit procedure on "globally viscous" problems, when "viscosity" vanishes within $\Omega_1$. In this respect, we will find the following interface conditions:

$$\begin{align*}
-\nu \frac{\partial w}{\partial n_2} + b \cdot n_2 w &= -b \cdot n_1 u \quad \text{on } \Gamma, \\
u \frac{\partial w}{\partial n_2} &= 0 \quad \text{on } \Gamma^{\text{in}},
\end{align*}$$

(1.9) (1.10)

Actually, we will exploit an equivalent form of these two conditions, namely:

$$\begin{align*}
-\nu \frac{\partial w}{\partial n_2} + b \cdot n_2 w &= -b \cdot n_1 u \quad \text{on } \Gamma^{\text{out}} \cup \Gamma^0, \\
u \frac{\partial w}{\partial n_2} &= 0 \quad \text{on } \Gamma^{\text{in}}, \quad \text{on } \Gamma^{\text{in}}.
\end{align*}$$

(1.11) (1.12) (1.13)

We point out that the two functions $u$ and $w$ exhibit a jump at $\Gamma^{\text{out}} \cup \Gamma^0$, in general. Actually, the interface conditions express the continuity of the flux along the whole interface, while the two solutions are required to join continuously only along $\Gamma^{\text{in}}$. However, the jump between $u$ and $w$ at $\Gamma^{\text{out}} \cup \Gamma^0$ turns out to be proportional to the value of $\nu$ at the interface. We notice that the decomposition we are dealing with is reasonable only if $\nu$ is extremely small in $\Omega_1$, hence at the interface. Within $\Omega_2$ the viscous term $\text{div}(\nu \nabla w)$ of equation (1.5) may be relevant either because the coefficient $\nu$ is "large" there or because of boundary layer effects. Our approach is well suited for the former situation, although also the latter might be recovered in some sense (with a scaling on the variables...).

Eventually, we state the complete problem we want to solve:

(P): find a pair of real valued functions $u, w$, defined in $\Omega_1$ and $\Omega_2$, respectively, which satisfy (1.4)-(1.8), (1.11)-(1.13).

To fix ideas, we regard problem (P) as the scalar version of a stationary linearized problem in fluid dynamics: in this framework, $\nu$ represents the kinematic viscosity, $b$ the transport field, $f$ the source term; the velocity field $(u, w)$ is prescribed on the inflow boundary $\Gamma^{\text{in}}$ of the "inviscid region" $\Omega_1$ and along $\Gamma^{\text{in}}$, while along $\Gamma^{\text{out}}$ a zero flux condition is expressed by (1.8). Problem (P) could also be viewed as a time discretization of an evolution advection-diffusion problem (namely, when a first order derivative with respect to time is added in the differential equations) by an implicit method. In such a case, $b_0$ is essentially the reciprocal of the time step. When dealing directly with the evolution problem, the interface conditions (1.11)-(1.13) are to be enforced at each time instant. Owing to the connection to the evolution problem, we term (1.4) a "hyperbolic equation" and consequently problem (P) is said to be a "coupled hyperbolic-elliptic problem", although actually it is a degenerate elliptic problem.

A thorough discussion is going to be carried out on problem (P). The main steps will concern:
(a) the theoretical justification of the interface conditions (1.11)-(1.13) and the analysis of the well posedness of problem (P);
(b) the numerical approximation of problem (P) by spectral collocation method;
(c) the setting of an effective iterative procedure yielding the solution of the problem as limit of a sequence of solutions of two independent sub-problems: a purely hyperbolic one within \( \Omega_1 \) and a purely elliptic one within \( \Omega_2 \);
(d) the adaptation of the above iterative procedure to the finite dimensional discretization of problem (P);
(e) the interpretation of the iterative procedure in terms of the Steklov-Poincaré interface operator.

For the sake of readability, we will state the main results concerning the previous steps in advance, postponing the detailed mathematical proofs until later. Therefore, the outline of the paper turns out to be the following.

In section 2 we deal with point (b); section 3 is devoted to the introduction of functional spaces and to face some of the questions raised in point (a): a precise statement of problem (P) is given, along with an existence and uniqueness theorem. Section 4 deals with points (c) and (d): in particular, the study of the iterative method is carried out by means of the investigation of the two separate sub-problems (hyperbolic in \( \Omega_1 \) and elliptic in \( \Omega_2 \)) for both the differential and the numerical case. Point (e) is developed in section 4.1.4. Section 4.2.4 contains some numerical results and their discussion. The proofs of the results stated in sections 3 and 4 are detailed in sections 5 and 6, respectively.

We end this introduction with the index of the paper.

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2. Numerical approximation of problem (P). In this section, we focus a very simple geometrical situation, see Figure 2.1. The loss of generality is motivated by our exigency of emphasizing the treatment of interface conditions. More general geometries can be reduced to this one, at least locally, by means of suitable mappings.

![Figure 2.1. The computational domain: arrows denote the local directions of the transport field b; shaded lines denote the inflow boundary Γ_1^{in}.

We are given an integer \( N > 0 \) and denote by \( P_N \) the space of algebraic polynomials of degree at most \( N \) with respect to each variable \( x \) and \( y \). We then denote by \( \Xi_N \) the set of \( (N+1)^2 \) collocation points of the Gauss-Lobatto formula within \( \Omega_i \), pertaining to either the Legendre or the Chebyshev weight function (see [7], Ch.2). We recall that a set \( (\Xi_N)^0 \) of \( (N-1)^2 \) of such points are internal to \( \Omega_i \), while the remaining \( 4N \) lie on the boundary of \( \Omega_i \) (precisely on each side of \( \Omega_i \) there are \( N+1 \) points, including the two extrema of the side).

If \( \Sigma \) is a subset of \( \partial \Omega_i \), we denote by \( \Sigma_N \) the intersection \( \Sigma \cap \Xi_N \) (this amounts to considering collocation points lying in \( \Sigma \)).

For each vector function \( \mathbf{v} = \{v_1, v_2\} \) defined on \( \Omega_i \) we denote by \( \tilde{I}_N^{ij} \) the vector \( (\tilde{v}_1, \tilde{v}_2) \) where \( \tilde{v}_j \) (\( j = 1, 2 \)) is the polynomial of \( P_N \) interpolating \( v_j \) at the collocation points of \( \Xi_N \) (\( i = 1, 2 \)).

Moreover, for \( P \in \Xi_N \) we denote by \( \omega_P^{(i)} \) the corresponding weight in the Gauss-Lobatto integration formula for rectangular regions. For \( P \in (\partial \Omega_i)_N \), we denote by \( \theta_P^{(i)} \) the corresponding weight in the one-dimensional Gauss-Lobatto integration formula referred to \( \partial \Omega_i \).

Finally, for \( \mathbf{v} \) defined in \( \Omega_i \), \( i = 1, 2 \), set

\[
I_N^{ij} \mathbf{v} = \frac{1}{2} [ \text{div} \, I_N^{ij} (b \mathbf{v}) + b \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \text{div} \, (I_N^{ij} b)] ;
\]

\( I_N^{ij} \) is the discrete skew-symmetric divergence operator (actually, \( I_N^{ij} \) is the skew-symmetric decomposition of \( \text{div} \, (b \mathbf{v}) \) relative to the discrete inner product associated with the Gauss-Lobatto integration formula (see section 4.2)).
Now we state the numerical approximation to problem (P) as follows. We look for a pair of polynomials $u_N \in P_N$ and $w_N \in P_N$, satisfying

$$L_N^1 u_N + b_0 u_N = f \quad \text{at } (\Xi_N^1)^0 \cup (\Gamma^\text{in} \cup \Gamma^\text{out})_N \cup (\Gamma^\text{out} \cup \Gamma^0)_N;$$ (2.2)

$$- \text{div} [L_N^2 (\nu \nabla w_N)] + L_N^2 w_N + b_0 w_N = f \quad \text{at } (\Xi_N^2)^0;$$ (2.3)

$$- [b \cdot n_1 (u_N - \phi)](P) = [L_N^1 u_N + b_0 u_N - f](P) \frac{\omega_P^{(1)}}{\theta_P^{(1)}} \quad \text{at } P \in (\Gamma_1^\text{in})_N;$$ (2.4)

$$w_N = \phi \quad \text{at } (\Gamma_2^\text{in})_N;$$ (2.5)

$$[-\nu \frac{\partial w_N}{\partial n_2}](P) = R_2 (P) \frac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma_{2}^{\text{out}})_N;$$ (2.6)

$$-[b \cdot n_1 (u_N - w_N)](P) = [L_N^1 u_N + b_0 u_N - f](P) \frac{\omega_P^{(1)}}{\theta_P^{(1)}} \quad \text{at } P \in (\Gamma^\text{in})_N;$$ (2.7)

$$[-\nu \frac{\partial w_N}{\partial n_2} + b \cdot n_2 (w_N - u_N)](P) = R_2 (P) \frac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma^\text{out})_N;$$ (2.8)

$$[-\nu \frac{\partial w_N}{\partial n_2}](P) = R_2 (P) \frac{\omega_P^{(2)}}{\theta_P^{(2)}} \quad \text{at } P \in (\Gamma^\text{in})_N;$$ (2.9)

where

$$R_2 = - \text{div} [L_N^2 (\nu \nabla w_N)] + L_N^2 w_N + b_0 w_N - f$$ (2.10)

is the residue coming from discrete integrations by parts.

**REMARK 2.1** We just note that (2.2) corresponds to (1.4); (2.3) to (1.5); (2.4) to (1.6); (2.5) to (1.7); (2.6) to (1.8); (2.7) to (1.13); (2.8) to (1.11); (2.9) to (1.12). The boundary and interface conditions in (2.4), (2.6)-(2.9) are imposed in a weak form related to a variational formulation (see section 4.2). The strong form is obtained just by replacing the right hand sides of these formulas with zero. Since the quotient $\omega_P^{(i)} [\theta_P^{(i)}]^{-1}$ is proportional to $N^{-2}$ (see, e.g., [7], Ch. 2), the weak form enforces the exact boundary and interface conditions up to the value of the residue of the equation times a constant that tends to zero as $N$ tends to infinity. We chose to adopt the weak form because this suits better the analysis we will carry out, while it is equivalent to the strong formulation from the point of view of accuracy. \(\square\)

The analysis of existence, uniqueness and stability estimates for the discrete problem (2.3)-(2.9) will be carried out under the following “coercivity assumption”, which is the finite dimensional analogue of (1.2)(ii):

$$\frac{1}{2} \text{div} [I_N^1 b(x)] + b_0(x) \geq \beta_N > 0, \ x \in \Omega_i, \ i = 1, 2.$$ (2.11)

Using standard interpolation error estimates, one can show that the constant $\beta_N$ can be bounded from below independently of $N$, provided $b$ is a smooth function, as a consequence of (1.2).

Under the assumptions (1.2)(i), (1.3) and (2.11), the hyperbolic-elliptic collocation problem (2.2)-(2.9) has one and only one solution. Uniqueness will be proved
in section 6 (Proposition 6.10); existence follows from uniqueness as the problem has finite dimension.

3. Functional framework; existence and uniqueness for problem (P).

In this section, we give a rigorous formulation of problem (P) and state the existence and uniqueness results for its solution. To this end, we must introduce some functional tools in advance.

Let \( A \) be an open, bounded subset of \( \mathbb{R}^2 \), with Lipschitz continuous boundary. If \( m \) is a positive integer, denote by \( \mathcal{H}^m(A) \) the Sobolev space of real valued functions belonging to \( L^2(A) \) along with all derivatives up to the order \( m \). \( \mathcal{H}^m(A) \) is a Hilbert space with norm

\[
\|v\|_{m, A} = \left[ \sum_{i+j=m} \left( \frac{\partial^i}{\partial x_1^i} \frac{\partial^j}{\partial x_2^j} \right) \|v\|_{0, A}^2 \right]^{\frac{1}{2}},
\]

where \( \cdot \|_{0, A} \) denotes the \( L^2(A) \) norm. Analogously, the \( \mathcal{L}^\infty(A) \) norm is denoted by \( \| \cdot \|_{\infty, A} \); \( \mathcal{W}^{1, \infty}(A) \) represents the space of Lipschitz continuous functions on \( A \), endowed with the usual norm

\[
\|v\|_{1, \infty, A} = \|v\|_{\infty, A} + \|\partial_1 v\|_{\infty, A} + \|\partial_2 v\|_{\infty, A}, \quad v \in \mathcal{W}^{1, \infty}.
\]

It is well known (see, for instance, [8]) that the value at the boundary \( \partial A \) of all elements of \( \mathcal{H}^m(A) \) can be given a meaning through a trace operator which maps linearly and continuously \( \mathcal{H}^m(A) \) onto a subset of \( L^2(\partial A) \), denoted by \( \mathcal{H}^{m-\frac{1}{2}}(\partial A) \) (a Hilbert space for the quotient norm \( \| \cdot \|_{m-\frac{1}{2}, \partial A} \)).

In the following, the dual space of \( \mathcal{H}^{\frac{1}{2}}(\partial A) \) will be called into play: this is denoted by \( \mathcal{H}^{-\frac{1}{2}}(\partial A) \) and endowed with the dual norm \( \| \cdot \|_{-\frac{1}{2}, \partial A} \). Eventually, we recall some notations and properties for vector valued functions. The space

\[
L^2_{\text{div}}(A) = \{ v \in L^2(A)^2 : \text{div} v \in L^2(A) \}
\]

is a Hilbert space with the graph norm:

\[
\|v\|_{L^2_{\text{div}}(A)} = \left[ \|v\|_{0, A}^2 + \|\text{div} v\|_{0, A}^2 \right]^{\frac{1}{2}}
\]

(we adopt the same notation for norms as in the scalar case). Next, consider a (relatively) open subset \( \Sigma \) of \( \partial A \) and define

\[
H^{\frac{1}{2}}(\Sigma) = \{ v \in L^2(\Sigma) : \tilde{v} \in H^{\frac{1}{2}}(\partial A) \},
\]

where \( \tilde{v} = \begin{cases} v & \text{in } \Sigma \\ 0 & \text{in } \partial A \setminus \Sigma \end{cases} \) is the trivial extension of \( v \) outside \( \Sigma \). If \( \Sigma = \partial A \), then \( H^{\frac{1}{2}}(\Sigma) \) coincides with \( H^{\frac{1}{2}}(\partial A) \). For \( v \in H^{\frac{1}{2}}(\Sigma) \) we set

\[
\|v\|_{H^{\frac{1}{2}}(\Sigma)} = \|\tilde{v}\|_{\frac{1}{2}, \partial A}
\]

instant \( \beta_N \) can function, as a elliptic colloca-vill be proved
and denote by $\langle \cdot, \cdot \rangle$ the pairing between $H_{00}^1(\Sigma)$ and its topological dual $(H_{00}^1(\Sigma))'$.

It is known (see, e.g., [8], [9]) that, if $v \in L^2_{\text{div}}(A)$, then $v \cdot n \in (H_{00}^1(\Sigma))'$, where $n$ is the outward unit vector normal to $\partial A$. Moreover Green's formula holds for all $v \in L^2_{\text{div}}(A)$ and all $w \in H^1(A)$, with $w|_{\partial A \setminus \Sigma} = 0$:

$$
\int_A w \, \text{div} \, v \, dx + \int_{\partial A} \nabla w \cdot v \, ds = \langle v \cdot n, w \rangle.
$$

(3.5)

In particular, if $A$ is an open subset of $\Omega$ and $b \in [W^{1,\infty}(\Omega)]^2$ is the assigned transport field, then (3.5) becomes:

$$
\int_A \nabla w \cdot b \, dx = - \int_A w \, \text{div} \, (b \cdot dx) + \langle b \cdot n, z \rangle.
$$

(3.6)

for all $z \in L^2(A)$ with $\text{div} \, (b \cdot dx) \in L^2(A)$ and all $w \in H^1(A)$ with $w|_{\partial A \setminus \Sigma} = 0$. Moreover, for such a $v$ the choice $z = w$ in (3.6) leads to

$$
\int_A \nabla w \cdot b \, dx = - \int_A w^2 \, \text{div} \, b \, dx - \int_A \nabla v \cdot b \, dx + \int_{\partial A} b \cdot n w^2 \, ds,
$$

whence

$$
\int_A \nabla w \cdot b \, dx = - \frac{1}{2} \int_A w^2 \, \text{div} \, b \, dx + \frac{1}{2} \int_{\partial A} b \cdot n w^2 \, ds.
$$

(3.7)

The previous formula suggests the following notation: if $v$ is a scalar function defined on a (relatively) open subset $\Sigma$ of $\partial\Omega \cup \Gamma$, we say that

$$
|v|_\Sigma < +\infty, \text{ whenever } \sqrt{|b \cdot n|} v \in L^2(\Sigma);
$$

(3.8)

in such a case, we set

$$
|v|_\Sigma^2 = \int_{\Sigma} |b \cdot n| v^2 \, ds.
$$

(3.9)

Notice that these formulas define a weighted norm in $L^2(\Sigma)$, at least when the weight $|b \cdot n|$ does not vanish on $\Sigma$. In such a case, we set

$$
L^2_b(\Sigma) = \{ v : \Sigma \to \mathbb{R} : \sqrt{|b \cdot n|} v \in L^2(\Sigma) \}:
$$

(3.10)

this is a Banach space for the norm $| \cdot |_\Sigma$ (actually, a Hilbert space).

By the way of dealing with multipliers, we state a lemma which will be used later.

**LEMMA 3.1** Let $A$ be an open, bounded subset of $\mathbb{R}^2$, with Lipschitz continuous boundary. Let $\Sigma$ be a (relatively) open subset $\partial A$. Let $\rho$ and $\varphi$ be such that

$$
\rho \in W^{1,\infty}(\Sigma), \quad \rho > 0 \text{ on } \Sigma; \quad \sqrt{\rho} \varphi \in L^2(\Sigma).
$$

(3.11)

(i) If $v$ satisfies $\rho v \in H_{00}^1(\Sigma)$ and $\rho v = \rho \varphi$ in the sense of $[H_{00}^1(\Sigma)]'$, then

$$
\sqrt{\rho} v \in L^2(\Sigma) \quad \text{and} \quad v = \varphi \text{ a.e. on } \Sigma.
$$

(3.12)

(ii) If $G$ belo

$$
\sqrt{\rho} \varphi \text{ in } L^2(\Sigma).
$$

Proof.}

$$
\rho^{-\frac{1}{2}} \varphi \in H_{00}^1(\Sigma)
$$

Cauchy-Sch

(3.13) gives $t$

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THEOREM

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(a) $u \in L^2(\Sigma)$

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(d) boundary

(e) boundary

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(ii) If \( G \) belongs to \( \left[ \text{H}^\frac{3}{2}_0(\Sigma) \right]' \) and \( z_n \) is a sequence such that \( \sqrt{\rho} z_n \in L^2(\Sigma) \), \( \sqrt{\rho} z_n \rightharpoonup \sqrt{\rho} z \) in \( L^2(\Sigma) \), \( \rho z_n \rightharpoonup G \) in \( \left[ \text{H}^\frac{3}{2}_0(\Sigma) \right]' \), then \( G = \rho z \).

**Proof.** Let \( \zeta \) be a smooth function with compact support in \( \Sigma \). By (3.11), it is \( \rho^{-\frac{1}{2}} \zeta \in \text{H}^\frac{3}{2}_0(\Sigma) \). To prove part (i), we note that

\[
\langle \rho v, \rho^{-\frac{1}{2}} \zeta \rangle = \langle \rho z, \rho^{-\frac{1}{2}} \zeta \rangle = \int_\Sigma \sqrt{\rho} z \zeta \, ds. \tag{3.13}
\]

Cauchy-Schwarz inequality in the last integral shows that \( \sqrt{\rho} v \in L^2(\Sigma) \) and then (3.13) gives the equality \( \sqrt{\rho} v = \sqrt{\rho} z \) a.e. on \( \Sigma \). Because of (3.11), this gives \( v = z \) a.e. on \( \Sigma \). To prove part (ii), we note that in the equality

\[
\langle \rho z_n, \rho^{-\frac{1}{2}} \zeta \rangle = \int_\Sigma \sqrt{\rho} z_n \zeta \, ds
\]

the left hand side converges to \( \langle G, \rho^{-\frac{1}{2}} \zeta \rangle \), while the right hand side converges to \( \int_\Sigma \sqrt{\rho} z \zeta \, ds \), whence \( \rho^{-\frac{1}{2}} G = \sqrt{\rho} z \) in \( L^2(\Sigma) \).\( \square \)

With the preceding notations, problem (P) may be given a precise mathematical formulation and the following result may be proved.

**THEOREM 3.2** Assume the following regularity properties on the data:

\[ \partial \Omega_1 \text{ and } \partial \Omega_2 \text{ are Lipschitz continuous, piecewise C}^{1,1}; \] \( \Gamma \) is of class C\(^{1,1} \); \( \nu \in L^\infty(\Omega_2), \ b \in [W^{1,\infty}(\Omega)]^2, \ b_0 \in L^\infty(\Omega), \ f \in L^2(\Omega); \) \( \phi \in H^{-\frac{1}{2}}(\partial \Omega), \ \phi|_{\Gamma^2} \in H^\frac{1}{2}(\Gamma^2), \ \phi|_{\Gamma^1} \in L^2(\Gamma^1). \tag{3.16} \]

Finally, assume (1.2) and (1.3) (which are meaningful, due to (3.14) and (3.15)). Then there is a unique pair \((u, w)\) which solves problem (P) in the following sense:

(a) \( u \in L^2(\Omega_1) \), \( w \in H^1(\Omega_2) \);

(b) equation (1.4) holds in the sense of distributions in \( \Omega_1 \);

(c) equation (1.5) holds in the sense of distributions in \( \Omega_2 \);

(d) boundary condition (1.6) holds a.e. on \( \Gamma^1 \);

(e) boundary condition (1.7) holds in \( H^\frac{1}{2}(\Gamma^2) \);

(f) boundary condition (1.8) holds in the sense of \( \left[ H^\frac{3}{2}_0(\Gamma^2) \right]' \);

(g) interface condition (1.11) holds in the sense of \( \left[ H^\frac{3}{2}_0(\Gamma^{\text{int}}) \right]' \);

(h) interface condition (1.12) holds in the sense of \( \left[ H^\frac{3}{2}_0(\Gamma^{\text{int}}) \right]' \);

(i) interface condition (1.13) holds a.e. on \( \Gamma^{\text{int}} \).

Finally, problem (P) is limit of a family of globally elliptic variational problems.\( \square \)

We will give two different proofs of existence: one in section 5, by building a family of globally elliptic approaching problems (in this way, we will also derive the interface conditions); the other proof of existence will be based on the iterative procedure introduced in section 4. Uniqueness will also be obtained in the framework of this
iterative procedure (see section 6.1.3). Here we point out that boundary and interface conditions defining problem (P) cannot be given a classical meaning, because of the irregularity of the solution in the general case.

REMARK 3.3. The regularity assumption made on \( \phi \) in (3.16) is natural, in some sense. Indeed, we will show in Theorem 4.1 that the solution \( u \) to a purely hyperbolic problem with transport field \( b \) is such that \( b \cdot n_1 u^2 \) is integrable along the outflow boundary. If we have in mind that the hyperbolic domain is an element of a decomposition of a larger hyperbolic region, then we realize that the inflow boundary of our domain is but the outflow boundary for another adjoining subdomain. Therefore, our assumption on the inflow data matches the expected regularity for the outflow coming from the adjoining domain. \( \square \)

4. Solution of the coupled problem (P) via an iterative procedure.

Our goal in this section is to exhibit the solution of problem (P) (see (1.4)-(1.8), (1.11)-(1.13)) as a limit of solutions of two subproblems within \( \Omega_1 \) and \( \Omega_2 \), respectively. This is done by attributing condition (1.13) to the hyperbolic problem in \( \Omega_1 \) and conditions (1.11), (1.12) to the elliptic problem in \( \Omega_2 \). Therefore, we consider the following iterative procedure.

Let \( u^0, w^0 \) be given on \( \Gamma^{in} \). We define a sequence \( (u^n, w^n) \), \( n \geq 1 \) by solving for each \( n \) the following hyperbolic problem within \( \Omega_1 \)

\[
\begin{align*}
(i) & \quad \text{div}(bu^n) + b_0 u^n = f \quad \text{in} \ \Omega_1 \\
(ii) & \quad u^n = \phi \quad \text{on} \ \Gamma_1^{in} \\
(iii) & \quad u^n = \psi^n \quad \text{on} \ \Gamma^{in}
\end{align*}
\]

and then the following elliptic problem within \( \Omega_2 \)

\[
\begin{align*}
(i) & \quad \text{div}(-\nu \nabla w^n + bw^n) + b_0 w^n = f \quad \text{in} \ \Omega_2 \\
(ii) & \quad w^n = \phi \quad \text{on} \ \Gamma_2^D \\
(iii) & \quad \nu \frac{\partial w^n}{\partial n_2} = 0 \quad \text{on} \ \Gamma_2^{Ne} \\
(iv) & \quad -\nu \frac{\partial w^n}{\partial n_2} + b \cdot n_2 w^n = -b \cdot n_1 u^n \quad \text{on} \ \Gamma^{out} \cup \Gamma^0 \\
(v) & \quad \nu \frac{\partial w^n}{\partial n_2} = 0 \quad \text{on} \ \Gamma^{in},
\end{align*}
\]

where

\[
\psi^n = \theta w^{n-1} + (1 - \theta) u^{n-1} \quad \text{on} \ \Gamma^{in}, \quad \theta > 0
\]

Formally, the limit of the sequence \( (u^n, w^n) \), if existing, solves the coupled problem (P).

The transformation of the original problem (P) into (4.1), (4.2) provides:

(a) a viable algorithm for finding the solution to the problem, which can be conveyed easily to finite dimensions (see section 4.2);

(b) an alternative way of proving the existence of a solution (see section 4.1);

(c) a proof of uniqueness (see section 4.1);
We point out that the splitting we have made carries several advantages from the computational point of view. First of all, the two subproblems can be faced by standard numerical methods for hyperbolic and elliptic problems, respectively. Furthermore, the two parts of the interface boundary play a separate role in the interaction mechanism between $\Omega_1$ and $\Omega_2$. Actually, the hyperbolic solution $u^n$ is influenced by the elliptic one $w^{n-1}$ through $\Gamma^\text{in}$ only, while, symmetrically, $u^n$ influences $w^n$ through $\Gamma^\text{out}$ solely. This makes the iterative procedure very effective, as documented by some numerical tests (see section 4.2.4).

The iterative procedure will be discussed both for the differential problem (4.1), (4.2) (see section 4.1) and for its finite dimensional approximation (see section 4.2). In both cases, we will face the question of solvability of the subproblems and prove some a priori estimates that will be used in the discussion of convergence of the iterative procedure. Finally, we will introduce the influence operator $S$ associated with the coupled problem $(P)$: this is precisely the pseudodifferential operator acting solely on the values of the solution at the interface. This operator is known as the Steklov-Poincaré operator in the differential problem and the capacitance matrix in the discrete approximation. The iterative algorithm can be interpreted by means of the operator $S$ as an iterative procedure acting solely on the interface variables.

4.1. The differential case. This subsection is divided into four parts: in the first one, we give the precise formulation of a hyperbolic problem of type (4.1) and state an a priori estimate for the solution; the second one contains the analogous arguments for an elliptic problem of type (4.2); the third one is devoted to the combination of the results of the two previous cases, in order to get a global estimate for the iterative scheme (4.1), (4.2). This will allow us to get a proof of convergence for the scheme, along with a new proof of existence of solutions to the coupled problem $(P)$; uniqueness will also follow in this framework. Finally, we will introduce the Steklov-Poincaré operator associated with the coupled problem $(P)$.

4.1.1. The hyperbolic problem in $\Omega_1$. With the usual notations, we consider the following problem:

\((P_H)\): $u \in L^2(\Omega_1)$ satisfies

\[
\text{div}(b u) + b_0 u = f \quad \text{in } \mathcal{D}'(\Omega_1),
\]

\[
u = \lambda \quad \text{a.e. on } \partial \Omega_1^\text{in},
\]

where $\lambda \in H^{-\frac{1}{2}}(\partial \Omega_1)$ is given, satisfying

\[
\lambda_{|\partial \Omega_1^\text{in}} \in L^2_b(\partial \Omega_1^\text{in}).
\]

Our aim is to find how the inflow data $\lambda$ propagates to $\partial \Omega_1^\text{out}$ through the solution of $(P_H)$. This influence is shown in the following theorem through an a priori estimates that will play an important role in what follows.

THEOREM 4.1 Under the same assumptions on the data $\Omega_1$, $b$, $b_0$ and $f$ as in Theorem 3.2, if (4.6) holds, then problem $(P_H)$ has a unique solution $u$. This satisfies

\[
\|u\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \|u\|_{\partial \Omega_1^\text{out}}^2 \leq \frac{1}{4}\|f\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \lambda_{|\partial \Omega_1^\text{in}}^2,
\]

\[
(\frac{b_0}{2} - \delta)\|u\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \|u\|_{\partial \Omega_1^\text{out}}^2 \leq \frac{1}{4}\|f\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \lambda_{|\partial \Omega_1^\text{in}}^2.
\]
for all $\delta > 0$. If $f = 0$, then (4.7) holds with $\delta = 0$, namely

$$
\rho_0 \|u\|_{\partial \Omega_1}^2 + \frac{1}{2} \|u\|_{\partial \Omega_1^\text{ext}}^2 \leq \frac{1}{2} \lambda_2 \|u\|_{\partial \Omega_1^\text{ext}}^2.
$$

(4.8)

Finally, problem (P_H) is limit of a family of elliptic problems.\)

The proof of this theorem will be given in section 6.1.1, still using a regularization argument. Here we state the estimate (4.7) for problem (4.1) as a corollary.

**COROLLARY 4.2** Under the same assumptions on the data as in Theorem 3.2, if

$$
\psi^n \in L^2_b(\Gamma^\text{in}),
$$

(4.9)

then problem (4.1) has a unique solution $u^n$. This satisfies $u^n_{\partial \text{ext}} \in L^2_b(\Gamma^\text{out})$ and

$$
(\rho_0^2 - \delta)\|u^n\|_{\partial \Omega_1}^2 + \frac{1}{2} \|u^n\|_{\partial \Omega_1^\text{ext}}^2 \leq \frac{1}{4\delta} \|f\|_{\partial \Omega_1}^2 + \frac{1}{2} \|\psi^n\|_{\partial \Omega_1^\text{ext}}^2.
$$

(4.10)

for all $\delta > 0$. Moreover, if $f = \phi = 0$, then (4.10) holds with $\delta = 0$, namely

$$
\rho_0 \|u^n\|_{\partial \Omega_1}^2 + \frac{1}{2} \|u^n\|_{\partial \Omega_1^\text{ext}}^2 \leq \frac{1}{2} \|\psi^n\|_{\partial \Omega_1^\text{ext}}^2.
$$

(4.11)

\)

Of course, $u^n$ solves problem (4.1) in a sense analogous to that of problem (P_H). The assumption (4.9) holds, provided the initial guesses $u^0$, $w^0$ are regular enough (which we may certainly assume).

4.1.2. The elliptic problem in $\Omega_2$. Keeping in mind the usual notations and geometry, we suppose that $\phi$ and $\mu$ are given, with

$$
\phi \in H^\frac{1}{2}(\Gamma^D_2), \quad \mu \in [H^\frac{1}{2}(\Gamma)]', \quad \mu_{\partial \text{ext}} \in L^2_b(\Gamma^\text{out}).
$$

(4.12)

Let $V$ be the following affine subspace of $H^1(\Omega_2)$:

$$
V = \{v \in H^1(\Omega_2) : v = \phi \text{ on } \Gamma^D_2\}.
$$

Consider the following variational problem in $\Omega_2$:

$$
(P_E) : \text{find } w \in V \text{ such that, for all } v \in V,
$$

$$
\int_{\Omega_2} (\nu \nabla w - bw) \cdot \nabla (w - v) \, dx + \int_{\Omega_2} b_0 w (w - v) \, dx + \int_{\Gamma^\text{ext} \cup \Gamma^D_2} b \cdot n_2 w (w - v) \, ds =
$$

$$
= \int_{\Omega_2} f(w - v) \, dx - \int_{\Gamma^\text{ext}} b \cdot n_2 \mu (w - v) \, ds.
$$

(4.13)
Problem \((P_E)\) is the variational formulation of an elliptic boundary value problem in \(\Omega_2\), analogous to (4.2). More precisely, \((P_E)\) is equivalent to finding \(w \in H^1(\Omega_2)\) such that

\[
\begin{align*}
\text{div} (-\nu \nabla w + b w) + b_0 w &= f & \text{in } \Omega_2, \\
w &= \phi & \text{on } \Gamma_D, \\
\nu \frac{\partial w}{\partial n_2} &= 0 & \text{in } [H^{\frac{3}{2}}(\Gamma_{\text{out}})]', \\
-\nu \frac{\partial w}{\partial n_2} + b \cdot n_2 w &= b \cdot n_2 \mu & \text{in } [H^{\frac{3}{2}}(\Gamma_{\text{out}} \cup \Gamma^0)]', \\
\nu \frac{\partial w}{\partial n_2} &= 0 & \text{in } [H^{\frac{3}{2}}(\Gamma_{\text{in}})]'.
\end{align*}
\]

(4.14) \hspace{1cm} (4.15) \hspace{1cm} (4.16) \hspace{1cm} (4.17) \hspace{1cm} (4.18)

The following result holds.

**Theorem 3.2,** if

\[
L^2_{\text{reg}}(\Gamma_{\text{out}}) \quad \text{and}
\]

\[
L^2_{\text{reg}}(\Gamma_{\text{in}}),
\]

namely

\[
\begin{align*}
\text{problem } (P_H) & \quad \text{and}
\end{align*}
\]

\[
\text{regular enough}
\]

\[
\text{estimations and geometrical}
\]

\[
(4.11)
\]

\[
(4.10)
\]

Again, we state the estimate (4.19) for problem (4.2) as a corollary.

**Corollary 4.4** Under the same assumptions on the data as in Theorem 3.2, if \((4.9)\) holds, then the solution to (4.2) satisfies

\[
\begin{align*}
(1 - \delta)|\nabla w|^2_{0,\Omega_2} + \{\beta_0 - \delta(\|b\|_{1,\infty,\Omega_2} + \|b_0\|_{\infty,\Omega_2})\}|w|^2_{0,\Omega_2} + \frac{1}{2} |\mu|^2_{\Gamma_{\text{out}}} & \leq \frac{1}{2} |\mu|^2_{\Gamma_{\text{out}}},
\end{align*}
\]

(4.20)

The comments on the validity of (4.2) and of (4.9) are analogous to those made for Corollary 4.2. The proof of Theorem 4.3 will be given in section 6.1.2.

### 4.1.3. Convergence of the iterative procedure: existence and uniqueness results for the coupled problem \((P)\)

We begin by giving an equivalent formulation of the convergence problem for the iterative scheme (4.1), (4.2).
Let \( \theta \in \mathbb{R} \) and \( \psi \in \mathbf{L}^2_b(\Gamma_{\text{in}}) \) be given. Let \( u^\psi, w^\psi \) be respectively the solutions to the two boundary value problems

\[
\begin{cases}
  (i) & \text{div} (bu^\psi) + b_0 u^\psi = f \quad \text{in } D'(\Omega_1) \\
  (ii) & u^\psi = \phi \quad \text{a.e. on } \Gamma_{\text{in}}^1 \\
  (iii) & u^\psi = \phi \quad \text{a.e. on } \Gamma_{\text{in}}^0; \\
  (iv) & \text{div} (-\nu \nabla w^\psi + bw^\psi) + b_0 w^\psi = f \quad \text{in } D'(\Omega_2) \\
  (v) & w^\psi = \phi \quad \text{on } \Gamma_2^D \\
  (vi) & \frac{\partial w^\psi}{\partial n_2} = 0 \quad \text{in } \left[ \mathbf{H}^{\frac{1}{2}}(\Gamma_{\text{out}}^N) \right]' \\
  (vii) & \frac{\partial w^\psi}{\partial n_2} = 0 \quad \text{in } \left[ \mathbf{H}^{\frac{1}{2}}(\Gamma_{\text{in}}^0) \right]'. 
\end{cases}
\] (4.23)

Define the application

\[
T_0 : \mathbf{L}^2_b(\Gamma_{\text{in}}) \to \mathbf{L}^2_b(\Gamma_{\text{in}}); \quad T_0 : \psi \mapsto T_0(\psi) = \theta w^\psi|_{\Gamma_{\text{in}}} + (1 - \theta)\psi. 
\] (4.25)

**PROPOSITION 4.5** The following two conditions are equivalent, for a given \( \psi \in \mathbf{L}^2_b(\Gamma_{\text{in}}) \).

1. The pair \((u^\psi, w^\psi)\) solves the coupled problem (P).

2. There exists \( \theta \in \mathbb{R} \) such that \( \psi \) is a fixed point for \( T_0 \).

Moreover, each fixed point for \( T_0 \) belongs to \( \mathbf{H}^{\frac{1}{2}}(\Gamma_{\text{in}}) \). Finally, if \( \psi \) is a fixed point for \( T_1 \), then it is a fixed point for \( T_0 \), for all \( \theta \in \mathbb{R} \). □

**LEMMA 4.6** Under the assumptions of Theorem 3.2, the solutions \( u_0^\psi, w_0^\psi \) to (4.23), (4.24), respectively, with \( f = \phi = 0 \), satisfy:

\[
\| \sqrt{\nu} |\nabla u_0^\psi|_{\partial \Omega_2}^2 + \frac{\beta_0}{2} |u_0^\psi|_{\partial \Omega_1}^2 + \beta_0 |w_0^\psi|_{\partial \Omega_2}^2 + \frac{1}{2} |w_0^\psi|_{\Gamma_{\text{in}}}^2 \|_{\Gamma_{\text{in}}}^2 \leq \frac{1}{2} |\psi|_{\Gamma_{\text{in}}}^2. 
\] (4.26)

□

In view of Proposition 4.5, we will search for fixed points of the application \( T_1 \).

**LEMMA 4.7** The application \( T_1 \) has a unique fixed point \( \psi \). □

The following estimate holds for the iterative scheme (4.1), (4.2) with homogeneous data.

**LEMMA 4.8** Under the assumptions of Theorem 3.2, if the initial guesses \( u^0, w^0 \) are such that \( \psi^0 \in \mathbf{L}^2_b(\Gamma_{\text{in}}) \), then the solutions \( u_n^0, w_n^0 \) to (4.1), (4.2) with \( f = \phi = 0 \) satisfy

\[
\| \sqrt{\nu} |\nabla u_n^0|_{\partial \Omega_2}^2 + \frac{\beta_0}{2} |u_n^0|_{\partial \Omega_1}^2 + \beta_0 |w_n^0|_{\partial \Omega_2}^2 + \frac{1}{2} |w_n^0|_{\Gamma_{\text{in}}}^2 \|_{\Gamma_{\text{in}}}^2 \leq \frac{1}{2} |\psi_n|_{\Gamma_{\text{in}}}^2. 
\] (4.27)

□

This estimate is obtained using the coerciveness properties of the bilinear form. Finally, define...
This estimate allows us to prove the following convergence result.

**LEMMA 4.9** There exists $\delta > 0$ such that if $\psi^0 \in L^2_0(\Gamma^{in})$ and $\theta \in [0, 1 + \delta]$, then the sequence $\psi^n$ defined in (4.3) converges in $L^2_0(\Gamma^{in})$ to the (unique) fixed point of $T_1$. □

Finally, we can state the convergence theorem for the iterative scheme, containing also the existence and uniqueness results for the coupled problem (P). The proof will be given in section 6.1.3.

**THEOREM 4.10** Under the assumptions of Theorem 3.2, if the initial guesses $u^0, w^0$ are such that $\psi^0 \in L^2_0(\Gamma^{in})$, then the sequence $(u^n, w^n)$ converges to a limit pair $(u, w)$, in the following sense:

\[
\begin{align*}
  u^n \to u & \quad \text{in } L^2(\Omega_1); \\
  w^n \to w & \quad \text{in } H^1(\Omega_2).
\end{align*}
\]

The limit pair provides the unique solution to the coupled problem (P). □

**REMARK 4.11** It will be clear from the proof of the theorem that the rate of convergence depends on $\nu$ and $b$, in principle. □

### 4.1.4. The Steklov-Poincaré operator associated with problem (P)

In this section, we introduce the Steklov-Poincaré pseudodifferential operator associated with problem (P) (see (1.4)-(1.8), (1.11)-(1.13)). This can be defined as an operator which permits to reduce the solution of the coupled problem to the solution of an equation involving only the interface values. For simplicity, we will detail only the homogeneous case $f = \phi = 0$.

Let $\psi \in H^{\frac{1}{2}}(\Gamma)$ be a given function. Correspondingly, denote by $U = U(\psi)$ the solution to the hyperbolic problem within $\Omega_1$

\[
\begin{align*}
  (i) & \quad \text{div}(bU) + b_0 U = 0 & \quad \text{in } \Omega_1 \\
  (ii) & \quad U = 0 & \quad \text{on } \Gamma^{in}_1 \\
  (iii) & \quad U = \psi & \quad \text{on } \Gamma^{in}_1.
\end{align*}
\]  

Next, denote by $W = W(\psi)$ the solution to the elliptic problem within $\Omega_2$

\[
\begin{align*}
  (i) & \quad \text{div}(-\nu \nabla W + bW) + b_0 W = 0 & \quad \text{in } \Omega_2 \\
  (ii) & \quad W = 0 & \quad \text{on } \Gamma^D_2 \\
  (iii) & \quad \frac{\partial W}{\partial n_2} = 0 & \quad \text{on } \Gamma^{Ne}_2 \\
  (iv) & \quad W = \psi & \quad \text{on } \Gamma.
\end{align*}
\]

Finally, define

\[
S^1\psi = \begin{cases} 
  b \cdot n_1 U & \quad \text{on } \Gamma^{out}, \\
  0 & \quad \text{on } \Gamma^{in},
\end{cases}
\]

\[
S^2\psi = \begin{cases} 
  -\nu \frac{\partial W}{\partial n_2} + b \cdot n_2 W & \quad \text{on } \Gamma^{out}, \\
  \frac{\partial W}{\partial n_2} & \quad \text{on } \Gamma^{in}
\end{cases}
\]

(Actually, $S^1\psi$ depends only on the values of $\psi$ on $\Gamma^{in}$).
The interface operator
\[ S = S^1 + S^2 : \mathcal{H}_{00}^1(\Gamma) \to [\mathcal{H}_{00}^1(\Gamma)]' \]  (4.32)
is the Steklov-Poincaré operator associated to problem (P). As a matter of fact, solving problem (P) (with homogeneous data \( f = \phi = 0 \)) is equivalent to the simultaneous solution of the two independent problems (4.28) and (4.29) along with the interface equation
\[ S \psi = 0 \]  (4.33)
(this equation is homogeneous because we are dealing with a homogeneous coupled problem). We can also give a sequential version of this procedure, which is strictly related to the iterative argument introduced at the beginning of this section 4. Indeed, the interface equation (4.33) may be equivalently reformulated by means of the operator \((S^2)^{-1}\), the inverse of \(S^2\) (this exists, because (4.29) is well posed). In this case, we end up with the equation
\[ -(S^2)^{-1}S^1 \psi = \psi, \]  (4.34)
that is, we look for fixed points of the application \(-(S^2)^{-1}S^1\). Now, it is readily seen that the restriction of this application to \(L^2_k(\Gamma^m)\) coincides with the map \(T_0\) defined in (4.25), when \(\theta = 1\) and \(f = \phi = 0\); dropping the restriction, we simply write
\[ -(S^2)^{-1}S^1 = T_1. \]  (4.35)
More generally, for any \(\theta\) (4.25) gives
\[ T_0 \psi = \theta T_1 \psi + (1 - \theta) \psi, \]  (4.36)
whence (4.34) is equivalent also to the equation
\[ T_0 \psi = \psi. \]  (4.37)
Thus, the interface equation is equivalent to the search for fixed points of the map \(T_0\) in the homogeneous case. The natural iterative procedure applied to (4.37) gives
\[ \psi^{n+1} = T_0 \psi^n. \]  (4.38)
On one hand, (4.35) and (4.36) allow us to read this iterative procedure as
\[ \psi^{n+1} = -\theta(S^2)^{-1}S^1 \psi^n + (1 - \theta) \psi^n: \]
the application of the map \(S^2\) and the definition (4.32) then give
\[ S^2(\psi^{n+1} - \psi^n) = -\theta S \psi^n, \]
which is nothing else than the Richardson iterative method applied to equation (4.33) with "preconditioner" \((S^2)^{-1}\) (the terminology is borrowed from numerical analysis). On the other hand, (4.3) and (4.25) show that the iterative procedure (4.38) is equivalent to the iterative algorithm (4.1), (4.2). Therefore, we conclude that the iterative algorithm (4.1), (4.2) is equivalent to the Richardson iterative method applied to equation (4.33) with "preconditioner" \((S^2)^{-1}\).
4.2. The discrete case. The iterative procedure (4.1), (4.2) can be realized numerically by the spectral collocation method as follows.

For each \( n \), we look for two polynomials \( u^N_n \in P_N, w^N_n \in P_N \) satisfying the following hyperbolic problem in \( \Omega_1 \):

\[
\begin{align*}
(i) & \quad L_N u^N_n + b_0 u^N_n = f \quad \text{at } (\Sigma^1_N)^0 \cup (\Gamma^\text{out}_1 \cup \Gamma^\text{in}_1)_N \cup (\Gamma^\text{out}_1 \cup \Gamma^0)_N, \\
(ii) & \quad [-b \cdot n_1(u^N_n - \phi)](P) = [L_N u^N_n + b_0 u^N_n - f](P)\frac{\omega^N_1}{\theta^1_P} \quad \text{at } P \in (\Gamma^\text{in}_1)_N, \\
(iii) & \quad [-b \cdot n_1(u^N_n - \psi^n)](P) = [L_N u^N_n + b_0 u^N_n - f](P)\frac{\omega^N_1}{\theta^1_P} \quad \text{at } P \in (\Gamma^\text{in}_N)_N;
\end{align*}
\]

and the following elliptic problem in \( \Omega_2 \):

\[
\begin{align*}
(i) & \quad \text{div}[\tau^2_N(\nu \nabla w^N_n)] + L^2_N w^N_n + b_0 w^N_n = f \quad \text{at } (\Sigma^2_N)^0, \\
(ii) & \quad w^N_n = \phi \quad \text{at } (\Gamma^\text{D}_2)_N, \\
(iii) & \quad [-\nu \frac{\partial w^N_n}{\partial n_2}](P) = R^2_P\frac{\omega^2_P}{\theta^2_P} \quad \text{at } P \in (\Gamma^\text{Nw}_2)_N, \\
(iv) & \quad [-\nu \frac{\partial w^N_n}{\partial n_2} + b \cdot n_2(w^N_n - u^N_n)](P) = R^2_P\frac{\omega^2_P}{\theta^2_P} \quad \text{at } P \in (\Gamma^\text{out}_2), \\
(v) & \quad [-\nu \frac{\partial w^N_n}{\partial n_2}](P) = R^2_P\frac{\omega^2_P}{\theta^2_P} \quad \text{at } P \in (\Gamma^\text{in}_N)_N.
\end{align*}
\]

The residue \( R^2_P \) was defined in (2.10); moreover, the initial values \( u^0_N \) and \( w^0_N \) have to be prescribed at the collocation points of \( \Gamma \). For \( n \geq 1 \), \( \psi^n \) is defined as follows, according to (4.3):

\[
\psi^n = \theta \psi^{n-1}_N + (1 - \theta) u^{n-1}_N, \quad \theta > 0
\]

Clearly, (4.39)-(4.40) is an iterative procedure that yields at the limit the solution to the finite dimensional coupled problem (2.2)-(2.9).

We define the following discrete inner products, associated with the Gauss-Lobatto collocation points:

\[
\begin{align*}
(i) & \quad (u, v)_{N, \Omega_i} = \sum_{P \in E^i_N} u(P)v(P)\omega^{(i)}_P, \\
(ii) & \quad (u, v)_{N, \Sigma} = \sum_{P \in E^\Sigma_N} u(P)v(P)\theta^{(i)}_P,
\end{align*}
\]

where \( \Sigma \) is any subset of \( \partial \Omega_i \) \( (\Sigma_N \text{ was defined at the beginning of section 2}) \) and \( u, v \) are defined in \( \Omega_i, i = 1, 2. \) Correspondingly, we introduce the norm associated to (4.42)(i):

\[
\|v\|_{N, \Omega_i} = (u, v)_{N, \Omega_i}^{\frac{1}{2}}, \quad i = 1, 2.
\]

Upon the space of algebraic polynomials of degree less than or equal to \( N \), this norm is equivalent to the continuous \( L^2 \) norm with constants independent of \( N \) (see, e.g., [7], Ch. 2). Precisely, there is a constant \( C_0 \) independent of \( N \) such that

\[
\|v\|_{0, \Omega_i} \leq \|v\|_{N, \Omega_i} \leq C_0\|v\|_{0, \Omega_i}, \quad i = 1, 2.
\]
for all $v \in P_N$. Finally, we introduce the following notation, in agreement with (3.9):
for all $v \in P_N$ and all subset $\Sigma$ of $\partial \Omega_1$, we set

$$|v|_{N,\Sigma} = (|b \cdot n_1|, v^2)_{N,\Sigma}^{1/2}. \quad (4.45)$$

We recall that $|v|_{N,\Sigma}$ is uniformly equivalent to the norm $|v|_\Sigma$ for all functions $v \in P_N$. In particular, there is a constant $C_1$ independent of $N$ such that

$$|v|_{N,\Sigma} \leq C_1 |v|_\Sigma. \quad (4.46)$$

The theoretical results we are going to give in the following three subsections will be derived in section 6.2 for the collocation method using Legendre-Gaussian points only (the case of Chebyshev points is not covered by our analysis).

4.2.1. The hyperbolic collocation problem in $\Omega_1$. The following stability result (which is the discrete analogue of (4.10)) holds for problem (4.39) (see section 6.2.1 for the proof).

**Lemma 4.12** Under the assumption (2.11), the solution $u_N^0$ to (4.39) satisfies

$$\beta_N u_N^0 \|_{N,\Omega_1}^2 + |u_N^0|_{N,\partial \Omega_1}^2 \leq \frac{1}{\beta_N} \|f\|_{N,\Omega_1}^2 + \|\phi\|_{N,\Gamma_1}^2 + \|\psi^0\|_{N,\Gamma_1}^2. \quad (4.47)$$

□

**Corollary 4.13** If in particular it is $f = 0$ and $\phi = 0$ on $\Gamma_1$, then the inequality (4.47) reduces to the following "outflow-inflow" estimate:

$$\beta_N u_N^0 \|_{N,\Omega_1}^2 + |u_N^0|_{N,\partial \Omega_1}^2 \leq \|\psi^0\|_{N,\Gamma_1}^2. \quad (4.48)$$

□

**Remark 4.14** (Boundary points in which $b \cdot n_1 = 0$). On the boundary subset $(\Gamma_1^b)_N$ we have collocated the differential equation, both in (2.2) and in (4.39). In other words, the collocation scheme here proposed treats $(\Gamma_1^b)_N$ as a part of the outflow boundary. In this way, the stability inequality (4.47) states that the discrete solution $u_N^0$ depends solely on the values of the "inflow data" on $\Gamma_1^b \cup \Gamma_1^b$. As a by-product, one deduces that $u_N^0 = 0$, whenever $f = 0$, $\phi = 0$ and $\psi^0 = 0$ (homogeneous case), i.e. the discrete hyperbolic problem has a unique solution. □

4.2.2. The elliptic collocation problem in $\Omega_2$. The following stability result holds for problem (4.40) (see section 6.2.2 for the proof).

**Lemma 4.15** Under the assumption (2.11), if $\phi = 0$ then the solution $u_N^0$ to problem (4.40) satisfies

$$2 \|\nabla u_N^0\|_{N,\Omega_2}^2 + \beta_N \|u_N^0\|_{N,\Omega_2}^2 + (|b \cdot n_2|, (u_N^0)^2)_{N,\Gamma_2} \leq \frac{1}{\beta_N} \|f\|_{N,\Omega_2}^2 + |u_N^0|_{N,\Gamma_2}^2. \quad (4.49)$$

□
The inequality (4.49) is the discrete counterpart of (4.19) for the homogeneous case.

4.2.3. Convergence of the discrete iterative procedure; existence and uniqueness results for the coupled collocation problem. To begin with, from the previous hyperbolic and elliptic stability estimates we deduce a global inequality (analogous to (4.27)) for the homogeneous case, which will be useful in proving convergence of the iterative scheme.

Under the same assumptions of Corollary 4.13 and Lemma 4.15, by a simple combination of (4.48) and (4.49) we get the following hyperbolic-elliptic inequality:

\[ 2\| \sqrt{N} w_N \|^2_{N, \Omega_2} + \beta_N \{ \| u_N \|^2_{N, \Omega_1} + \| w_N \|^2_{N, \Omega_2} \} + \| w_N \|^2_{N, \Omega_2} \leq \| \psi_N \|^2_{N, \Omega_2}. \]  

(4.50)

As a consequence of this estimate, we will prove the existence and uniqueness of the solution to the coupled collocation problem (2.2)-(2.9). Let us define the following error functions:

\[ e_1^n = u_N - u^n_N, \quad e_2^n = w_N - w^n_N, \]  

(4.51)

where \( u_N, w_N \) is the solution to problem (2.2)-(2.9), while \( u^n_N, w^n_N \) are the solutions to (4.39) and (4.40), respectively. The following theorem states the convergence of \( (u_N, w_N) \) to \( (u_N, w_N) \). The proof will be given in section 6.2.3.

THEOREM 4.16 There exists \( \delta > 0 \) such that for all \( \theta \in (1-\delta, 1+\delta) \) we have

\[ e_1^n \to 0, \quad e_2^n \to 0, \quad \text{as} \quad n \to +\infty. \]

Therefore, the sequence \( (u_N, w_N) \) converges to the solution \( (u_N, w_N) \) of the coupled collocation problem (2.2)-(2.9). Moreover, the rate of convergence is independent of \( N \).

REMARK 4.17 An argument analogous to that of Remark 4.11 can be repeated in the present case as well. \( \square \)

4.2.4. Some numerical results. Several numerical experiments based on the spectral collocation method (2.2)-(2.9) are reported in [2]. They show that the numerical solution is in very good agreement with the differential solution, for the commonly used case of collocation points of Chebyshev type, even without using the (expensive) skew-symmetric decomposition (2.1). Here we make some additional remarks about the convergence properties of our iterative procedure (4.39), (4.40).

There are situations in which the differential problem \( P \) decouples in a natural way into two problems, one of them being independent of the other. This is the case, for instance, when either \( \Gamma_{out} = \emptyset \), or \( \Gamma_{in} = \emptyset \) (see Figure 4.1).

Figure 4.1. Examples of convergence in one iteration: the orientation of the characteristic lines is shown in \( \Omega_1 \).
In these situations, our iterative procedure converges just in one iteration (in the former case, the elliptic problem is solved first, while in the latter the hyperbolic problem must be solved in advance). We note that this feature occurs in the first case because we used (1.12) instead of (1.9). This choice is helpful also in a more general context, namely when no characteristic lines exist entering \( \Omega_1 \) across \( \Gamma^{in} \) and leaving \( \Omega_1 \) across \( \Gamma^{out} \) (see Figure 4.2). In this case, the hyperbolic problem must be solved first and a successive resolution in \( \Omega_2 \) and \( \Omega_1 \) provides the exact solution.

![Figure 4.2. Examples of convergence in two iterations: the orientation of the characteristic lines is shown in \( \Omega_1 \).](image)

Here we report some results obtained for situations in which the coupling of the two problems is more severe. In all cases, we choose the following data:

\[
\begin{align*}
\Omega &= (-1,1) \times (-1,1), \quad \Omega_1 = (-1,0) \times (-1,1), \quad \Omega_2 = (0,1) \times (-1,1); \\
f &= 1, \quad b_0 = 1, \quad b = \alpha B.
\end{align*}
\]

\((N+1)^2\) Chebyshev collocation points are considered within each subdomain and the skew-symmetric decomposition (2.1) is not used. We denote by \( n_{it} \) the minimum value of the integer \( n \) such that

\[
\max \frac{|u^n - u^{n-1}|}{|u^n|} < 10^{-8},
\]

the maximum being taken on all grid points lying on \( \Gamma \).

Table 4.1 shows the values of \( n_{it} \) for various choices of \( \nu \), \( N \) and \( \alpha \) in the case \( B = (y,1)^T \).

Tables 4.2 and 4.3 deal with the case \( B = (y,-x)^T \). Table 4.2 shows the values of \( n_{it} \) for various choices of \( \nu \) and \( \alpha \), with \( N = 8 \). Table 4.3 shows the dependence of \( n_{it} \) on \( N \) and \( \alpha \), with \( \nu = 10^{-3} \).

In all of these cases, the relaxation parameter \( \theta \) appearing in (4.41) was chosen dynamically so as to minimize the interface error at each step.

For the case \( B = (y,1)^T \), Table 4.1 shows that the rate of convergence is independent of the number of grid points, while it has a mild dependence on the size of \( \nu \) and \( |b| \), in agreement with the results of our investigation.

For the other case \( B = (y,-x)^T \), Table 4.2 shows that the rate of convergence is still mildly depending on the size of \( \nu \) and \( b \), while Table 4.3 shows a dependence also on the number of grid points, when \( \alpha = 100 \). We conjecture that this pathology is due to the ill-conditioning of the problem (the rate \( |b|\nu^{-1} \) ranges from 0 to \( 10^5 \) on the computational domain).
one iteration (in the hyperbolic case in the first case in a more general setting) leaving the minimization in the last. Table 4.1.

<table>
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<th>$\alpha$</th>
<th>$N$</th>
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Table 4.2.

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Table 4.3.

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5. Vanishing viscosity approximation to problem (P). This section is devoted to the proof of existence of a solution to problem (P), stated in Theorem 3.2, via a "vanishing viscosity" argument. Throughout the section we shall assume (1.2), (1.3), (3.14), (3.15) and (3.16) without any further explicit mention.

Let $\epsilon > 0$ be a small parameter and choose sequences of $C^\infty$ functions $\{\phi_\epsilon\}$, $\{b_{0\epsilon}\}$, $\{(\nu_\epsilon)\}$ such that

\[
\phi_\epsilon \in C^\infty(\partial \Omega), \quad \phi_\epsilon|_{\Gamma^D} \rightarrow \phi|_{\Gamma^D} \text{ in } H^{\frac{3}{2}}(\Gamma^D), \quad \phi_\epsilon|_{\Gamma^{1*}} \rightarrow \phi|_{\Gamma^{1*}} \text{ in } L^2(\Gamma^{1*}); \quad (5.1)
\]

\[
b_{0\epsilon} \in C^\infty(\Omega), \quad \nu_\epsilon \in C^\infty(\Omega), \quad b_{0\epsilon} \rightharpoonup b_0 \text{ in } L^\infty(\Omega), \quad \nu_\epsilon \rightharpoonup \nu \text{ in } L^\infty(\Omega_2); \quad (5.2)
\]

as $\epsilon \to 0$ (in (5.2) denotes the weak* convergence): the reason for such a regularization on the data will be clear later on. As a consequence of (1.2) and (5.2), we see...
that
\[ \nu_\epsilon(x) \geq \frac{1}{2} \nu_0, \quad x \in \Omega_2, \quad \frac{1}{2} \text{div} b(x) + b_0(x) \geq \frac{1}{2} \beta_0, \quad x \in \Omega, \]
for \( \epsilon \) small enough: form now on we shall consider only values of \( \epsilon \) such that (5.3) holds. Set
\[ a_\epsilon(x) = \begin{cases} \epsilon & \text{if } x \in \Omega_1, \\ \nu_\epsilon(x) & \text{if } x \in \Omega_2. \end{cases} \]
(8.4)
Consider the closed, linear affine subspace of \( \mathbf{H}^1(\Omega) \):
\[ V_\epsilon = \{ v \in \mathbf{H}^1(\Omega) : v|_{\Gamma_2} = \phi_\epsilon \} \]
and the following variational problem:
\( (P_\epsilon) \): to find \( z_\epsilon \in V_\epsilon \) such that, for all \( v \in V_\epsilon \),
\[ \int_\Omega (a_\epsilon \nabla z_\epsilon - b z_\epsilon) \cdot (z_\epsilon - v) \, dx + \int_\Omega b_0 z_\epsilon (z_\epsilon - v) \, dx + \int_{\Gamma_1^{\text{out}}} |b \cdot n| z_\epsilon (z_\epsilon - v) \, ds = \]
\[ = \int_\Omega f (z_\epsilon - v) \, dx - \int_{\Gamma_1} b \cdot n_1 \phi_\epsilon (z_\epsilon - v) \, ds. \]
(5.6)
Note that in the last integral on the left hand side it is \( |b \cdot n| = b \cdot n_1 \) on \( \Gamma_1^{\text{out}} \) and \( |b \cdot n| = b \cdot n_2 \) on \( \Gamma_2^{\text{ext}} \) (by (1.3)).
As we shall see in a moment, \( (P_\epsilon) \) is equivalent to a boundary value problem quite similar to \( (P) \), the difference lying in the regularization on the data and in the presence of an elliptic singular perturbation (which affects also the boundary and interface conditions). To state this equivalence, we introduce the following notation which will be adopted from now on: if \( z_\epsilon \) is a solution to \( (P_\epsilon) \), we denote by
\[ u_\epsilon = z_\epsilon|_{\Omega_1} \text{ and } w_\epsilon = z_\epsilon|_{\Omega_2} \]
the restrictions of \( z_\epsilon \) to \( \Omega_1 \) and \( \Omega_2 \), respectively.

**Lemma 5.1** Let \( z_\epsilon \) be a solution to \( (P_\epsilon) \). Then
\[ -\epsilon \Delta u_\epsilon + \text{div} (b u_\epsilon) + b_0 u_\epsilon = f \quad \text{in } \Omega_1, \]
\[ \text{div} (-\nu_\epsilon \nabla w_\epsilon + bw_\epsilon) + b_0 w_\epsilon = f \quad \text{in } \Omega_2, \]
\[ -\epsilon \frac{\partial u_\epsilon}{\partial n_1} + b \cdot n_1 u_\epsilon = \frac{b \cdot n_1 \phi_\epsilon}{2} \quad \text{in } [\mathbf{H}^{\frac{1}{2}}(\Gamma_1^{\text{in}})]', \]
\[ \frac{\partial u_\epsilon}{\partial n_1} = 0 \quad \text{in } [\mathbf{H}^{\frac{1}{2}}(\Gamma_1^{\text{out}} \cup \Gamma_1^{\text{in}})]', \]
\[ w_\epsilon = \phi_\epsilon \quad \text{on } \Gamma_1^{\text{in}}, \]
\[ \nu_\epsilon \frac{\partial w_\epsilon}{\partial n_2} = 0 \quad \text{in } [\mathbf{H}^{\frac{1}{2}}(\Gamma_2^{\text{ext}})]', \]
\[ -\epsilon \frac{\partial w_\epsilon}{\partial n_1} + b \cdot n_1 w_\epsilon = \nu_\epsilon \frac{\partial w_\epsilon}{\partial n_2} - b \cdot n_2 w_\epsilon \quad \text{in } [\mathbf{H}^{\frac{1}{2}}(\Gamma)]', \]
\[ u_\epsilon = w_\epsilon \quad \text{on } \Gamma. \]
(5.10)
(5.11)
(5.12)
(5.13)
(5.14)
(5.15)

**Proof.** Let \( z_\epsilon \) solve \( (P_\epsilon) \) and write (5.6) with \( v = z_\epsilon + \psi \), where \( \psi \in \mathcal{D}(\Omega) \) (that is, a smooth function with compact support). It follows
\[ \text{div} (-a_\epsilon \nabla z_\epsilon + b z_\epsilon) + b_0 z_\epsilon = f \quad \text{in } \mathcal{D}'(\Omega), \]
whence (5.8)
Setting
(5.10) 
\( \psi = 0 \) in \( \Omega_2 \).
are obvious. Applying Green's

**Remark 5.2** Effective in a lot of cases, (3.16) is not stable under the assumptions of

**Theorem 5.3**

**Proof.**

\[ a(u, v) \]
is coercive in \( u, v \in V_\epsilon \),

\[ a(u - v, \]
Therefore, La property
whence (5.8) and (5.9) are obtained by localization. Setting
\[ F_\varepsilon = -a_\varepsilon \nabla z_\varepsilon + b z_\varepsilon \] (5.17)
(the flux corresponding to \( z_\varepsilon \)), by (5.16) we see that \( F_\varepsilon \in L^2_{\text{div}}(\Omega) \). Hence, Green’s formula (3.5) (with \( A = \Omega \) and \( \Sigma = \Gamma_1 \)) gives, for all \( \psi \in H^1(\Omega) \) with \( \psi = 0 \) on \( \Gamma_2 \),
\[ \int_\Omega \psi \text{div} F_\varepsilon dx + \int_\Omega F_\varepsilon \cdot \nabla \psi dx = \langle F_\varepsilon \cdot n_1, \psi \rangle. \]

Now, by the regularity assumption on \( b \), it is
\[ \langle F_\varepsilon \cdot n_1, \psi \rangle = -\langle \varepsilon \frac{\partial z_\varepsilon}{\partial n_1}, \psi \rangle + \int_{\Gamma_1} b \cdot n_1 u_\varepsilon \psi ds. \]

Thus, (5.10) and (5.11) follow easily by taking \( v = z_\varepsilon + \psi \) in (5.6), with \( \psi \in C^\infty(\tilde{\Omega}) \), \( \psi = 0 \) in \( \Omega_2 \). An analogous argument shows (5.13). Since \( z_\varepsilon \in V_\varepsilon \), (5.12) and (5.15) are obvious. Finally, (5.14) may be proved by a standard argument which consists in applying Green’s formula separately to \( \Omega_1 \) and \( \Omega_2 \), then using (5.8) and (5.9). \( \square \)

**REMARK 5.2** The way we have chosen for regularizing problem (P) has proven effective in a low regularity framework such as ours: remind that the original inflow data satisfies (3.16), hence a pure Dirichlet condition on \( \Gamma_1^{\text{in}} \) for the approaching problems is not stable as \( \varepsilon \to 0 \). Moreover, the a priori estimates we are going to obtain are strongly based on the type of regularization we adopted. \( \square \)

Existence and uniqueness of the solution to problem \( (P_\varepsilon) \) is a consequence of our assumptions (in particular, of (5.3) and (1.3)).

**THEOREM 5.3** For all \( \varepsilon > 0 \), problem \( (P_\varepsilon) \) has one and only one solution \( z_\varepsilon \).

**Proof.** The bilinear, continuous form appearing in (5.6)
\[ a(u, v) = \int_{\Omega} (a_\varepsilon \nabla u - b_\varepsilon u) \cdot \nabla v \, dx + \int_{\Gamma^{\text{in}}_1 \cup \Gamma^{\text{out}}_2} b \cdot n_1 |uv| \, ds + \int_{\Omega} b_{0\varepsilon} uv \, dx \]
is coercive in \( V_\varepsilon - V_\varepsilon \) (algebraic difference). In fact, Green’s formula (3.7) gives, for all \( u, v \in V_\varepsilon \),
\[ a(u - v, u - v) = \int_{\Omega} a_\varepsilon |\nabla (u - v)|^2 \, dx + \int_{\Omega} \left( \frac{1}{2} \text{div} b + b_{0\varepsilon} \right) (u - v)^2 \, dx + \]
\[ + \frac{1}{2} \int_{\Gamma^{\text{in}}_1 \cup \Gamma^{\text{out}}_2} |b \cdot n_1 (u - v)|^2 \, ds - \frac{1}{2} \int_{\Gamma^{\text{in}}_1} b \cdot n_1 (u - v)^2 \, ds \geq \]
\[ \geq \min(\varepsilon, \frac{1}{2} \nu_0) \int_{\Omega} |\nabla (u - v)|^2 \, dx + \frac{1}{2} \beta_0 \int_{\Omega} (u - v)^2 \, dx + \]
\[ + \frac{1}{2} \int_{\Gamma^{\text{in}}_1 \cup \Gamma^{\text{out}}_2} |b \cdot n| (u - v)^2 \, ds \geq \min(\varepsilon, \frac{1}{2} \nu_0, \frac{1}{2} \beta_0) \|u - v\|_{L^2(\Omega)}^2. \]

Therefore, Lax-Milgram Lemma gives existence and uniqueness of the solution. These properties are not uniform in \( \varepsilon \), since the coerciveness constant vanishes as \( \varepsilon \to 0 \). \( \square \)
REMকা Remark 5.4 Assumption (1.3) has proven convenient for coerciveness, although it is not necessary. For instance, one can do without it in the coerciveness computation, whenever the integral on $\Gamma^N_{N'}$ can be absorbed by the $H^1$ norm near $\Gamma^N_{N'}$. This requires a balance between $u_\epsilon$, $b$ and $b_{0e}$ we do not detail here. □

Now we are in a position to begin our asymptotic analysis, as $\epsilon \to 0$.

**Lemma 5.5** There is a constant $C > 0$ such that, for all $\epsilon > 0$,

\begin{align}
\|z_\epsilon\|_{0, \Omega} & \leq C, \\
\sqrt{\epsilon}\|\nabla u_\epsilon\|_{0, \Omega} & \leq C, \\
\|\nabla w_\epsilon\|_{0, \Omega} & \leq C.
\end{align}

(5.18) (5.19) (5.20)

**Proof.** For all $\epsilon > 0$, let $u_\epsilon \in H^1(\Omega)$ be a function such that

$$v_\epsilon|_{\Gamma^N_{N'}} = \phi_\epsilon, \quad \|u_\epsilon\|_{1, \Omega} \leq C\|\phi_\epsilon\|_{1, \Gamma^N_{N'}}$$

(there are such $v$'s). By (5.1), it follows that

$$\|v_\epsilon\|_{1, \Omega} \leq M,$$

(5.21)

for some $M > 0$. Write (5.6) for $v = v_\epsilon$ and apply Green's formula (3.7):

$$\int_{\Omega} a_\epsilon \nabla z_\epsilon \cdot \nabla z_\epsilon \, dx + \int_{\Omega} \left( \frac{1}{2} \div b + b_{0e} \right) z_\epsilon^2 \, dx + \frac{1}{2} \int_{\Gamma^N_{N'}} b \cdot n |z_\epsilon|^2 \, ds =$$

$$\int_{\Omega} a_\epsilon \nabla z_\epsilon \cdot \nabla v_\epsilon \, dx - \int_{\Omega} (b_0 \cdot v_\epsilon) \nabla v_\epsilon \, dx + \int_{\Omega} b_{0e} z_\epsilon v_\epsilon \, dx + \frac{1}{2} \int_{\Gamma^N_{N'}} b \cdot n_2 \phi_\epsilon^2 \, ds +$$

$$+ \int_{\Gamma^N_{N'}} b \cdot n |z_\epsilon|_1 \, ds - \int_{\Gamma^N_{N'}} b \cdot n_1 \phi_\epsilon (u_\epsilon - v_\epsilon) \, ds + \int_{\Omega} f (z_\epsilon - v_\epsilon) \, dx.$$

This inequality, together with (5.3), (1.3) and the well known algebraic inequality $ab \leq \delta a^2 + \frac{1}{4\delta} b^2$, valid for all $a, b \geq 0$ and all $\delta > 0$, gives

$$\epsilon \|\nabla u_\epsilon\|_{0, \Omega}^2 + \frac{1}{2} \|\nabla w_\epsilon\|_{0, \Omega}^2 + \frac{1}{2} \|\phi_\epsilon\|_{0, \Omega}^2 + \frac{1}{2} \|u_\epsilon\|_{1, \Omega}^2 + \frac{1}{2} |u_\epsilon|_{\Gamma^N_{N'}}^2 + \frac{1}{2} |w_\epsilon|_{\Gamma^N_{N'}}^2 \leq$$

$$\leq \delta \epsilon \|\nabla u_\epsilon\|_{0, \Omega}^2 + \delta \|\nabla w_\epsilon\|_{0, \Omega}^2 + \frac{1}{4\delta} \|w_\epsilon\|_{0, \Omega}^2 +$$

$$+ \{ \delta \|z_\epsilon\|_{0, \Omega}^2 + \frac{1}{4\delta} \|v_\epsilon\|_{1, \Omega}^2 \} \{ \|b\|_{1, \infty, \Omega} + \|b_{0e}\|_{\infty, \Omega} \} + \delta \|\phi_\epsilon\|_{0, \Omega} +$$

$$\|v_\epsilon\|_{1, \Omega} + (1 + \frac{1}{4\delta}) \|\phi_\epsilon\|_{1, \Omega} + \delta |u_\epsilon|_{\Gamma^N_{N'}}^2 + \frac{1}{2\delta} |\phi_\epsilon|_{\Gamma^N_{N'}}^2 + \delta |v_\epsilon|_{\Gamma^N_{N'}}^2 +$$

$$+ \delta |w_\epsilon|_{\Gamma^N_{N'}}^2 + \frac{1}{4\delta} |v_\epsilon|_{\Gamma^N_{N'}}^2 + \frac{1}{2} |\phi_\epsilon|_{\Gamma^N_{N'}}^2 + \frac{1}{4\delta} |v_\epsilon|_{\Gamma^N_{N'}}^2 + \delta |u_\epsilon|_{\Gamma^N_{N'}}^2.$$

(5.22)

Hence, (5.17)-(5.20) follow by (5.1), (5.2) and (5.21). □

We note here that (5.22) gives an additional information on some boundary integrals that will be exploited in section 6.

Lemma 5.5 implies the following Corollary immediately.

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THEOREM 5

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\[ (5.2), (5.24) \]

\[ (5.24). \]

\[ \text{Morec} \]

\[ \left[ H^1_{00}(\Gamma) \right]' \an\]
COROLLARY 5.6 There exist
\[ u \in L^2(\Omega_1), \quad w \in H^1(\Omega_2) \] (5.23)
such that, possibly taking subsequences,
\[ u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(\Omega_1); \quad w_\varepsilon \rightharpoonup w \text{ weakly in } H^1(\Omega_2). \] (5.24)
Moreover,
\[ \varepsilon \nabla u_\varepsilon \to 0 \text{ strongly in } L^2(\Omega_1). \] (5.25)
\[ \square \]

Analogous results hold for the flux \( F_\varepsilon \) defined in (5.17).

PROPOSITION 5.7 There is a constant \( C > 0 \) such that, for all \( \varepsilon > 0 \),
\[ \| F_\varepsilon \|_{L^2_{\text{div}}(\Omega)} \leq C. \] (5.26)

Consequently, on a subsequence, we find that
\[ F_\varepsilon \rightharpoonup F \text{ weakly in } L^2_{\text{div}}(\Omega), \] (5.27)
where
\[ F = \begin{cases} bu & \text{in } \Omega_1, \\ -\nu \nabla w + bw & \text{in } \Omega_2, \end{cases} \] (5.28)
u, w being the functions provided by Corollary 5.6.

**Proof.** The estimate (5.26) follows easily by Lemma 5.5. Consequently, there exists some \( F \in L^2_{\text{div}}(\Omega) \) such that (5.27) holds. By uniqueness of the \( L^2(\Omega) \) weak limit, (5.2), (5.24) and (5.25) give formula (5.28) for \( F \). \( \Box \)

The limit functions \( u \) and \( w \) satisfy differential equations and boundary and interface conditions. A first result is provided by next theorem.

THEOREM 5.8 The functions \( u \) and \( w \) provided by Corollary 5.6 satisfy:
\[ \text{div} (bu) + b_0 u = f \quad \text{in } D'(\Omega_1), \] (5.29)
\[ \text{div} (-\nu \nabla w + bw) + b_0 w = f \quad \text{in } D'(\Omega_2), \] (5.30)
\[ b \cdot n_1 u = b \cdot n_1 \phi \quad \text{in } \left[ H^\frac{1}{2}_0(\Gamma_1^\text{in}) \right] ', \] (5.31)
\[ w = \phi \quad \text{on } \Gamma_2^D, \] (5.32)
\[ \nu \frac{\partial w}{\partial n_2} = 0 \quad \text{in } \left[ H^\frac{1}{2}_0(\Gamma_2^\text{ne}) \right] ', \] (5.33)
\[ -b \cdot n_1 u = -\nu \frac{\partial w}{\partial n_2} + b \cdot n_2 w \quad \text{in } \left[ H^\frac{1}{2}_0(\Gamma) \right] '. \] (5.34)

**Proof.** Let \( \psi_i \in D(\Omega_i), i = 1, 2 \), and choose \( v = z_\varepsilon + \psi_i \) in (5.6). Thanks to (5.2), (5.24) and (5.25), we may take the limit in (5.6), obtaining the two differential equations (5.29) and (5.30). The boundary condition (5.32) follows by (5.1), (5.12) and (5.24). Moreover, (5.27) implies the weak convergence of \( F_\varepsilon \cdot n \) to \( F \cdot n \) in \( \left[ H^\frac{1}{2}_0(\Gamma^\text{in}) \right] ', \left[ H^\frac{1}{2}_0(\Gamma) \right] ' \) and \( \left[ H^\frac{1}{2}_0(\Gamma^\text{ne}) \right] ' \), so that (5.28), (5.10), (5.1) and Lemma 3.1, part (ii),
give (5.31). Again (5.28), (5.24) and (5.13) imply (5.33), while (5.34) follows by (5.28) and (5.14). Finally, assumption (3.16) on $\phi$ and Lemma 3.1, part (i), (with $A = \Omega$, $\Sigma = \Gamma^m$, $\rho = |b \cdot n|$, $v = u$ and $z = \phi$) show that $u|_{\Gamma^m} \in L^2(\Gamma^m)$ and that the boundary condition (5.31) gives $u = \phi$ a.e. on $\Gamma^m$, which is the sense stated in Theorem 3.2. □

To complete the proof of Theorem 3.2, we must show that $u$ and $w$ satisfy a further condition on $\Gamma^m$. This needs a preliminary study of higher regularity for $u_\varepsilon$ and $w_\varepsilon$.

**Proposition 5.9** The solution $z_\varepsilon$ to $(P_\varepsilon)$ is such that $a_\varepsilon \nabla z_\varepsilon \in H^1(\Omega \setminus \Lambda)$, where $\Lambda$ is an arbitrarily small neighborhood of $\partial \Omega$.

**Proof.** Since the local regularity result is well known in the interior of $\Omega_\varepsilon$, here we check the local regularity near a point $P \in \Gamma$, where the transmission condition is enforced.

After a localization and flattening of $\Gamma$, we are left with the problem of finding $\tilde{z}_\varepsilon \in H^1(\mathbb{R}^2)$ such that, for all $v \in H^1(\mathbb{R}^2)$,

$$
\int_{\mathbb{R}^2} \left( \tilde{a}_\varepsilon \nabla \tilde{z}_\varepsilon - \tilde{b}_\varepsilon \tilde{z}_\varepsilon \right) \cdot \nabla v \, dx + \int_{\mathbb{R}^2} \tilde{b}_0 \varepsilon \tilde{z}_\varepsilon v \, dx = \int_{\mathbb{R}^2} \tilde{f} v \, dx
$$

(tilde's take into account that the original solution has been multiplied by a localizing factor and that a change of variables has been made in order to flatten the interface; note that $\tilde{z}_\varepsilon$ shows a jump across $\{x_2 = 0\}$ and that it behaves like $\varepsilon$ on $\{x_2 < 0\}$).

We note that (5.35) is equivalent to the differential equation

$$
\nabla \cdot \left( -a_\varepsilon \nabla \tilde{z}_\varepsilon + b \tilde{z}_\varepsilon \right) + \tilde{b}_0 \varepsilon \tilde{z}_\varepsilon = \tilde{f} \quad \text{in} \; \mathcal{D}'(\mathbb{R}^2).
$$

To shorten notations, we will drop the tildas, the subscript $\varepsilon$ and the domain of integration $\mathbb{R}^2$ in the rest of this computation. Moreover, we will denote by $C, C(\varepsilon), \ldots$ possibly different constants.

With the aim of applying a difference quotients technique, we fix $h > 0$ and set, for a given $v$ defined in $\mathbb{R}^2$,

$$
v_+(x_1, x_2) = v(x_1 + h, x_2), \quad (x_1, x_2) \in \mathbb{R}^2; \quad \delta_h v = v_+ - v.
$$

Plug $v = \delta_h z$ in (5.35): it follows

$$
\int (a \nabla z - bx) \cdot \nabla (\delta_h z) \, dx + \int b_0 z (\delta_h z) \, dx = \int f (\delta_h z) \, dx.
$$

Next, make the change of variables $(x_1, x_2) \mapsto (x_1 + h, x_2)$ in (5.35), then choose $v_+ = \delta_h z$ in the transformed equation: it follows

$$
\int (a_+ \nabla x_1 + b_+ x_2) \cdot \nabla (\delta_h z) \, dx + \int b_{0+} z_+ (\delta_h z) \, dx = \int f_+ (\delta_h z) \, dx.
$$

Taking the difference between (5.37) and (5.38), observing that $\nabla$ and $\delta_h$ commute and recalling the identity $\delta_h (uv) = u \delta_h v + v_+ \delta_h u$, we find

$$
- \int a_+ |\nabla (\delta_h z)|^2 \, dx - \int \delta_h a \nabla z \cdot \nabla (\delta_H z) \, dx + \int b_+ \delta_h z \cdot \nabla (\delta_H z) \, dx +
\int z \delta_h b \cdot \nabla (\delta_h z) \, dx - \int b_{0+} |\delta_h z|^2 \, dx - \int \delta_h b_0 z \delta_h z \, dx = - \int \delta_h f \delta_h z \, dx.
$$

Therefore,

$$
\int a_+ |\nabla (\delta_h z)|^2 \, dx + \int b_+ \delta_h z \cdot \nabla (\delta_H z) \, dx + \int z \delta_h b \cdot \nabla (\delta_h z) \, dx
$$

where $\| \cdot \|_{-1}$ and, for $h$ small enough,

$$
\| \delta_h a \|.
$$

Thus, (5.39) is true because $a$ is not a local function of $\varepsilon$.

Taking the limit as $h \to 0$ and recalling that (5.40) is true,

so that (5.40)

that is

Note that the variables, (5.46) proof is complete.

**Remark 5.** which will be the solution $r$ compatible.

The local estimate for $z$

**Lemma 5.11**

*a neighborhood*
Therefore, 
\[
\int a_+ |\nabla (\delta_h z)|^2 dx \leq \| \delta_h a\|_\infty \| \nabla z\|_0 + \| b\|_\infty \| \delta_h z\|_0 + \| b_0\|_\infty \| \delta_h z\|_0^2 + \\
+ \| \delta_h b\|_\infty \| z\|_0 \| \nabla (\delta_h z)\|_0 + \| b_0\|_\infty \| \delta_h z\|_0^2 + \\
+ \| \delta h b_0\|_\infty \| z\|_0 \| \delta h z\|_0 + \| \delta_h f\|_{-1} \| \delta_h z\|_1,
\]  
(5.39)

where \( \| \cdot \|_{-1} \) denotes the norm of the dual space of \( \mathcal{H}^1(\Omega) = \mathcal{H}^1_0(\Omega) \).

Now, 
\[
\| z\delta_h z\|_0 \leq \| z\|_0 \| \delta_h z\|_0, \quad \| \delta_h z\|_0^2 = \| \delta_h z\|_0^2 + \| \nabla (\delta_h z)\|_0^2
\]
and, for \( h \) small,
\[
\| \delta_h a\|_\infty \leq Ch \| \frac{\partial a}{\partial x_1} \|_\infty, \quad \| \delta_h b\|_\infty \leq C h \| b\|_1, \quad \| \delta h b_0\|_0 \leq C h \| b_0\|_1, \\
\| \delta_h f\|_{-1} \leq C h \| f\|_0, \quad \| \delta h z\|_0 \leq C h \| z\|_1.
\]
Thus, (5.39) gives in a standard way
\[
\| \nabla (\delta_h z)\|_0^2 \leq C(\epsilon) h^2 \{ \| a\|_{L_\infty}^2 + \| b\|_{L_\infty}^2 + \| b_0\|_1^2 + \| f\|_0^2 + \| z\|_1^2 \}.
\]

Taking the limit as \( h \to 0 \), we find
\[
\frac{\partial}{\partial x_1} (a \nabla z) \in L^2(\Omega) \quad \text{and} \quad \| \frac{\partial}{\partial x_1} (a \nabla z)\|_0 \leq C(\epsilon),
\]  
(5.40)

because \( a \) is regular in the \( x_1 \) direction. By (5.36), we know that
\[
\frac{\partial}{\partial x_2} (a \nabla z) = -\frac{\partial}{\partial x_1} (a \nabla z) + \text{div} (bz) + b_0 x - f,
\]
so that (5.40) implies
\[
\| \frac{\partial}{\partial x_2} (a \nabla z)\|_0 \leq C(\epsilon),
\]
that is
\[
\| (a \nabla z)\|_1 \leq C(\epsilon).
\]  
(5.41)

Note that the constant \( C(\epsilon) \) is unbounded as \( \epsilon \to 0 \). Going back to the original variables, (5.41) gives the local \( \mathcal{H}^1 \) regularity of \( a_\epsilon \nabla z_\epsilon \) near the point \( P \), whence the proof is complete. \( \square \)

REMARK 5.10 We did not care about the regularity near the fixed boundary \( \partial \Omega \), which will be of no use in the sequel. We just warn the reader that \( \mathcal{H}^2 \) regularity of the solution near a point of \( \partial \Omega \) may not hold, even for very smooth data, unless some compatibility condition depending also on the geometry of \( \Omega \) is satisfied. \( \square \)

The local regularity provided by Proposition 5.9 enables us to get an a priori estimate for \( z_\epsilon \).

LEMMA 5.11 Let \( \psi \) be a nonnegative, smooth function defined in \( \Omega \), vanishing in a neighborhood of \( \partial \Omega \cup \Gamma_{\text{out}} \). There is a constant \( C_\psi > 0 \) such that, for all \( \epsilon \),
\[
\int \psi |\text{div} (bz_\epsilon)|^2 dx \leq C_\psi.
\]  
(5.42)
Proof. Let \( \psi \) be such a function and take the \( L^2(\Omega) \) scalar product of (5.16) by \( \psi \) \( \text{div} (b z_c) \). Recalling (5.7), it follows:

\[
- \epsilon \int_{\Omega_1} \Delta u_c \psi b \cdot \nabla u_c \, dx - \epsilon \int_{\Omega_1} \Delta u_c \psi \text{div} \, b \, u_c \, dx - \int_{\Omega_2} \text{div} (\nu_c \nabla w_c) \psi b \cdot \nabla w_c \, dx - \int_{\Omega_2} (\nu_c \nabla w_c) \psi \text{div} \, b \, w_c \, dx + \\
+ \int_{\Omega} \psi |\text{div} (b z_c)|^2 \, dx + \int_{\Omega} \psi b_{0c} \cdot \nabla \text{div} (b z_c) \, dx = \int_{\Omega} \psi f \text{div} (b z_c) \, dx. \tag{5.43}
\]

Proposition 5.9 allows us to apply Green's formula separately to \( \Omega_1 \) and \( \Omega_2 \), since the restriction of \( \psi \) \( \text{div} (b z_c) \) to \( \Omega_i \) belongs to \( H^1(\Omega_i) \), \( i = 1, 2 \). Therefore, we find

\[
- \epsilon \int_{\Omega_1} \Delta u_c \psi b \cdot \nabla u_c \, dx = \epsilon \int_{\Omega_1} \nabla u_c \cdot \nabla \psi b \cdot \nabla u_c + \\
\epsilon \int_{\Omega_1} \psi \nabla u_c \cdot (\nabla b \cdot \nabla u_c) \, dx - \epsilon \int_{\partial \Omega_1} \psi \frac{\partial u_c}{\partial n_1} b \cdot \nabla u_c \, ds. \tag{5.44}
\]

The second integral on the right hand side of (5.44) may be worked out as follows:

\[
\epsilon \int_{\Omega_1} \psi \nabla u_c \cdot (\nabla b \cdot \nabla u_c) \, dx = \frac{\epsilon}{2} \int_{\Omega_1} \psi b \cdot \nabla (|\nabla u_c|^2) \, dx + \epsilon \int_{\Omega_1} \psi \nabla u_c \cdot (\nabla u_c \cdot \nabla b) \, dx = \\
= - \frac{\epsilon}{2} \int_{\Omega_1} \text{div} (\psi b) |\nabla u_c|^2 \, dx + \frac{\epsilon}{2} \int_{\partial \Omega_1} \psi \cdot b \cdot n_1 |\nabla u_c|^2 \, ds + \epsilon \int_{\Omega_1} \psi \nabla u_c \cdot (\nabla u_c \cdot \nabla b) \, dx.
\]

Therefore, the first integral in (5.43) becomes:

\[
- \epsilon \int_{\Omega_1} \Delta u_c \psi b \cdot \nabla u_c \, dx = \epsilon \int_{\Omega_1} \nabla u_c \cdot \nabla \psi b \cdot \nabla u_c + \\
+ \epsilon \int_{\Omega_1} \psi \nabla u_c \cdot (\nabla u_c \cdot \nabla b) \, dx - \epsilon \int_{\partial \Omega_1} \psi \frac{\partial u_c}{\partial n_1} b \cdot \nabla u_c \, ds. \tag{5.45}
\]

Analogously, the third integral in (5.43) becomes:

\[
- \int_{\Omega_2} \text{div} (\nu_c \nabla w_c) \psi b \cdot \nabla w_c \, dx = \int_{\Omega_2} \nu_c \nabla w_c \cdot \nabla \psi b \cdot \nabla w_c + \\
+ \int_{\Omega_2} \nu_c \psi \nabla w_c \cdot (\nabla w_c \cdot \nabla b) \, dx - \int_{\partial \Omega_2} \nu_c \psi \frac{\partial w_c}{\partial n_2} b \cdot \nabla w_c + \\
- \frac{1}{2} \int_{\partial \Omega_2} \nu_c \psi \frac{\partial w_c}{\partial n_2} b \cdot \nabla w_c + \frac{1}{2} \int_{\partial \Omega_2} \nu_c \psi b \cdot n_2 |\nabla w_c|^2 \, ds. \tag{5.46}
\]

Using (5.16), second and fourth integrals in (5.43) become respectively:

\[
- \epsilon \int_{\Omega_1} \Delta u_c \psi \text{div} \, b \, u_c \, dx = \int_{\Omega_1} (f - \text{div} (b u_c) - b_{0c} u_c) \psi u_c \text{div} \, b \, dx, \tag{5.47}
\]

Taking (5.45)

\[
\int_{\Omega} \psi |\text{div} (b z_c)|^2 \, dx - \int_{\partial \Omega_3} \psi \left( \frac{1}{2} |\nabla z_c|^2 - \frac{1}{2} b_{0c}^2 |\nabla w_c|^2 \right) \, ds \leq C(\psi)
\]

for a constant

\[
\{\}
\]

where \( C(\psi) \) is

Now we recall that on the support to \( \Gamma^{in} \), orient \( t_1 = -t_2 \). The along the dire

Moreover, the

Therefore, the

Each of the tv \( b \cdot n_1 < 0 \) on 1

left hand side

REMARK 5.1

used only to a
\[
- \int_{\Omega_2} \text{div}(\nu \nabla w_\epsilon) \psi \text{div} \, b \, w_\epsilon \, dx = \int_{\Omega_2} (f - \text{div} (bw_\epsilon) - b_0 \cdot w_\epsilon) \psi u_\epsilon \text{div} \, b \, dx. \tag{5.48}
\]

Taking (5.45)-(5.48) into account, (5.43) becomes

\[
\int_{\Omega} \psi |\text{div} (b \cdot w_\epsilon)|^2 \, dx - \epsilon \int_{\partial \Omega_1} \psi \frac{\partial u_\epsilon}{\partial n_1} b \cdot \nabla u_\epsilon \, ds + \frac{\epsilon}{2} \int_{\partial \Omega_1} \psi b \cdot n_1 |\nabla u_\epsilon|^2 \, ds - \int_{\partial \Omega_2} \psi \nu \epsilon \frac{\partial w_\epsilon}{\partial n_2} b \cdot \nabla w_\epsilon \, ds + \frac{1}{2} \int_{\partial \Omega_2} \psi \nu \epsilon \cdot b \cdot n_2 |\nabla w_\epsilon|^2 \, ds \leq C(\psi) \left\{ \|b\|_{1, \infty, \Omega} \left[ \|\sqrt{\epsilon} \nabla u_\epsilon\|_{0, \Omega}^2 + \|\sqrt{\epsilon} \nabla w_\epsilon\|_{0, \Omega}^2 \right] + \|\sqrt{\epsilon} \text{div} (b \cdot w_\epsilon)\|_{0, \Omega} + \|b\|_{1, \infty, \Omega} \|z_\epsilon\|_{0, \Omega} \right\} \tag{5.49}
\]

for a constant \(C(\psi)\) independent of \(\epsilon\). Recalling (5.2), (5.24) and (5.25), we have that

\[
\{\text{right hand side of (5.49)}\} \leq C(\psi) \{1 + \|\sqrt{\epsilon} \text{div} (b \cdot w_\epsilon)\|_{0, \Omega}\}, \tag{5.50}
\]

where \(C(\psi)\) is a new constant, depending on \(\Omega, b, b_0, \nu, f, \phi\) and \(\psi\), but not on \(\epsilon\). Now we deal with the boundary integrals on the left hand side of (5.49). To begin with, we recall that the only nontrivial contribution is along \(\Gamma^{in}\), because of the assumption on the support of \(\psi\). Next, denote by \(t_1\) (respectively, \(t_2\)) the unit vector tangent to \(\Gamma^{in}\), oriented in the usual way with respect to \(n_1\) (respectively, \(n_2\)); note that \(t_1 = -t_2\). Thanks to Proposition 5.9, the gradient of \(z_\epsilon\) on \(\Gamma^{in}\) may be decomposed along the directions \(n_1\) and \(t_1\), that is

\[
\nabla w_\epsilon = \frac{\partial w_\epsilon}{\partial n_2} n_2 + \frac{\partial w_\epsilon}{\partial t_2} t_2, \quad \nabla u_\epsilon = \frac{\partial u_\epsilon}{\partial n_1} n_1 + \frac{\partial u_\epsilon}{\partial t_1} t_1 \text{ on } \Gamma^{in}. \tag{5.51}
\]

Moreover, the transmission condition (5.14) gives

\[
\frac{\partial u_\epsilon}{\partial n_2} = -\epsilon \frac{\partial u_\epsilon}{\partial n_1} \quad \text{on } \Gamma^{in}. \tag{5.52}
\]

Therefore, the boundary integrals on the left hand side of (5.49) become

\[
- \epsilon \int_{\partial \Omega_1} \psi \frac{\partial u_\epsilon}{\partial n_1} b \cdot \nabla u_\epsilon \, ds + \frac{\epsilon}{2} \int_{\partial \Omega_1} \psi b \cdot n_1 |\nabla u_\epsilon|^2 \, ds - \int_{\partial \Omega_2} \psi \nu \epsilon \frac{\partial w_\epsilon}{\partial n_2} b \cdot \nabla w_\epsilon \, ds + \frac{1}{2} \int_{\partial \Omega_2} \psi \nu \epsilon \cdot b \cdot n_2 |\nabla w_\epsilon|^2 \, ds = \frac{1}{2} \left[ \int_{\Gamma^{in}} (\nu - \epsilon) \psi b \cdot n_1 \frac{\partial u_\epsilon}{\partial t_1}^2 \, ds + \int_{\Gamma^{in}} \epsilon (1 - \frac{1}{\nu}) \psi b \cdot n_1 \frac{\partial u_\epsilon}{\partial t_1}^2 \, ds \right]. \tag{5.56}
\]

Each of the two boundary integrals between square brackets is nonpositive, for \(\epsilon\) small, because \(b \cdot n_1 < 0\) on \(\Gamma^{in}_1 \cup \Gamma^{in}_2\). Therefore, the contribution of the boundary integrals on the left hand side of (5.49) is nonnegative and (5.42) follows easily by (5.50). \(\square\)

REMARK 5.12 We note that the local regularity result of Proposition 5.9 has been used only to apply Green's formula, to give the boundary integrals in (5.49) a meaning...
and to decompose the gradient along the normal and the tangent directions. In particular, we need not the $H^1$ local estimates be uniform in $\varepsilon$ (which cannot be expected). □

As a consequence of Lemma 5.11, we get a convergence result.

**PROPOSITION 5.13** Let $\Omega'$ be any open subset of $\Omega_1$, such that $\Omega' \cap [\Gamma_1 \cup \Gamma_{\text{out}}] = \emptyset$. Possibly taking a subsequence, $b_{u_\varepsilon}$ converges to $b_u$ weakly in $L^2_{\text{div}}(\Omega')$, as $\varepsilon \to 0$ ($u$ is the function found in Corollary 5.6).

**Proof.** By (5.17) and (5.42), $b_{u_\varepsilon}$ is bounded in $L^2_{\text{div}}(\Omega')$, uniformly in $\varepsilon$. Therefore, a subsequence must converge weakly in $L^2_{\text{div}}(\Omega')$. Possibly taking a further subsequence, by (5.24), the limit must be $b_u$. □

We are in a position to find the remaining interface condition fulfilled by the limit functions $u$ and $w$.

**PROPOSITION 5.14** The functions $u$, $w$ defined in Corollary 5.6 satisfy

$$-b \cdot n_1 u = b \cdot n_2 w \quad \text{in} \quad \left[ H^\frac{1}{2}_{00}(\Gamma^{in}) \right]' .$$  \hfill (5.51)

**Proof.** Let $\Gamma'$ be any (relatively) open subset of $\Gamma^{in}$, with $\overline{\Gamma'} \subset \Gamma^{in}$. Let $\Omega'$ be the intersection of $\Omega_1$ with a small neighborhood of $\Gamma'$. By Proposition 5.13, $b_{u_\varepsilon}$ converges to $b_u$ weakly in $L^2_{\text{div}}(\Omega')$, whence $b \cdot n_1 u_\varepsilon$ converges to $b \cdot n_1 u$ weakly in $H^{-\frac{1}{2}}(\partial \Omega')$. Choosing test functions in $H^\frac{1}{2}_{00}(\Gamma')$, it follows that the convergence is weak in $\left[ H^\frac{1}{2}_{00}(\Gamma') \right]'$. Recalling (3.15) and (5.15), we get that

$$-b \cdot n_1 u = b \cdot n_2 w \quad \text{in} \quad \left[ H^\frac{1}{2}_{00}(\Gamma') \right]' .$$

Since $\Gamma'$ is arbitrary, this gives (5.51). Again, Lemma 3.1, part (i), allows us to read condition (5.51) as $u = w$ a.e. on $\Gamma^{in}$, which is the sense stated in Theorem 3.2. □

The proof of the existence part of Theorem 3.2 is complete. Clearly, the very procedure we have followed gives (P) as a limit of globally elliptic problems, as stated in the existence theorem. The uniqueness part will be proved by means of a different argument in section 6.1.3.

6. The iterative procedure of section 4: the proofs. In this section we prove the results stated in section 4.

6.1. The differential case. We begin by detailing the differential problem first, distinguishing between the hyperbolic and the elliptic subproblems.

6.1.1. Proofs of the results of section 4.1.1. Our goal is to prove Theorem 4.1. The strategy of approaching $(P_H)$ by regularized problems has proven effective also in this case, yielding the a priori estimate (4.7). As a by-product, we will also get a direct proof of existence and uniqueness of a solution to $(P_H)$.

Let $\varepsilon > 0$ be a small parameter and choose sequences of functions $\{\lambda_\varepsilon\}$, $\{b_{0_\varepsilon}\}$ satisfying

$$\lambda_\varepsilon \in C^\infty(\partial \Omega_1), \quad \lambda_\varepsilon|_{\partial \Omega^{in}_1} \to \lambda|_{\partial \Omega^{in}_1} \quad \text{in} \quad L^2(\partial \Omega^{in}_1),$$ \hfill (6.1)

$$b_{0_\varepsilon} \in C^\infty(\Omega_1), \quad b_{0_\varepsilon} - b_0 \in L^\infty(\Omega_1).$$ \hfill (6.2)

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Consider $(P_{H\varepsilon})$: to find

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As in the preceding section, we may assume that (5.3) holds for all values of $\epsilon$ which will appear in the sequel.

Consider the variational problem

$$(PH_\epsilon) : \text{to find } u_\epsilon \in H^1(\Omega_1) \text{ such that, for all } v \in H^1(\Omega_1),$$

$$\int_{\Omega_1} (\epsilon \nabla u_\epsilon - b u_\epsilon) \cdot \nabla (u_\epsilon - v) \, dx + \int_{\Omega_1} b_{0\epsilon} u_\epsilon (u_\epsilon - v) \, dx + \int_{\partial \Omega_1}^\ast b \cdot n_1 u_\epsilon (u_\epsilon - v) \, ds =$$

$$= \int_{\Omega_1} f (u_\epsilon - v) \, dx - \int_{\partial \Omega_1}^\ast b \cdot n_1 \lambda_\epsilon (u_\epsilon - v) \, ds. \quad (6.3)$$

Problem $(PH_\epsilon)$ is an elliptic regularization to the hyperbolic problem $(PH)$, analogous to the one introduced in the previous section for the coupled problem $(P)$. Therefore, the techniques developed in section 5 still apply to the present case and give a number of properties, which we summarize as follows.

**PROPOSITION 6.1** For all $\epsilon > 0$, problem $(PH_\epsilon)$ has a unique solution $u_\epsilon$, which satisfies:

$$-\epsilon \Delta u_\epsilon + \text{div} (b u_\epsilon) + b_{0\epsilon} u_\epsilon = f \quad \text{in } D'(\Omega_1), \quad (6.4)$$

$$-\epsilon \frac{\partial u_\epsilon}{\partial n_1} + b \cdot n_1 u_\epsilon = b \cdot n_1 \lambda_\epsilon \quad \text{in } H^1_0(\partial \Omega_1^{in}), \quad (6.5)$$

$$\frac{\partial u_\epsilon}{\partial n_1} = 0 \quad \text{in } H^1_0(\partial \Omega_1^{out}); \quad (6.6)$$

$$\epsilon \nabla u_\epsilon \to 0 \quad \text{strongly in } L^2(\Omega_1), \quad \text{as } \epsilon \to 0. \quad (6.7)$$

Moreover, there is a function $u \in L^2(\Omega_1)$ such that (possibly taking a subsequence)

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega_1), \quad (6.8)$$

$$b \cdot n_1 u_\epsilon \rightharpoonup b \cdot n_1 u \quad \text{weakly in } H^1_0(\partial \Omega_1^{in}), \quad (6.9)$$

$$b \cdot n_1 u_\epsilon \rightharpoonup b \cdot n_1 u \quad \text{weakly in } H^1_0(\partial \Omega_1^{out}). \quad (6.10)$$

**Finally, the limit function satisfies (4.4) and (4.5).**

**Proof.** The proof of (6.4)-(6.9) is obtained by repeating the corresponding arguments of section 5. We just note that the boundary integrals of (5.49) (in the proof of Lemma 5.11) do not contain any interface contribution: now the whole boundary $\partial \Omega_1$ behaves like $\Gamma_1$ did in the preceding case. To show (6.10), let us repeat the proof of Lemma 5.11, this time choosing a nonnegative, smooth $\psi$ vanishing in a neighborhood of $\partial \Omega_1^{in}$. All steps in the proof of the lemma can be followed again, up to the treatment of the boundary integrals in (5.49). Now, we may exploit (6.6) to see that the two integrals on $\partial \Omega_1$ (presently, there is no contribution from $\partial \Omega_2$) reduce to

$$\frac{\epsilon}{2} \int_{\partial \Omega_1^{in}} ^\ast \psi b \cdot n_1 (\frac{\partial u_\epsilon}{\partial \Gamma_1})^2 \, ds. \quad (6.11)$$

This is a nonnegative quantity added to the left hand side of (5.49), hence it can be dropped in the subsequent computation. As a consequence, we get that Lemma 5.11 still holds with this $\psi$, hence $b \cdot n_1 u_\epsilon \rightharpoonup b \cdot n_1 u$ in $H^1_0(\partial \Omega_1^{out})$. This proves (6.10). □
REMARC 6.2 More precisely, the strategy adopted in section 5 permits to obtain (6.10) along $\partial \Omega^\text{out}_1$, except at singular points. We do not stick to this matter, because we will exploit this convergence along $\Gamma^\text{out}$, which is regular.\(\square\)

In addition to the previous properties, the following estimate is fulfilled.

PROPOSITION 6.3 For all $\epsilon > 0$ and all $\delta > 0$,

\[
\epsilon \|\nabla u_c\|_{\delta, \Omega_1}^2 + \left(\frac{3\delta}{2} - \delta\right)\|u_c\|_{\delta, \Omega_1}^2 + \frac{1}{2} \|u_c\|_{\partial \Omega_1^*}^2 \leq \frac{1}{4\delta} \|f\|_{\delta, \Omega_1}^2 + \frac{1}{2} \|\lambda\|_{\partial \Omega_1^*}^2.
\] (6.11)

Moreover, if $f = 0$ then (6.11) holds with $\delta = 0$.

Proof. Choose $v = 0$ in (6.3): Green's formula (3.7) gives

\[
\epsilon \int_{\Omega_1} |\nabla u_c|^2 dx + \int_{\Omega_1} \frac{1}{2} \text{div} b + b_{\nu_b} u_c^2 dx + \frac{1}{2} \int_{\partial \Omega_1^*} b \cdot n_1 u_c^2 ds - \frac{1}{2} \int_{\partial \Omega_1^*} b \cdot n_1 u_c^2 ds = \int_{\Omega_1} f u_c dx - \int_{\partial \Omega_1^*} b \cdot n_1 u_c \lambda dx.
\]

Owing to (5.3), we get (6.11) easily. Note that the small parameter $\delta$ comes into play just in order to treat the first term on the right hand side of the previous equality. Therefore, one can forget it whenever $f = 0$.\(\square\)

Proof of Theorem 4.1. For all $\sigma > 0$, set

\[(\partial \Omega_1)_\sigma = \{x \in \partial \Omega^\text{out}_1 : (b \cdot n_1)(x) > \sigma\}.\] (6.12)

Obviously, $\partial \Omega^\text{out}_1 = \cup_{\sigma > 0} (\partial \Omega_1)_\sigma$; moreover, (6.11) and (6.1) imply that

\[
\sigma \int_{(\partial \Omega_1)_\sigma} u_c^2 ds \leq C,
\]

for all $\sigma > 0$, with a constant $C$ independent of $\epsilon$ and $\sigma$. Therefore, for all $\sigma > 0$ there is $g_\sigma \in L^2((\partial \Omega_1)_\sigma)$ such that $u_c \rightharpoonup g_\sigma$ weakly in $L^2((\partial \Omega_1)_\sigma)$, as $\epsilon \to 0$ (possibly taking a subsequence). Thus,

\[
b \cdot n_1 u_c \rightharpoonup b \cdot n_1 g_\sigma \text{ weakly in } L^2((\partial \Omega_1)_\sigma).\] (6.13)

by (6.10), $g_\sigma$ does not depend on the above subsequence and

\[g_\sigma = u \text{ on } (\partial \Omega_1)_\sigma, \text{ whence } u \in L^2((\partial \Omega_1)_\sigma),\] (6.14)

for all $\sigma > 0$. By (6.1), (6.7), (6.13), (6.14) and by the lower semicontinuity of $| \cdot |$, the limit of (6.11) as $\epsilon \to 0$ gives

\[
\left(\frac{3\delta}{2} - \delta\right)\|u\|_{\delta, \Omega_1}^2 + \frac{1}{2} \|u\|_{(\partial \Omega_1)_\sigma}^2 \leq \frac{1}{4\delta} \|f\|_{\delta, \Omega_1}^2 + \frac{1}{2} \|\lambda\|_{\partial \Omega_1^*}^2.
\] (6.15)

The family of functions $\{G_\sigma\}$ defined by

\[
G_\sigma = \begin{cases} b \cdot n_1 u_2^2, & \text{on } (\partial \Omega_1)_\sigma, \\ 0, & \text{on } \partial \Omega^\text{out}_1 \backslash (\partial \Omega_1)_\sigma \end{cases}
\]

is such that: on $\partial \Omega^\text{out}_1$, this (6.15). Thus, a.e. pointwise of the corresp (4.7). It remain

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\[ |\lambda_c|^2_{\partial \Omega^{\ast}_1}. \quad (6.11) \]

\[ \delta \text{ comes into play previous equality.} \quad (6.12) \]

\[ 2 ds - n_1 u_{\lambda_c} ds. \]

is such that: $G_{\sigma} \geq 0$ on $\partial \Omega^{\ast}_{1,0}$, for all $\sigma > 0$; $G_{\sigma_1} \geq G_{\sigma_2}$, if $\sigma_1 \leq \sigma_2$; $G_{\sigma}$ is integrable on $\partial \Omega^{\ast}_{1,0}$, this integral equals $|u|^2_{(\partial \Omega^{\ast}_1,s)}$, hence it is uniformly bounded with $\sigma$, due to (6.15). Thus, Beppo Levi’s theorem on monotone convergence applies and gives the a.e. pointwise convergence of $G_{\sigma}$ to some $G_0 \in L^1(\partial \Omega^{\ast}_{1,0})$, along with the convergence of the corresponding integrals. It follows that $b \cdot n_1 u^2 \in L^1(\partial \Omega^{\ast}_{1,0})$ and (6.15) yields (4.7).

It remains to discuss uniqueness: by linearity, it is enough to do this in the homogeneous case $f = \lambda = 0$. With this assumption, (4.7) gives immediately that $u = 0$ a.e. in $\Omega_1$ and a.e. on $\partial \Omega^{\ast}_{1,0}$ (whence uniqueness), but only for the solution obtained as limit of solutions to the regularized problems $(P_{H\varepsilon})$. Next Proposition shows that uniqueness holds independently of the regularization. Introduce the space

\[ W = \{ v \in L^2(\Omega_1) : b \cdot \nabla v \in L^2(\Omega_1), v = 0 \text{ on } \partial \Omega^{\ast}_{1,0} \}, \quad (6.16) \]

endowed with the norm

\[ \left[ \| v \|_{0, \Omega_1}^2 + \| b \cdot \nabla v \|_{0, \Omega_1}^2 \right]^\frac{1}{2}, \quad v \in W. \]

The vanishing of the elements of $W$ on $\partial \Omega^{\ast}_{1,0}$ is intended in the following sense: $b \cdot n_1 v = 0 \text{ in } \left[ H^3_{00}(\partial \Omega^{\ast}_{1,0}) \right]' \text{ (which is meaningful, since } bv \text{ belongs to } L^2_{\div}(\Omega_1))$, or even a.e. on $\partial \Omega^{\ast}_{1,0}$, due to Lemma 3.1, part (i).

**PROPOSITION 6.4** Let $u \in W$ be such that

\[ b \cdot \nabla u + (b_0 + \div b) u = 0 \quad a.e. \text{ in } \Omega_1. \quad (6.17) \]

Then, $u = 0$ a.e. in $\Omega_1$ and a.e. on $\partial \Omega^{\ast}_{1,0}$. In particular, problem $(P_H)$ with homogeneous data $f = \lambda = 0$ has only the trivial solution, whence the nonhomogeneous problem has a unique solution.

**Proof.** It is easy to check that the space $W$ is complete. Moreover, the family $Z = \{ v \in C^1(\Omega_1) : v|_{\partial \Omega^{\ast}_1} = 0 \}$ is dense in $W$. This can be shown by means of the same technique used to prove the density of the smooth functions in Sobolev spaces (see [8]). As a consequence, we claim that Green’s formula (3.7) holds for all $v \in W$, namely

\[ \frac{1}{2} \int_{\partial \Omega^{\ast}_{1,0}} b \cdot n v^2 ds = \int_{\Omega_1} v b \cdot \nabla v dx + \frac{1}{2} \int_{\Omega_1} v^2 \div b dx. \quad (6.18) \]

Indeed, let $\{ v_n \}$ be a sequence in $Z$, converging to $v$ in $W$: in particular, $bv_n \rightarrow bv$ in $L^2_{\div}(\Omega_1)$, whence

\[ b \cdot n v_n \rightarrow b \cdot n v \text{ in } \left[ H^3_{00}(\partial \Omega^{\ast}_{1,0}) \right]' \quad (6.19) \]

Now, (3.7) gives

\[ \frac{1}{2} \int_{\partial \Omega^{\ast}_{1,0}} b \cdot n v_n^2 ds = \int_{\Omega_1} v_n b \cdot \nabla v_n dx + \frac{1}{2} \int_{\Omega_1} v_n^2 \div b dx, \quad (6.20) \]

for all $n$. The right hand side of this equality converges to the right hand side of (6.18), because $v_n \rightarrow v$ in $W$. Therefore, the left hand side of (6.20) has a limit, which turns out to be equal to the left hand side of (6.18) (this can be shown by the same argument
used at the beginning of the proof of Theorem 4.1, recalling also (6.19)). Thus, (6.18) holds. Now, take the $L^2(\Omega_1)$ scalar product of (6.17) by $u$ and apply (6.18): it follows

$$\frac{1}{2} \int_{\partial \Omega_1^{\text{out}}} b \cdot n u^2 \, ds + \int_{\Omega_1} u^2 \left( \frac{1}{2} \text{ div } b + b_0 \right) \, dx = 0.$$ 

Hence, (1.2) gives $u = 0$ a.e. in $\Omega_1$ and a.e. on $\partial \Omega_1^{\text{out}}$. The consequent application to problem $(P_{\text{H}})$ is trivial, because (6.17) is nothing else than (4.4) when $f = 0$. □

**Remark 6.5** Proposition 6.4 does not provide any information on the behavior of the solution to the homogeneous problem along $\partial \Omega_0$, where $\mathbf{b} \cdot n_1 = 0$. Clearly, this lack of information is not relevant for "regular" solutions. □

By now, the proof of Theorem 4.1 is complete.

### 6.1.2. Proofs of the results of section 4.1.2: Proof of Theorem 4.3.

By Green's formula (3.7), the variational equality (4.13) gives

$$\int_{\Omega_2} \nu |\nabla w|^2 \, dx + \int_{\Omega_2} \frac{1}{2} \text{ div } b + b_0 \right) w^2 \, dx - \frac{1}{2} \int_{\Gamma^{\text{in}} \cup \Gamma^{\text{ext}}} b \cdot n_2 w^2 \, ds +$$

$$+ \frac{1}{2} \int_{\Gamma^{\text{in}} \cup \Gamma^{\text{ext}}} b \cdot n_2 w^2 \, ds = \int_{\Omega_2} \nu \nabla w \cdot \nabla v \, dx - \int_{\Omega_2} (b w) \cdot \nabla v \, dx +$$

$$+ \int_{\Omega_2} b_0 w v \, dx + \int_{\Gamma^{\text{in}} \cup \Gamma^{\text{ext}}} b \cdot n_2 w v \, ds + \int_{\Omega_2} f (w - v) \, dx - \int_{\Gamma^{\text{ext}}} b \cdot n_2 \mu (w - v) \, ds.$$ 

Recalling (1.2), (1.3) and noticing that $b \cdot n_2$ is negative on $\Gamma^{\text{out}}$ and positive on $\Gamma^{\text{in}}$, for all $\delta > 0$ we get

$$\left| \sqrt{\nu} |\nabla w|^2 \right|_{0, \Omega_2} + \beta_0 ||v||_{0, \Omega_2} + \left( \frac{1}{2} - \delta \right) ||w||_{\Gamma^{\text{cut}}} + \delta ||\nabla w||_{0, \Omega_2} + \frac{1}{4\delta} ||v||_{1, \Omega_2}^2 +$$

$$+ \{ \delta ||w||_{0, \Omega_2} + \frac{1}{2\delta} ||v||_{1, \Omega_2} \} \{ ||b||_{1, \infty, \Omega_2} + ||b_0||_{\infty, \Omega_2} \} +$$

$$+ \delta ||w||_{0, \Omega_2} + ||v||_{0, \Omega_2} + \left( 1 + \frac{1}{4\delta} \right) ||f||_{0, \Omega_2}^2 + \frac{1}{2} ||\phi||_{L^2}^2 +$$

$$+ \left( \frac{1}{2} - \delta \right) ||\mu||_{\Gamma^{\text{cut}}}^2 + \frac{1}{4\delta} ||v||_{0, \Omega_2}^2 \right|_{\Gamma^{\text{cut}}}.$$ 

(6.21)

Now, we may choose a test function $v$ such that $||v||_{1, \Omega_2} \leq C ||\phi||_{L^2}^2$, hence (6.21) gives

$$\left| \sqrt{\nu} |\nabla w|^2 \right|_{0, \Omega_2} + \beta_0 ||v||_{0, \Omega_2} + \left( \frac{1}{2} - \delta \right) ||w||_{\Gamma^{\text{cut}}} +$$

$$+ \delta ||w||_{0, \Omega_2} \{ ||b||_{1, \infty, \Omega_2} + ||b_0||_{\infty, \Omega_2} \} + \delta ||w||_{0, \Omega_2}^2 + \left( 1 + \frac{1}{4\delta} \right) ||f||_{0, \Omega_2}^2 +$$

$$+ \frac{1}{2} ||\phi||_{L^2}^2 + \left( \frac{1}{2} + \delta \right) ||\mu||_{\Gamma^{\text{cut}}}^2 + C (\Omega_2, b, b_0; \delta) ||\phi||_{L^2}^2,$$

for a suitable positive constant $C (\Omega_2, b, b_0; \delta)$ depending only on its argument. This proves (4.19). Finally, if $f = \phi = 0$, then we may choose $v = 0$ in (4.13): $\delta = 0$ is then allowed in (6.21) and (4.20) follows. □

### 6.1.3. Proof of Theorem 4.4

Given in $L^2(\Gamma^{\text{cut}})$

$$(1) \Rightarrow (2).$$

(4.23), it is $u^\psi$:

$$T^\psi \psi = \theta (w^\psi)_{\Gamma^{\text{cut}}}$$

follows that the

$$(2) \Rightarrow (1).$$

$\theta (w^\psi)_{\Gamma^{\text{cut}}} + (1 - \text{the pair } (u^\psi, w^\psi))$

**Proof of Lemma 4.4** estimates (4.8) at

**Proof of Lemma 4.4**

this end, let $\psi_1, \psi_2, \psi_3 \in (4.24)$ with $f = 0$ holds. Now, if all $\psi \in H^1(\Omega_2)$.

$$||\sqrt{\nu} |\nabla w^\psi||_{0, \Omega_2}^2 ||\psi||_{0, \Omega_2}^2 \left| \sqrt{\nu} |\nabla w||_{0, \Omega_2}^2 \right|_{0, \Omega_2} + \beta_0 ||v||_{0, \Omega_2} +$$

$$+ \left( \frac{1}{2} - \delta \right) ||\mu||_{\Gamma^{\text{cut}}}^2 + \frac{1}{4\delta} ||v||_{0, \Omega_2}^2 \right|_{\Gamma^{\text{cut}}}.$$ 

where

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Therefore, $T^\psi$

**Remark 6.6**

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**Proof of Theorem 4.3**

holds for all $n$. The estimates

**Proof of Theorem 4.3**

where $u^n, w^n$

see that $U^n, V$
6.1.3. Proofs of the results of section 4.1.3. Proof of Proposition 4.5. Let \( \psi \) be given in \( L^2_b(\Gamma^{in}) \).

(1) \( \Rightarrow \) (2). If \((u^\psi, w^\psi)\) solves problem (P), then it is \( u^\psi = w^\psi \) a.e. on \( \Gamma^{in} \). By (4.23), it is \( u^\psi = \psi \) a.e. in \( \Gamma^{in} \), whence \( w^\psi = \psi \) a.e. on \( \Gamma^{in} \). Therefore, for all \( \theta \in \mathbb{R} \),
\[
T_\theta \psi = \theta (w^\psi)|_{\Gamma^{in}} + (1 - \theta) \psi = \psi, \quad \text{that is,} \quad \psi \text{ is a fixed point for } T_\theta.
\]
In particular, it follows that the fixed point \( \psi \) belongs to \( H^1(\Gamma^{in}) \) and that it does not depend on \( \theta \).

(2) \( \Rightarrow \) (1). Let \( \psi \) be a fixed point for \( T_\theta \), for some \( \theta \in \mathbb{R} \). Then, \( \psi = T_\theta \psi = \theta (w^\psi)|_{\Gamma^{in}} + (1 - \theta) \psi \), that is, \( w^\psi = \psi \) a.e. on \( \Gamma^{in} \). Thus, \( u^\psi = \psi = w^\psi \) a.e. in \( \Gamma^{in} \) and the pair \((u^\psi, w^\psi)\) solves the coupled problem (P).

Proof of Lemma 4.6. The proof is immediate, by a simple combination of the estimates (4.8) and (4.20) applied to (4.23), (4.24). \( \Box \)

Proof of Lemma 4.7. Our aim is to show that the map \( T_1 \) is a contraction. To this end, let \( \psi_1, \psi_2 \) be given in \( L^2_b(\Gamma^{in}) \). By linearity, \( T_1 \psi_1 - T_1 \psi_2 = T_1 (\psi_1 - \psi_2) = (w^\psi_1 - w^\psi_2)|_{\Gamma^{in}} \) (notation of Lemma 4.6), since we have to solve the two problems (4.23), (4.24) with \( f = \phi = 0 \) and \( \psi = \psi_1 - \psi_2 \). Therefore, Lemma 4.6 applies and (4.26) holds. Now, if \( \gamma \) denotes the trace constant in \( \Omega_2 \) (that is, \( ||v||_{0, \partial \Omega_2} \leq \gamma ||v||_{1, \Omega_2} \), for all \( v \in H^1(\Omega_2) \)), we have that
\[
||\sqrt{\nu} \nabla w^\psi_0 ||^2_{0, \Omega_2} + \beta_0 ||w^\psi_0 ||^2_{0, \Omega_2} \geq \min(\nu_0, \beta_0) \gamma^{-2} ||w^\psi_0 ||^2_{0, \partial \Omega_2} \geq C \frac{1}{2} ||w^\psi_0 ||^2_{1, \Omega_2}, \quad (6.22)
\]
where
\[
C = \frac{2 \gamma^{-2} \min(\nu_0, \beta_0)}{\max_{\Gamma^{in}} |b-n_1|} > 0.
\]
Thus, a combination of (4.26) and (6.22) gives
\[
||w^\psi_1 ||^2_{1, \Omega_2} \leq \frac{1}{C + 1} ||\psi_1 ||^2_{1, \Omega_2},
\]
namely
\[
||T_1 \psi_1 - T_1 \psi_2 ||^2_{1, \Omega_2} \leq \frac{1}{C + 1} ||\psi_1 - \psi_2 ||^2_{1, \Omega_2}. \quad (6.23)
\]
Therefore, \( T_1 \) is a contraction in \( L^2_b(\Gamma^{in}) \), hence it admits a unique fixed point. \( \Box \)

Remark 6.6 Since fixed points for \( T_\theta \) do not depend on \( \theta \), Lemma 4.7 provides the existence and uniqueness of the fixed point for \( T_\theta \), for all \( \theta \in \mathbb{R} \). Actually, a convenient choice of \( \theta \) might diminish the value of the contraction constant appearing in (6.23), thus improving the rate of convergence in numerical approximations. \( \Box \)

Proof of Lemma 4.8. By the assumption on \( \psi \), Corollary 4.2 implies that (4.9) holds for all \( n \). After this remark, the proof is immediate, by a simple combination of the estimates (4.11) and (4.22). \( \Box \)

Proof of Lemma 4.9. Let \( \psi \) be the fixed point of \( T_1 \). For all \( n \), set
\[
d^n = \psi^n - \psi, \quad u^n = u^n - u^\psi, \quad W^n = w^n - w^\psi, \quad (6.24)
\]
where \( u^n, w^n \) solve (4.1), (4.2), while \( u^\psi, w^\psi \) solve (4.23), (4.24). It is immediate to see that \( U^n, W^n \) solve the two boundary value problems (4.1), (4.2) with \( f = \phi = 0 \).
and $\psi^n$ replaced by $d^n$. Therefore, we can read (4.27) with $u_0^n$, $w_0^n$ and $\psi^n$ replaced by $U^n$, $W^n$ and $d^n$, respectively: it follows

$$\|\sqrt{\nu} \nabla W^n\|_{0,\Omega_1}^2 + \frac{\beta_0}{2} \|U^n\|_{0,\Omega_1}^2 + \beta_0 \|W^n\|_{0,\Omega_2}^2 + \frac{1}{2} \|W^n\|_{1,\Omega_1}^2 \leq \frac{1}{2} \|d^n\|_{1,\Omega_1}^2. \quad (6.25)$$

This inequality and a trick analogous to (6.22) give

$$|W^n|_{1,\Omega_1}^2 \leq K^2 |d^n|_{1,\Omega_1}^2, \quad (6.26)$$

for a suitable constant $K < 1$. Since $(w^\psi)|_{\Gamma_{in}} = T_1 \psi = \psi$, by (6.24) and (4.3) it follows that $d^{n+1} = \theta W^n + (1 - \theta) d^n$, whence

$$|d^{n+1}|_{\Gamma_{in}} \leq |\theta| |W^n|_{\Gamma_{in}} + |1 - \theta| |d^n|_{\Gamma_{in}} \leq (|\theta| K + |1 - \theta|) |d^n|_{\Gamma_{in}}. \quad (6.27)$$

Therefore, if $\theta$ belongs to the interval $|0, \frac{2}{1+K}|$ (which includes the value 1), then the constant $|\theta| K + |1 - \theta|$ is smaller than 1 and (6.27) entails that $|d^n|_{\Gamma_{in}} \to 0$, as $n \to +\infty$. It follows that $\psi^n \to \psi$ in $L^2_0(\Gamma_{in})$. \[\square\]

Proof of Theorem 4.10. Again, let $\psi$ be the fixed point of $T_1$ and denote by $u^\psi, w^\psi$ the corresponding solutions to (4.23), (4.24). By Lemma 4.9, the sequence $d^n$ (defined in (6.24)) converges to $0$ in $L^2_0(\Gamma_{in})$. By (6.25), the sequences of functions $U^n, W^n$ (still defined in (6.24)) satisfy $U^n \to 0$ in $L^2(\Omega_1)$ and $W^n \to 0$ in $H^1(\Omega_2)$. Therefore, $u^n \to u^\psi$ in $L^2(\Omega_1)$ and $w^n \to w^\psi$ in $H^1(\Omega_2)$. By Proposition 4.5, we know that $(u^\psi, w^\psi)$ is a solution to the coupled problem (P).

Finally, we discuss uniqueness. If $\psi$ is the unique fixed point of $T_1$, the solutions $u^\psi, w^\psi$ to (4.23), (4.24) are such that $u^\psi$ is uniquely determined, in the sense of Proposition 6.4, and consequently $w^\psi$ is unique as a solution to (4.24). By Proposition 4.5, $(u^\psi, w^\psi)$ provides the unique solution (in the above sense) to the coupled problem (P). \[\square\]

6.2. The discrete case. The analysis we are going to carry out in this section is concerned solely with the collocation method using the Legendre-Gaussian points. In particular, it does not apply to the case in which the collocation points pertain to the Chebyshev-Lobatto formula.

**Lemma 6.7** Within the domain $\Omega_i$, $i = 1, 2$, the operator $L_N^i$ defined by (2.1) satisfies the following property:

$$(L_N^i v, v)_{\Omega_i} = \frac{1}{2} \left( \text{div} \left( I_N^i b \right), v^2 \right)_{\Omega_i} + \frac{1}{2} (b \cdot n_i, v^2)_{\Omega_i} \quad (6.28)$$

for all polynomial $v$ of degree $N$ (the discrete inner products are defined in (4.42)).

**Proof.** Since the one-dimensional Gauss-Lobatto quadrature formula is exact when the integrand is a one-dimensional algebraic polynomial of degree less than or equal to $2N - 1$ (see, e.g., [10]), it is easy to show that the following discrete rule of integration by parts holds:

$$(\text{div} g, v)_{\Omega_i} = -(g, \nabla v)_{\Omega_i} + (g \cdot n_i, v)_{\Omega_i} \quad (6.29)$$

for all polyn...
for all polynomials \( g, v \) such that \( g_i v \in P_{2N}, \ i = 1, 2 \). From (6.29) it follows:

\[
(\text{div}(I_N^i b v), v)_{N, \Omega_i} = -(b v, \nabla v)_{N, \Omega_i} + (b \cdot n_i, v^2)_{N, \partial \Omega_i}.
\] (6.30)

The inequality (6.28) follows from (6.30) and from the definition (2.1). □

Let us now define the bilinear form \( c_N^i : P_N \times P_N \to \mathbb{R} \),

\[
c_N^i(u, v) = (I_N^i u, v)_{N, \Omega_i} + (b \cdot n_i)^+ u, v)_{N, \partial \Omega_i}, \quad i = 1, 2,
\] (6.31)

here and in the following we denote by \( u^+ \) the positive and negative parts of the function \( u \), respectively. From (6.28) we easily deduce that

\[
c_N^i(v, v) = \frac{1}{2} (\text{div}(I_N^i b), v^2)_{N, \Omega_i} + \frac{1}{2} ((b \cdot n_i)^+, v^2)_{N, \partial \Omega_i} + \frac{1}{2} ((b \cdot n_i)^-, v^2)_{N, \partial \Omega_i}, \quad i = 1, 2
\] (6.32)

for all polynomial \( v \in P_N \).

6.2.1. Proofs of the results of section 4.2.1.

**LEMMA 6.8** The hyperbolic problem (4.39) can be written equivalently as follows:

find \( u_N^1 \in P_N \) such that for all \( v \in P_N \)

\[
c_N^1(u_N^1, v) + (b_0 u_N^1, v)_{N, \Omega_i} = (f, v)_{N, \Omega_i} + ((b \cdot n_i)^- \phi, v)_{N, \Gamma_i^1} + ((b \cdot n_i)^- \psi_n, v)_{N, \Gamma_i^1}.
\] (6.33)

**Proof.** The finite dimensional problem (6.33) can be equivalently reformulated by restricting test functions to the \((N + 1)^2\) Lagrangean functions \( v^{(k)} \in P_N \) such that \( v^{(k)}(P_j) = \delta_{kj} \) for all \( P_j \in \Xi_k^1 \). If \( P_k \) is an internal point and \( v^{(k)} \) is the associated Lagrangean function, then it is \( v^{(k)} \equiv 0 \) on \( \partial \Omega_i \). Hence (6.33) with \( v = v^{(k)} \) gives (4.39)(i) immediately, owing to (6.31).

Next, write (6.33) with \( v = v^{(j)} \), where \( P_j \in \partial \Omega_1 \). Using (4.39)(i), (4.42) as well as (6.31), we find

\[
[I_N^1 u_N^1 + b_0 u_N^1 - f](P_j)\omega_j^{(1)} = [b_0(1)(\chi - u_N^1)](P_j)\theta_j^{(1)},
\] (6.34)

where

\[
\chi = \begin{cases}
\phi \text{ at } \Gamma_i^1, \\
\psi_n \text{ at } \Gamma_i^1.
\end{cases}
\] (6.35)

Therefore, (4.39)(ii) and (4.39)(iii) follow easily by (6.34). □

**Proof of Lemma 4.12.** We use the equivalent variational formulation (6.33). Taking \( v = u_N^1 \) and using (6.32) along with definitions (4.43) and (4.44), by (2.11) we obtain

\[
\beta_N \| u_N^1 \|_{N, \Omega_1}^2 + \frac{1}{2} | u_N^1 |_{N, \partial \Omega_1}^2 + \frac{1}{2} | u_N^1 |_{N, \partial \Omega_1}^2 = (f, u_N^1)_{N, \Omega_1} + ((b \cdot n_i)^- \chi, u_N^1)_{N, \partial \Omega_1}.
\] (6.36)
By Cauchy-Schwarz inequality we have
\[
(f, u_N^\alpha)_{N,\Omega_1} \leq \frac{1}{2\beta_N} \|f\|^2_{N,\Omega_1} + \frac{\beta_N}{2} \|u_N^\alpha\|^2_{N,\Omega_1},
\]
(6.37)
\[
((b \cdot n_1)^- v, u_N^0)_{N,\partial\Omega_1} \leq \frac{1}{2} |\phi_{N,\partial\Omega_1}^2 + \frac{1}{2} |\psi_{N,\partial\Omega_1}^2 + \frac{1}{2} |u_N^0|_{N,\partial\Omega_1}^2.
\]
(6.38)
The inequality (4.47) follows by (6.36)-(6.38). □

6.2.2. Proofs of the results of section 4.2.2. We turn now to the elliptic problem (4.40). Let us define the following polynomial subset:
\[
P^D_N \overset{\text{def}}{=} \{ v \in P^D : v = 0 \text{ on } \Gamma_1^2 \}.
\]
(6.39)
This is precisely the set of algebraic polynomials of degree less than or equal to \( N \) that vanish on the portion of \( \partial\Omega_1 \) where a Dirichlet condition is prescribed. We start proving the following equivalence statement.

**Lemma 6.9** Assume \( \phi = 0 \) on \( \Gamma_1^2 \). The collocation problem (4.40) is equivalent to the following discrete variational problem:

find \( w_N^0 \in P^D_N \) such that for all \( v \in P^D_N \)
\[
(\nu \nabla w_N^0, \nabla v)_{N,\Omega_2} + c_N^2 (w_N^0, v)_{N,\Omega_2} + (b_0 w_N^0, v)_{N,\partial\Omega_2} = (f, v)_{N,\Omega_2} + ((b \cdot n_1)^+ u_N^0, v)_{N,\Gamma}.
\]
(6.40)

**Proof.** We first note that the same argument used to get (6.29) gives the discrete integration by parts formula:
\[
(\nu \nabla w_N^0, \nabla v)_{N,\Omega_2} = -(\text{div} [f_N^2 (\nu \nabla w_N^0), v])_{N,\Omega_1} + (\nu \frac{\partial w_N^0}{\partial n_2}, v)_{N,\partial\Omega_2}.
\]
(6.41)
As we noticed in the proof of Lemma 6.8, problem (6.40) can be equivalently reformulated by letting the test functions \( v \) range within the space of Lagrangean functions \( v^{(k)} \in P^D_N \) associated with the collocation points of \( \Xi^0_N \cap (\Omega_2 \setminus \Gamma_1^2) \).

(i) If \( v^{(k)} \) is associated with a collocation point \( P_k \) of \( (\Xi_1^0)^0 \), then \( v^{(k)} \) vanishes on \( \partial\Omega_2 \), hence all boundary terms involving \( v^{(k)} \) can be dropped. Therefore, from (6.40), (6.41) and (6.31) it follows that the set of equations (4.40)(i) holds.

(ii) Let \( v^{(k)} \) be now the Lagrangean function associated with a collocation point \( P_k \) of \( (\Gamma_2^\infty)_N \). Taking \( v = v^{(k)} \) in (6.40) and using (6.41) and (6.31) we obtain
\[
R_2(P_k) \omega_{P_k}^{(2)} - |(b \cdot n_2) - u_N^0||P_k| \theta_{P_k}^{(2)} = ((b \cdot n_1)^+ u_N^0, v^{(k)})_{N,\Gamma} - |(b \cdot n_1)^+ u_N^0| |(P_k)| \theta_{P_k}^{(2)}.
\]
(6.42)
The second term in the left hand side vanishes because of (1.3). Moreover, the first term in the right hand side vanishes, since \( v^{(k)} \) is zero on \( \Gamma \). Therefore, we obtain (4.40)(iii).

(iii) Now, take \( v = v^{(k)} \) in (6.40), with \( v^{(k)} \) associated to a point \( P_k \) of \( (\Gamma^{in})_N \). Relation (6.42) holds also in the present case, but now we have that
\[
((b \cdot n_1)^+ u_N^0, v^{(k)})_{N,\Gamma} = ((b \cdot n_1)^+ u_N^0, v^{(k)})_{P_k} \theta_{P_k}^{(2)}.
\]
This term vanishes along with the first term in the right hand side, because \((b \cdot n_2)^- = (b \cdot n_1)^+ = 0\) on \(\Gamma^{in}\). Thus, the set of relations \((4.40)(v)\) holds.

(iv) Finally, let \(u^{(k)}\) be the Lagrangean function associated with a collocation point \(P_k\) of \((\Gamma^{out})_n\). Taking \(v = u^{(k)}\) in \((6.40)\) and using \((6.42)\) we obtain

\[
R_2(P_k)\omega_k^{(2)} = [-\nu \frac{\partial w^N_n}{\partial n_2} - (b \cdot n_2)^- w^N_n + (b \cdot n_1)^+ u^N_n](P_k)\theta_k^{(2)}.
\]

Noticing that \((b \cdot n_2)^- = (b \cdot n_1)^+ = -b \cdot n_2\) on \(\Gamma^{out}\), the set of relations \((4.40)(iv)\) follows.\(\square\)

We are in a position to prove the stability estimate \((4.49)\).

Proof of Lemma 4.15. Take \(v = w^N_n\) in \((6.40)\) and use \((2.11)\) and \((6.32)\). It follows

\[
(\nu \nabla w^N_n, \nabla w^N_n)_{\Omega_3} + \beta_N \|w^N_n\|^2_{\Omega_3} + \frac{1}{2}((b \cdot n_2)^+, (w^N_n)^2)_{\Omega_3} + \frac{1}{2}((b \cdot n_2)^-, (w^N_n)^2)_{\Omega_3} = (f, w^N_n)_{\Omega_3} + ((b \cdot n_1)^+ u^N_n, w^N_n)_{\Omega_3}.
\]

(6.43)

We notice that

\[
((b \cdot n_2)^-, (w^N_n)^2)_{\Omega_3} = ((b \cdot n_2)^-, (w^N_n)^2)_{\Omega_3},
\]

(6.44)

since \(w^N_n = 0\) on \(\Gamma^D\) and \((b \cdot n_2)^- = 0\) on \(\Gamma^S\). Moreover, using once again the identity \((b \cdot n_1)^+ = (b \cdot n_2)^-\) on \(\Gamma\) and the Cauchy-Schwarz inequality, it follows

\[
((b \cdot n_1)^+ u^N_n, w^N_n)_{\Omega_3} \leq \frac{1}{2}((b \cdot n_2)^-, (w^N_n)^2)_{\Omega_3} + \frac{1}{2}((b \cdot n_2)^-, (w^N_n)^2)_{\Omega_3},
\]

(6.45)

Owing to \((6.44)\) and \((6.45)\), we get \((4.49)\) easily from \((6.43)\).\(\square\)

6.2.3. Proofs of the results of section 4.2.3. Owing to the previous stability inequalities we can deduce the uniqueness result for the collocation coupled problem \((2.2)-(2.9)\), stated in Theorem 4.16.

PROPOSITION 6.10 The collocation coupled problem \((2.2)-(2.9)\) has a unique solution.

Proof. It is enough to prove that the homogeneous problem (namely the one with \(f = \phi = 0\)) has the only trivial solution \(u_N = w_N = 0\). Let us introduce the unknown function

\[
\psi_N = (w^N)_{\Gamma^{in}}.
\]

(6.46)

As soon as \(\psi_N\) is available on \(\Gamma^{in}\), we can solve the hyperbolic problem \((2.2), (2.4), (2.7)\) first. Next we can solve the elliptic problem \((2.3), (2.5), (2.6), (2.8), (2.9)\). We can therefore read the stability estimate \((4.50)\) with \(\psi^N\) replaced by \(\psi_N, u^N_N\) by \(u_N\) and \(w^N_N\) by \(w_N\), obtaining

\[
2\|\sqrt{\nu} \nabla w_N\|^2_{\Omega_3} + \beta_N \|u_N\|^2_{\Omega_3} + \|w_N\|^2_{\Omega_3} + \|w_N\|^2_{\Omega_3} \leq \|\psi_N\|^2_{\Omega_3}.
\]
By (6.46), we may drop the two boundary terms in the previous inequality and conclude that \( u_N = w_N = 0 \).

Now we turn to the proof of the convergence results of Theorem 4.16.

By simply taking the difference between equations (2.2)-(2.9) and the corresponding equations (4.39), (4.40), we see immediately that the following statement holds.

**PROPOSITION 6.11** The error functions defined in (4.51) satisfy the same equations (4.39), (4.40), with \( w_N^\varepsilon \) and \( w_N^\theta \) replaced respectively by \( e_N^\varepsilon \) and \( e_N^\theta \), \( f = 0 \), \( \phi = 0 \) and

\[
\psi^n = \theta e^\theta_{N-1} + (1 - \theta)e^\varepsilon_{N-1}, \quad \theta > 0. \tag{6.47}
\]

In view of the above statement, we can write the inequality (4.50) for \( e_N^\varepsilon \) and \( e_N^\theta \), obtaining:

\[
2\|\sqrt{\nu}\nabla e^\varepsilon_{N,\Omega_2} + \beta_N \|e^\varepsilon_{N,\Omega_1} + \|e^\varepsilon_{N,\Omega_2}\| + \|e^\varepsilon_{N,\Gamma_{1*}}\| \leq \|\psi^n\|_{N,\Gamma_{1*}}^2. \tag{6.48}
\]

Now, (1.2), (4.44), (4.46) and (6.22) give

\[
2\|\sqrt{\nu}\nabla e^\theta_{N,\Omega_2} + \beta_N \|e^\varepsilon_{N,\Omega_1} + \|e^\varepsilon_{N,\Omega_2}\| \geq K \|e^\theta_{N,\Gamma_{1*}}\|, \tag{6.49}
\]

where

\[
K = \frac{\gamma^{-2} \min(\nu_0, \beta_N)}{C^2_{\max_{\Gamma_{1*}} |b_n^N|}} > 0.
\]

Thus, a combination of (6.48) and (6.49) gives

\[
|e^\varepsilon_{N,\Gamma_{1*}}|^2 \leq \frac{1}{K+1} |\psi^n|^2_{N,\Gamma_{1*}}. \tag{6.50}
\]

This inequality expresses a contraction of the input error \( \psi^n \) on \( \Gamma_{1*} \). More precisely, let us define a discrete interface operator as follows (see (4.25))

\[
T_N : P_N(\Gamma_{1*}) \to P_N(\Gamma_{1*}), \quad \psi^n \mapsto T_N \psi^n = (e^\varepsilon_{N,\Gamma_{1*}}). \tag{6.51}
\]

The inequality (6.50) reads equivalently in terms of this operator:

\[
|T_N \psi^n|^2_{N,\Gamma_{1*}} \leq \frac{1}{K+1} |\psi^n|^2_{N,\Gamma_{1*}}, \quad \forall \psi^n \in P_N(\Gamma_{1*}). \tag{6.52}
\]

If \( \theta = 1 \) in (6.47), then it is \( (e^\varepsilon_{N,\Gamma_{1*}}) = \psi^{n+1}, \forall n \). Thus, (6.52) yields

\[
|\psi^{n+1}|^2_{N,\Gamma_{1*}} \leq \frac{1}{K+1} |\psi^n|^2_{N,\Gamma_{1*}}, \quad \forall n \geq 2, \tag{6.53}
\]

hence the operator \( T_N \) is a contraction and \( \psi^n \to 0 \) as \( n \to +\infty \). As shown in the proof of Proposition 6.10, this (trivial) limit value at \( \Gamma_{1*} \) allows us to reconstruct the whole solution to the coupled homogeneous collocation problem (2.2)-(2.9), which turns out to be identically zero. Therefore, we get \( e^\varepsilon_{N,\Gamma_{1*}} \to 0 \), as \( n \to +\infty \). In view of the definition (4.51), we infer that the sequence \( (u^N_N, w^N_N) \) converges to the (unique) solution of the collocation problem (2.2)-(2.9), in the case \( \theta = 1 \).
If $0 < \theta \neq 1$, then we define

$$T_{N,\theta} = \theta T_N + (1 - \theta)I,$$

(6.54)

where $I$ is the identity operator. From (6.47) we deduce that

$$\psi^{n+1} = T_{N,\theta} \psi^n, \quad \forall n \geq 2.$$

Since $T_{N,1} = T_N$, for $\theta$ sufficiently close to 1 the operator $T_{N,\theta}$ is still a contraction (with a contraction constant possibly smaller than $(K + 1)^{-1}$).

REFERENCES


