On the Schwarz Alternating Method III: A Variant for Nonoverlapping Subdomains

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Abstract.

We continue here a systematic investigation of convergence properties of the Schwarz alternating method and related domain decomposition methods. Our study here concerns a new variant of the Schwarz method, adapted to the situation of an arbitrary number of nonoverlapping subdomains. We present this iterative method in the "continuous" situation and analyse its convergence in self-adjoint and nonself-adjoint cases.

I. Introduction.

This paper is a sequel of [**36**] and [**54**], and part III of a series devoted to the mathematical study of various decomposition methods (domain decomposition methods) for various linear or nonlinear partial differential equations. In the recent years, the applications of iterative methods (and their study) solving subproblems or problems in subdomains to

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the numerical analysis of boundary value problems have received a lot of attention and a partial list of contributions to this general theme can be found in the bibliography (and of course in this volume).

Parts I and II were devoted to the classical Schwarz alternating method where, roughly speaking, one approximates the solution of Laplace's equation by solving successively the same equation in two subdomains keeping at each step the values on the internal boundaries as boundary conditions for the next step - passing "Dirichlet" data from one subdomain to the other through the respective internal boundaries. This method requires some overlapping of the two subdomains. We reviewed in Parts I and II the convergence analysis of the classical Schwarz method and showed, in particular, that there are two reasons for its convergence namely some variational reason (iterated projections in an Hilbert space) (cf. Part I [**36**]) and some "maximum principle type" reason (cf. Part II [**54**]). This explains why Schwarz method can be applied or extended to a wide variety of situations and equations. Let us also mention that the arguments of [**54**] can be conveniently adapted to yield some rather striking convergence results in the case of special geometries - see T. Chan, T. Hou and P.L. Lions [**55**].

However, as we said above, the Schwarz method requires that the subdomains overlap and this may be a severe restriction - without speaking of the obvious or intuitive waste of efforts in the region shared by the two subdomains-. This is why the search for iterative methods allowing to treat geometrical decompositions with non-overlapping subdomains has received a lot of attention - at least, once decomposition methods were realized to be useful or potentially useful. Several methods have been proposed and we refer to (for instance) Q.V. Dinh, R. Glowinski and J. Périaux [**11**], P. Bjorstad and O. Widlund [**34**] or, L.D. Marini and A. Quarteroni [**45**], D. Funaro, A. Quarteroni and P. Zanolli [**46**] or J.F. Bourgat, R. Glowinski, P. Le Tallec and M. Vidrascu [**56**]...

We present here a domain decomposition method that we believe is new and that is directly inspired by the original Schwarz alternating method. Thus, it is related to the methods referred to above and, in particular, is of the same type of the one introduced in [**45**], [**46**] and developed in [**47**]-[**52**]. Our method is described in section II for the model case of Laplace's equation with homogeneous Dirichlet boundary conditions. At each step, we solve the same Laplace's equation in each subdomain "passing from each subdomain to the others a convex combination of Neumann and Dirichlet data" : in particular, this yields a Robin (or Fourier) type boundary condition on each interface. Let us emphasize at this point the method allows an arbitrary number of arbitrary non-overlapping subdomains (in fact, they might even overlap if necessary) and, furthermore, all subdomains are treated in a parallel way. In particular, "interior subdomains" are allowed.

We analyse the convergence of this method in section III in the model case and we prove its convergence through somewhat delicate "energy" estimates. In section IV, we consider another model case namely Laplace's equation with convection terms and we show that the method still converges in this case. Let us also point out that the parameters present in this method on each interface (the convex weights of Neumann and Dirichlet data) do not need to be restricted because of the convection terms.

Finally, in section V, we present various extensions, adaptations of this method and we list some of the equations which can be treated with our iterative method.

II. Presentation of the method.

Let Ω be a bounded, smooth open set in $\mathbb{R}^N$. We are going to consider a decomposition of Ω into an arbitrary number m (≥ 2) of subdomains $\Omega_1, \ldots, \Omega_m$ i.e. we assume that

$$(1) \qquad \Omega = \Omega_1 \cup \cdots \cup \Omega_m \cup \Sigma \quad , \quad \Sigma = \bigcup_{1 \leq i \neq j \leq m} \gamma_{ij}$$

where Ω_i are disjoint open sets in $\mathbb{R}^N$, $\gamma_{ij} = \gamma_{ji}$ is the interface between Ω_i and Ω_j i.e. $\gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ ($1 \leq i \neq j \leq m$). In order to simplify the presentation, we will make the unnecessary assumptions that each Ω_i is connected, $\overline{\Omega}_i \cap \overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$ for all $1 \leq i \neq j \neq k \leq m$, γ_{ij} is the trace on Ω of a smooth manifold intersecting $\partial\Omega$ orthogonally for all $1 \leq i \neq j \leq m$. When $m = 2$, the two figures below show typical decompositions of Ω.

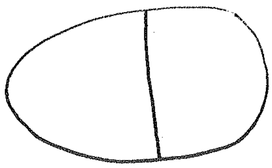

Figure 1.

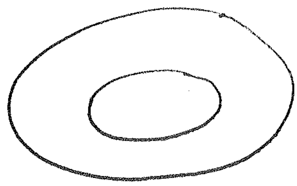

Figure 2.

Then, the model equation we consider is

$$(2) \qquad -\Delta u = f \quad \text{in } \Omega \quad , \quad u = 0 \quad \text{on } \partial\Omega = \Gamma$$

where f, say, is given in $L^2(\Omega)$, so the unique solution of (2) belongs to $H_0^1(\Omega)$.

Given arbitrary initial guesses $(u_i^0)_{1 \leq i \leq m}$ in $H^2(\Omega_i) \cap H_\Gamma^1(\Omega_i)$ where $H_\Gamma^1(\Omega_i) = \{u \in H^1(\Omega_i)/u = 0 \text{ on } \partial\Omega_i \cap \Gamma\}$ ($1 \leq i \leq m$), we build inductively sequences $(u_i^n)_{1 \leq i \leq m}$ solving for all $n \geq 0$ and all $1 \leq i \leq m$

$$(3) \qquad -\Delta u_i^{n+1} = f \quad \text{in } \Omega_i \quad , \quad u_i^{n+1} \in H^1_\Gamma(\Omega_i)$$

$$(4) \quad \frac{\partial u_i^{n+1}}{\partial n_{ij}} + \lambda_{ij}\, u_i^{n+1} = \frac{\partial u_j^n}{\partial n_{ij}} + \lambda_{ij}\, u_j^n \quad \text{on } \gamma_{ij}\,,\ \forall\, 1 \le j \le m\,,\, j \ne i\,,$$

where n_{ij} $(= -n_{ji})$ is the unit outward normal to $\partial\Omega_i$ on γ_{ij}, and $\lambda_{ij} = \lambda_{ji} > 0$ for all $1 \le i \ne j \le m$.

In fact, some explanations might be useful in order to understand the precise meaning of (3)-(4) : by induction, we see that we only have to explain the meaning of the following problem (for each $1 \le i \le m$)

$$(5) \qquad -\Delta v = f \quad \text{in } \Omega_i \quad , \quad v \in H^1_\Gamma(\Omega_i)$$

$$(6) \qquad -\frac{\partial v}{\partial n_{ij}} = g_{ij} \in L^2(\gamma_{ij}) \quad \text{on } \gamma_{ij}\,, \quad \text{for all } 1 \le j \le m\,,\ j \ne i\,.$$

And this is nothing but the usual variational formulation : $v \in H^1_\Gamma(\Omega_i)$ satisfies

$$(7) \qquad \int_{\Omega_i} \nabla v \cdot \nabla \varphi\, dx + \sum_{\substack{1 \le j \le m \\ j \ne i}} \int_{\gamma_{ij}} g_{ij}\, \varphi\, dS = \int_{\Omega_i} f\, \varphi\, dx\,, \quad \text{for all } \varphi \in H^1_\Gamma(\Omega_i)\,.$$

Notice also that, for $n \ge 1$, we may deduce from (3)-(4) for all i :

$$(8) \qquad -\Delta(u_i^{n+1} - u_i^{n-1}) = 0 \quad \text{in } \Omega_i \quad , \quad u_i^{n+1} - u_i^{n-1} \in H^1_\Gamma(\Omega_i)$$

$$(9) \qquad \frac{\partial}{\partial n_{ij}}(u_i^{n+1} - u_i^{n-1}) = \lambda_{ij}\{2u_j^n - u_i^{n+1} - u_i^{n-1}\} \quad \text{on } \gamma_{ij} \text{ for all } 1 \le j \le m\,,\ j \ne i$$

Notice also that (9) may be written

$$\frac{\partial}{\partial n_{ij}}(u_i^{n+1} - u_i^{n-1}) + \lambda_{ij}(u_i^{n+1} - u_i^{n-1}) = 2\lambda_{ij}(u_j^n - u_i^{n-1}) \qquad \text{on } \gamma_{ij} . \tag{10}$$

At this stage, it might be worth mentioning the differences between the above iterative method and other known methods. First of all, for overlapping subdomains, the classical Schwarz algorithm consists in, roughly speaking, passing from one subdoamin to the neighboring ones some "Dirichlet data" on the relevant interfaces. In fact, one might also pass "Neumann data" or convex combinations of both (see section IV for comments on these possibilities) and (4) is nothing but the illustration of this last possibility when the "overlapping goes to 0".

Next, we might also compare with the methods in [**11**] or [**56**] : it is not hard to see that the main change in (4) is that we have u_i^{n+1} instead of, say, u_i^n. Thus, in some sense, the above algorithm is an implicit variant of the afore mentioned ones.

Finally, if we compare with [**45**]-[**46**], we see that the "combination of Dirichlet and Neumann problems" is done here a priori instead of a posteriori in [**45**]-[**46**]. This allows a more symmetric treatment of all subdomains and interior subdomains.

Let us finally mention that it is possible to view the above method in the light of augmented Lagrangian techniques (P. Le Tallec [**61**], M. Fortin [**62**] - where some numerical experiments can also be found).

III. Convergence analysis in the self-adjoint case.

Our main convergence result is

Theorem 1. *For all $1 \leq i \leq m$, u_i^n converges weakly to $u_{|\Omega_i}$ in $H^1_\Gamma(\Omega_i)$ and in particular $u^n_{i|\gamma_{ij}}$ converges to $u_{|\gamma_{ij}}$ weakly in $H^{1/2}$ for all $j \neq i$, as n goes to $+\infty$. Furthermore, $\frac{1}{2}(u_i^{n+1} + u_j^n)$ converges to $u|_{\gamma_{ij}}$ in $H^{1/2}(\gamma_{ij})$ as n goes to $+\infty$ for all $j \neq i$.*

Remark: The same analysis applies to approximations of (2) by, say, finite elements methods and, in fact, yields a convergence uniform in the mesh size. For a finite dimensional version of the problem, one can in fact show a geometric convergence of the method. However, in general, we do not know of any estimate on the rate of convergence of the method.

Proof: The proof will be divided in three steps and is based upon a careful use of "energy type" estimates.

Step 1: We multiply (8) by $(u_i^{n+1} - u_i^{n-1})$, integrate by parts over Ω_i, use (9) and find for all i, n

$$\begin{aligned} &\int_{\Omega_i} |\nabla(u_i^{n+1} - u_i^{n-1})|^2 \, dx + \sum_{j \neq i} \lambda_{ij} \int_{\gamma_{ij}} |u_i^{n+1} - u_i^n|^2 \, dS \\ &\qquad = \sum_{j \neq i} \lambda_{ij} \int_{\lambda_{ij}} |u_i^n - u_i^{n-1}|^2 \, dS \, . \end{aligned} \tag{13}$$

Then, summing over i and over n, we find

$$\sum_{n \geq 1} \sum_{i=1}^{m} \int_{\Omega_i} |\nabla(u_i^{n+1} - u_i^{n-1})|^2 \, dx \; \leq \; C \tag{14}$$

$$(15) \qquad \int_{\gamma_{ij}} |u_i^{n+1} - u_i^n|^2 \, dS \leq 0 \qquad \text{for all} 1 \leq i \neq j \leq m \,, \; n \geq 0$$

where C denote various constants depending only on the geometry, $(u_i^0)_i$and $(\lambda_{ij})_{ij}$. Then, if $\partial\Omega_i \cap \partial\Omega$ has a nomptempty relative interior, we deduce from (14) and Poincaré's inequality

$$\sum_{n\geq 1} \int_{\Omega_i} |u_i^{n+1} - u_i^{n-1}|^2 \, dx \; \leq \; C \, .$$

At this stage, we need to introduce some notations : we denote by $I_1 = \{i \in \{1, \ldots, m\}$ such that $\partial\Omega_i \cap \partial\Omega$ has a nonempty relative interior$\}$ and then by $I_2 = \{i \in \{1, \ldots, m\}$ such that $\partial\Omega_i \cap \partial\Omega_j$ has a nonempty relative interior for some $j \in I_1\}$ and so on ... For some $k_0 \in \{1, \ldots, m\}$ we have $I_{k_0} = \{1, \ldots, m\}$. In the case of Figure 1, $I_1 = \{1, 2\}$, while in the case of Figure 2, $I_1 = \{1\}$, $I_2 = \{1, 2\}$. With these notations, we have proved above

$$(16) \qquad \sum_{n\geq 1} \sum_{i \in I_1} \int_{\Omega_i} |u_i^{n+1} - u_i^{n-1}|^2 \, dx \; \leq \; C \, .$$

We next want to show a similar bound for $i \in I_2$ or equivalently for $i \in I_2 - I_1$: for such an i, we pick some $j \in I_1$ such that $\partial\Omega_i \cap \partial\Omega_j$ has a nonempty relative interior and we remark that because of (8), (14) and (16) we have

$$\sum_{n\geq 2} \| \frac{\partial}{\partial n_{ji}} (u_j^n - u_j^{n-2}) \|^2_{H^{-1/2}(\gamma_{ij})} + \| (u_j^n - u_j^{n-2}) |^2_{H^{-1/2}(\gamma_{ij})} \; \leq \; C \, .$$

then, (4) yields

$$\sum_{n\geq 1} \| \frac{\partial}{\partial n_{ij}} (u_i^{n+1} - u_i^{n-1}) + \lambda_{ij} (u_i^{n+1} - u_i^{n-1}) \|^2_{H^{-1/2}(\gamma_{ij})} \; \leq \; C$$

and we deduce

$$\sum_{n\geq 1} \int_{\Omega_i} |u_i^{n+1} - u_i^{n-1}|^2 \, dx \; \leq \; C$$

in view of the following lemma.

Lemma 2. *Let $\mathcal{O}$ be a bounded, open, smooth domain in $\mathbb{R}^N$, let $\lambda > 0$ and let γ_0 be an open (relative to $\partial\mathcal{O}$) subset of $\partial\Omega$. Then, there exists a positive constant C such that*

$$\|u\|_{L^2(\mathcal{O})} \leq C\{\|\nabla u\|_{L^2(\mathcal{O})} + \|\frac{\partial u}{\partial \nu} + \lambda u\|_{H^{-1/2}(\gamma_0)}\} \tag{19}$$

for all $u \in H^1(\mathcal{O})$ satisfying : $\Delta u = 0$ in $\mathcal{O}$, where ν denotes the outward unit normal to $\partial\mathcal{O}$.

Proof of Lemma 2: By contradiction, we consider $u_n \in H^1(\mathcal{O})$ such that $\Delta u_n = 0$ in $\mathcal{O}$ and $\|u_n\|_{L^2(\mathcal{O})} = 1$ while $\nabla u_n \underset{n}{\to} 0$ in $L^2(\mathcal{O})$, $\frac{\partial u_n}{\partial \nu} + \lambda u_n \underset{n}{\to} 0$ in $H^{-1/2}(\gamma_0)$. Clearly, we may assume without loss of generality that u_n converges weakly in $H^1(\mathcal{O})$ to some u which thus satisfies

$$\Delta u = 0 \text{ in } \mathcal{O} \quad , \quad \nabla u = 0 \text{ in } \mathcal{O} \quad , \quad \frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \gamma_0$$

Thus, u is constant and $\|u\|_{L^2(\mathcal{O})} = 1$ since u_n converges strongly in $L^2(\mathcal{O})$ to u. Therefore, $u \not\equiv 0$. Then, we choose $\varphi \in C^1(\overline{\mathcal{O}})$ such that its support is contained in $\mathcal{O} \cup \gamma_0$, $\varphi \geq 0$ and $\varphi \not\equiv 0$ on γ_0. Since u is contant, we have

$$0 = \int_{\partial\mathcal{O}} \frac{\partial u}{\partial \nu}\varphi \, dS = \int_{\gamma_0} \frac{\partial u}{\partial \nu}\varphi \, dS = -\lambda \int_{\gamma_0} u\varphi \, dS$$

and we easily reach a contradiction since u is constant.

One also sees immediately that the argument which led to (18) for an arbitrary $i \in I_2$ can be iterated and eventually leads to

$$\sum_{n\geq 1} \sum_{i=1}^{m} \int_{\Omega_i} |u_i^{n+1} - u_i^{n-1}|^2 \, dS \leq C \, . \tag{20}$$

Step 2: We multiply (8) by $(u_i^{n+1} - u_i^{n-1})$, integrate by parts over Ω_i, use (9) and find for all i, n

$$\begin{aligned} &\int_{\Omega_i} \nabla(u_i^{n+1} - u_i^{n-1}) \cdot \nabla u_i^{n+1} \, dx + \\ &\quad + \sum_{j\neq i} \lambda_{ij} \int_{\gamma_{ij}} (u_i^{n+1} - u_i^{n-1} - 2u_j^n) u_i^{n+1} \, dS = 0 . \end{aligned} \tag{21}$$

We then remark that we may write

$$\nabla(u_i^{n+1} - u_i^{n-1}) \cdot \nabla u_i^{n+1} = \frac{1}{2}|\nabla u_i^{n+1}|^2 + \frac{1}{2}|\nabla(u_i^{n+1} - u_i^{n-1})|^2 - \frac{1}{2}|\nabla u_i^{n-1}|^2$$

and

$$\begin{aligned} (u_i^{n+1} - u_i^{n-1} - 2u_j^n) u_i^{n+1} &= \frac{1}{2}|u_i^{n+1}|^2 + \frac{1}{2}|u_i^{n-1}|^2 - |u_j^n|^2 + \\ &\quad |u_i^{n+1} - u_j^n|^2 - \frac{1}{2}|u_i^{n+1} - u_i^{n-1}|^2 . \end{aligned}$$

Inserting these formulas in (21), summing the resulting expressions with respect to i and n, we find in view of (20)

$$\int_{\Omega_i} |\nabla u_i^{n+1}|^2 \, dx + \int_{\partial\Omega_i} |u_i^{n+1}|^2 \, dS \leq C \tag{22}$$

$$\sum_{n\geq 0} \sum_{j\neq i} \int_{\gamma_{ij}} |u_i^{n+1} - u_i^n|^2 \, dS \leq C . \tag{23}$$

Of course, (22) immediately implies that $(u_i^n)_n$ is bounded in $H^1(\Omega_i)$ for all $1 \leq i \leq m$.

Step 3: In view of (14), (20), (23), we see that $u_i^{n+1} - u_i^{n-1} \underset{n}{\rightarrow} 0$ in $H^{1/2}(\gamma_{ij})$, and that $u_i^{n+1} - u_i^n \underset{n}{\rightarrow} 0$ in $L^2(\gamma_{ij})$ for all $1 \leq i \neq j \leq m$. Because of (22), we just have to consider a weakly convergent in $H^1(\Omega_i)$ $(\forall\, i)$ subsequence $u_i^{n'}$ and we denote by u_i its limit. Of course, u_i satisfies (3) and in view of the above convergences and (4) we have

$$(24) \qquad u_i = u_j \qquad \text{on } \gamma_{ij} \text{ for all } 1 \le i \ne j \le m$$

and thus

$$(25) \qquad \frac{\partial u_i}{\partial n_{ij}} = \frac{\partial u_j}{\partial n_{ij}} \qquad \text{in } H^{-1/2}(\gamma_{ij}) \text{ for all } 1 \le i \ne j \le m .$$

This is enough to ensure that $u_i \equiv u_{|\Omega_i}$ for all $1 \le i \le m$.

The remaining assertion is more complicated and in order to simplify the presentation we will explain the idea of the proof in the (very) particular case when $m = 2$, $\gamma = \gamma_{12} = \gamma_{21}$ is a piece of an hyperplane, that $\partial\Omega$ is also flat in a neighborhood of $\overline{\gamma} \cap \partial\Omega$ and that $\partial\Omega$ and $\overline{\gamma}$ intersect orthogonally. Then, we may assume without loss of generality that $\gamma = \{x_1 = 0\} \cap \Omega$ and we introduce $v_1^{n+1} = u - u_1^{n+1}$,

$$\hat{v}_2^n(x_1, x') = (u - u_k^n)(-x_1, x') \qquad \text{for all } x = (x_1, x') \in \Omega_1^0$$

where Ω_1^0 is a rectangular domain of the form $\{-\varepsilon < x_1 < 0\} \cap \Omega$ for ε small enough and where we assume that $\Omega_1 \subset \{x_1 < 0\}$ (for instance) - see figure 3 below.

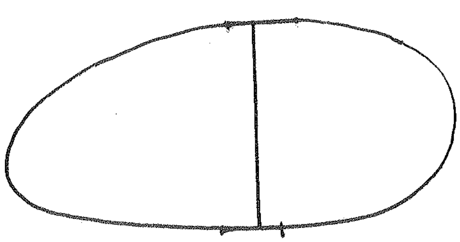

Figure 3.

Then, we have in view of Step 2 and by straightforward considerations

(26)
$$
\begin{aligned}
&-\Delta \hat{v}_2^n = -\Delta v_1^{n+1} = 0 \quad \text{in } \Omega_1^0 \quad , \\
&\frac{\partial}{\partial n_{12}} (\hat{v}_2^n + v_1^{n+1}) \underset{n}{\rightarrow} 0 \quad \text{in } L^2(\gamma) \quad .
\end{aligned}
$$

while of course $\hat{v}_2^n$, v_1^{n+1} are bounded in $H^1(\Omega_1^0)$ and converge weakly to 0. By standard elliptic considerations, one deduces that

$$
v_1^{n+1} + \hat{v}_2^n \underset{n}{\rightarrow} 0 \qquad \text{in } H^{1/2}(\gamma)
$$

therefore

$$
\frac{1}{2} (u_1^{n+1} + u_2^n) \underset{n}{\rightarrow} u \qquad \text{in } H^{1/2}(\gamma_{12}) \, .
$$

IV. Convergence analysis in the general case.

We now replace the Laplace operator in (2), (3), (5), (8) ... by a more general second-order elliptic operator of the form

$$A = -\Delta + b(x) \cdot \nabla + c(x) \tag{27}$$

where b, c are smooth (for instance). Our main assumption will be a coercivity assumption on each subdomain Ω_i

$$\begin{aligned} &\exists\, \nu > 0\,,\ \forall\, 1 \leq i \leq m\,,\ \forall\, w \in H^1_\Gamma(\Omega_i) \\ &\int_{\Omega_i} |\nabla w|^2 + b(x) \cdot \nabla w\, w + cw^2\, dx \geq \nu \int_{\Omega_i} |\nabla w|^2\, dx\,. \end{aligned} \tag{28}$$

This of course allows to consider the same iterative method as in section II.

Theorem 3. *Under the above assumption, u_i^n converges weakly in $L^2(\partial\Omega_i)$ to $u_{|\partial\Omega_i}$ and u_i^n converges in $L^2(\Omega_i)$ to $u_{|\Omega_i}$.*

Remarks: 1) One can make the same remark as the one we made after Theorem 1.

2) We do not know whether u_i^n remains bounded in $H^1(\Omega_i)$ but the proof below will show that u_i^n remains bounded in $H^{1/2}(\Omega_i)$.

Proof: One first observes that, because of (28), the proof made in step 1 of the proof of Theorem 1 still applies and thus yields (14),(15) and (18).

Next, we want to prove that $u^n_{i|\partial\Omega_i}$ remain bounded for all $1 \leq i \leq m$. We then argue by contradiction : let n'' be a subsequence such that

$$\max_{1 \leq i \leq m} \|u_i^{n''}\|_{L^2(\partial\Omega_i)} \underset{n''}{\rightarrow} +\infty$$

and let $i \in \{1, \ldots, m\}$, let n' be a subsequence of n'' such that

$$\|u_i^{n'}\|_{L^2(\partial\Omega_i)} = \max_{1\leq i\leq m} \|u_i^{n'}\|_{L^2(\partial\Omega_i)} \underset{n'}{\to} +\infty .$$

Without loss of generality we may assume that for some $j \neq i$

$$\|u_i^{n'}\|_{L^2(\gamma_{ij})} \overset{n'}{\to} +\infty \quad , \quad \|u_i^{n'}\|_{L^2(\gamma_{ij})} \geq \frac{1}{\sqrt{m}}\|u_i^{n'}\|_{L^2(\partial\Omega_i)} .$$

We then define $w_k^n = u_k^n \|u_i^n\|^{-1}_{L^2(\partial\Omega_i)}$ for all k, n and we claim that $w_j^{n'-1}$ is bounded in $L^2(\partial\Omega_j)$. First of all, because of (15), $w_j^{n'-1}$ is bounded in $L^2(\gamma_{ij})$. Now, recalling that for all k

$$\|u_k^{n'}\|_{L^2(\partial\Omega_k)} \leq \|u_i^{n'}\|_{L^2(\partial\Omega_i)}$$

we see that, in view of (15), we have

$$\begin{aligned}\|u_j^{n'-1}\|_{L^2(\partial\Omega_j)} &\leq \sum_{k\neq j} \|u_j^{n'-1}\|_{L^2(\gamma_{jk})} \leq C + \sum_{k\neq j} \|u_k^{n'}\|_{L^2(\gamma_{jk})} \\ &\leq C + \|u_i^{n'}\|_{L^2(\partial\Omega_i)}\end{aligned}$$

where C denotes various constants independent of $n, i, j, k \ldots$

Therefore, $w_j^{n'-1}$ is bounded in $L^2(\partial\Omega_j)$. At this stage, we simplify the presentation and avoid tedious arguments by assuming that $m = 2$, $i = 1$, $j = 2$. Then, we deduce from elliptic estimates that $w_1^{n'}$ and $w_2^{n'-1}$ are respectively bounded in $H^{1/2}(\Omega_1)$, $H^{1/2}(\Omega_2)$. Thus, assuming without loss of generality that they converge, say in L^2, to some $w_1, w_2 \in H^{1/2}$, we deduce easily that

$$Aw_1 = 0 \quad \text{in } \Omega_1 \qquad , \qquad Aw_2 = 0 \quad \text{in } \Omega_2 \quad ,$$

and that the traces of w_1, w_2 on γ $(= \gamma_{12})$ make sense, belong to $L^2(\gamma)$ and coincide because of (15). Furthermore, by standard uses of Green's formula, one checks that the traces of $\frac{\partial w_1^{n'}}{\partial \nu_1}$, $\frac{\partial w_2^{n'-1}}{\partial \nu_2}$ make sense and are bounded in $H^{-1}(\partial\Omega_1)$, $H^{-1}(\partial\Omega_2)$ respectively and that $\frac{\partial w_1}{\partial n_{12}} = \frac{\partial w_2}{\partial n_{12}}$ on γ (in $H^{-1}(\gamma)$), where ν_1, ν_2 denote the unit outward normal to $\partial\Omega_1$, $\partial\Omega_2$ (so that $\nu_1 = n_{12} = -\nu_2$ on γ). Of course, one also has $w_1 = 0$ on $\partial\Omega_1 \cap \partial\Omega$, $w_2 = 0$ on $\partial\Omega_1 \cap \partial\Omega$. Combining all these informations, we deduce finally $w_1 \equiv w_2 \equiv 0$.

Next, a similar proof to step 3 of the proof of Theorem 1 shows that $w_1^{n'} + w_2^{n'-1}$ is bounded in $H^{1/2}(\gamma)$, therefore

$$w_1^{n'} + w_2^{n'-1} \xrightarrow{n'} 0 \qquad \text{in } L^2(\gamma)\,,$$

while we know that $w_1^{n'} - w_2^{n'-1} \xrightarrow{n'} 0$ in $L^2(\gamma)$ because of (15). But

$$\|w_1^{n'}\|_{L^2(\gamma_{12})} = \frac{\|u_1^{n'}\|_{L^2(\gamma_{12})}}{\|u_1^{n'}\|_{L^2(\partial\Omega_1)}} \geq \frac{1}{\sqrt{m}}$$

by the choice of i, j. This contradiction shows the desired bound.

Then, we deduce that u_i^n is bounded in $H^{1/2}(\Omega_i)$. In order to conclude, one observes that, because of (14) and (18), we have

$$\sum_{n\geq 1}\sum_i \|\frac{\partial(u_i^{n+1} - u_i^{n-1})}{\partial\nu_i}\|^2_{H^{-1/2}(\partial\Omega_i)} < \infty\,.$$

Hence using once more (14), (18)

$$\sum_{n\geq 1}\sum_{i\neq j} \|u_i^{n+1} - u_j^n\|_{H^{-1/2}(\gamma_{ij})} < \infty\,.$$

Then, it is not difficult to conclude the convergence proof.

Remark: The proof above also yields a bound on $u_i^{n+1} + u_j^n$ in $H^{1/2}(\gamma_{ij})$.

V. Remarks

We begin with a few remarks concerning the choice of parameters (γ_{ij}) : first of all, it is possible to replace, say in the case $m = 2$, $\lambda = \lambda_{12}$ and $\mu = \lambda_{21}$ by two arbitrary constants (i.e. we do not need to assume $\lambda = \mu$) or even by two proportional functions on $\gamma = \gamma_{12} = \gamma_{21}$, or even by local or nonlocal operators ($-\Delta_s$ or $(-\Delta_s)^{1/2}$ where $-\Delta_s$ stands for the Laplace-Beltrami operator on γ ...). One can also consider sequences λ_{ij}^n ...

Next, it is worth discussing the effective choice of λ_{ij} : let us first indicate that this is by large an open problem. However, some examples may be illuminating : for instance, in one dimension, when $\Omega = (0,1)$, $\Omega_1 = (0,h)$, $\Omega_2 = (h,1)$ (thus $m = 2$) one gets exact convergence for the following two values of $\lambda = \lambda_{12} = \lambda_{21}$ namely $\lambda = \frac{1}{h}$ or $\lambda = \frac{1}{1-h}$. More generally, still in the above setting, if one replaces $-\frac{d^2}{dx^2}$ by $-\frac{d}{dx}((a(x)\frac{d}{dx})$ for some $a \in C([0,1])$, $a > 0$ in $[0,1]$, exact convergence still holds for $\lambda = \frac{1}{a(h)}\left(\int_h^1 \frac{1}{a(s)}\,ds\right)^{-1}$ or $\lambda = \frac{1}{a(h)}\left(\int_0^h \frac{1}{a(s)}\,ds\right)^{-1}$. Similarly, if Ω is the ball of radius R_0 in $\mathbb{R}^3$, Ω_1 is the ball of radius h and $\Omega_2 = \Omega - \overline{\Omega}_1$, then the iterative method for radial initial choices converges exactly for the value of λ : $\lambda = \frac{R_0}{h(R_0-h)}$.

We now make some remarks about the convergence proofs : our methods easily extend to a wide class of equations like general second-order elliptic equations, systems like linear elasticity or the Stokes problem, non selfadjoint problems like linearized Navier-Stokes equations, or even higher-order problems and time-dependent problems. It might be worth mentioning that it also applies to analogous iterative methods with overlapping subdomains (the classical "Schwarz setting" but with Neumann or our combinations of Neumann-Dirichlet conditions on the relevant interfaces). Of course, general boundary conditions are possible on $\partial\Omega$ and we can also consider transmission problems i.e. different elliptic equations in Ω_1 and Ω_2 and general relations between say the normal derivatives on each side (general oblique vectorfields are also possible).

References.

[1] H.A. Schwartz. *Über einige Abbildungsaufgaben.* Ges. Math. Abh., **11** (1869), 65–83.

[2] S.L. Sobolev. *The Schwarz algorithm in the theory of elasticity.* Dokl. Acad. N. USSR, **IV** 1936, 236–238 (in Russian).

[3] S.G. Michlin. *On the Schwarz algorithm.* Dokl. Acad. N. USSR, **77** (1951), 569–571 (in Russian).

[4] M. Prager. *Scharzüv algoritmus pro polyharmonické funcke.* Aplikace matematiky, **3** (1958), 2, 106–114.

[5] D. Morgenstern. *Begründung des alternierenden Verfahrens durch Orthogonalprojektion.* ZAMM, **36** (1956), 7–8.

[6] I. Babuska. *On the Schwarz algorithm in the theory of differential equations of Mathematical Physics.* Tchecosl. Math. J., **8** (83) (1958), 328–342 (in Russian).

[7] R. Courant and D. Hilbert. Methoden der matematischen Physik, Vol. 2, Berlin, 1937.

[8] F.E. Browder. *On some approximation methods for solutions of the Dirichlet problem for linear elliptic equations of arbitrary order.* J. Math. Mech., **7** (1958), 69–80.

[9] P.L. Lions. Unpublished notes, 1978.

[10] R. Glowinsky, J. Périaux and Q.V. Dinh. *Domain decomposition methods for nonlinear problems in fluid dynamics.* Rapport INRIA, **147**, (1982), Comp. Meth. Appl. Mech. Eng., **40** (1983), 27–109.

[11] Q.V. Dinh, R. Glowinski and J. Périaux. *Solving elliptic problems by decomposition methods with applications.* In Elliptic Problem Solvers II, Academic Press, New York, 1982.

[12] R. Glowinski. *Numerical solution of partial differential equation problems by domain decomposition. Implementation on an array processors system.* In Proceeding of International Symposium on Applied Mathematics and Information Science, Kyoto University, 1982.

[13] A. Fischler. *Résolution du problème de Stokes par une méthode de décomposition de domaines. Application à la simulation numérique d'écoulements de Navier-Stokes de fluides incompressibles en éléments finis.* Thèse de 3e cycle, Univ. P. et M. Curie, 1985, Paris.

[14] Q.V. Dinh, A. Fischler, R. Glowinski and J. Périaux. *Domain decomposition method for the Stokes problem. Application to the Navier-Stokes equations.* In Numeta 85, Swansea, 1985.

[15] Q.V. Dinh, J. Périaux, G. Terrasson and R. Glowinski. *On the coupling of incompressible viscous flows and incompressible potential lows via domain decomposition.* In ICNMFD, Pékin, 1986.

[16] J.M. Frailong and J. Pakleza. *Resolution of a general partial differential equation on a fixed size SIMD:MIMD large cellular processor.* In Proceedings of the IMACS International Congress, Sorente, 1979.

[17] Q.V. Dinh. *Simulation numérique en éléments finis d'écoulements de fluides visqueux incompressibles par une méthode de décomposition de domaines sur processeurs vectoriels.* Thèse de 3e cycle, Univ. P. et M. Curie, 1983, Paris.

[18] P.E. Bjorstad and O.B. Widlung. *Solving elliptic problems on regions partitioned into substructures.* In Elliptic Problem Solvers II, Academic Press, New York, 1982.

[19] P. Lemonnier. *Résolution numérique d'équations aux dérivées partielles par décomposition et coordination.* Rapport INRIA, 1972.

[20] L. Cambier, W. Ghazzi, J.P. Veillot and H. Viviand. *A multidomain approach for the computation of viscous transonic flows by unsteady methods.* In Recent advances in numerical methods in fluids. Vol. 3, Pineridge, Swansea, 1984.

[21] P. Anceaux, G. Gay, R. Glowinski, J. Périaux. *Résolution spectrale du problème de Stokes et coordination.* Euromech. 159, Spectral methods in computational fluid mechanics, Nice, 1982.

[**22**] J.P. Benque, J.P. Grégoire, A. Hauguel and M. Maxant. *Application des méthodes de décomposition aux calculs numériques en hydraulique industrielle.* In 6e Colloque International sur les méthodes de Calcul Scientifique et Technique, Versailles, 1983.

[**23**] Q.V. Dinh, R. Glowinski, B. Mantel, J. Périaux and P. Perrier. *Subdomain solutions of nonlinear problems in fluid dynamics on parallel processors.*In 5th International Symposium on computational methods in applied sciences and engineering, Versailles, 1981, North-Holland.

[**24**] G.I. Marchuk, Yu. A. Kuznetsov and A.M. Matsokin. *Fictitious domain and domain decomposition methods.* Sov. J. Numer. Anal. Math. Modelling, **1** (1986), 3–35.

[**25**] M. Dryja. *Domain decomposition method for solving variational difference systems for elliptic problems.* In Variatsionnonaznostnye metody v matematicheskoi fizike, Eds. N.S. Bakhvalov and Yu. A. Kuznetsov. Otdel Vichislitel'noi Matematiki Akad. Nauk SSSR, Moscow, 1984 (in Russian).

[**26**] M. Dryja. *A finite element - capacitance matrix method for the elliptic problem.* SIAM J. Numer. Anal., **20** (1983), 671–680.

[**27**] A.M. Matsokin. *Fictitious component method and the modified difference analogue of the Schwarz method.* In Vychislitel'nye Metody Lineinoi Algebry, Ed. G.I. Marchuk, Vychisl. Tsentr. Sib. Otdel. Akad. Nauk. SSSR, Novosibirsk, 1980 (in Russian).

[**28**] A.M. Matsokin and S.V. Nepomnyashchikh. *On the convergence of the alternating subdomain Schwarz method without intersections.* In Metody Interpolyatsii i Approksimatsii, ed. Yu. A. Kuznetsov. Vychisl. Tsentr. Sib. Otdel. Akad. Nauk SSSR, Novosibirsk, 1981 (in Russian).

[**29**] J.B. Baillon and P.L. Lions. *Convergence de suites de contractions dans un espace de Hilbert.* Rapport Ecole Polytechnique, Palaiseau, 1977.

[**30**] J. Céa. *Optimisation. Théorie et algorithmes.*. Dunod, Paris, 1971.

[**31**] Z. Opial. *Weak convergence of the sequence of successive ap-*

proximations for non-expansive mappings. Bull. Amer. Math. Soc. **73** (1967), 591–597.

[**32**] Z. Cai and S. Mac Cormick. *Computational complexity of the Schwarz alternating procedure.* Preprint.

[**33**] Z. Cai and B. Cuo. *Error estimate of the Schwarz alternating procedure on L -shape domain.* Appl. Math. Comp., to appear.

[**34**] P. Bjorstad and O. Widlund. *Iterative methods for the solution of elliptic problems on regions partitioned into substructures.* SIAM Num. Anal., **23** (1986), 1097–1120.

[**35**] J. Olger, W. Shamarock and W. Tang. *Convergence analysis and acceleration of the Schwarz alternating method.* Technical report, Stanford Univ., Center for Large Scale Scientific Computation, 1986.

[**36**] P.L. Lions. *On the Schwarz alternating method. I.* In First International Symposium on Domain Decomposition methods for Partial Differential Equations. SIAM, Philadelphia, 1988.

[**37**] R. Dautray and J.L. Lions. *Analyse mathématique et calcul numérique pour les sciences et techniques.* Tome 1. Masson, Paris, 1985.

[**38**] P.L. Lions. *Interprétation stochastique de la méthode alternée de Schwarz.* C.R. Acad. Sci. Paris, **268** (1978), 325–328.

[**39**] O.B. Widlund. *Iterative substructuring methods : algorithms and theory for elliptic problems in the plane.* In Domain Decomposition Methods for Partial Differential Equations. I. R. Glowinski, G. Golub and J. Périaux, eds., SIAM, Philadelphia, 1988.

[**40**] J. Bramble, J. Pasciak and G. Schatz. *The construction of preconditioners for elliptic problems by substructuring. I.* Math. Comp., **47** (1986), 103–134.

[**41**] J. Bramble, J. Pasciak and G. Schatz. *The construction of preconditioners for elliptic problems by substructuring. II.* Preprint.

[**42**] M. Dryja and O. Widlund. *An additive variant of the Schwarz alternating method for the case of many subregions.* To appear in

Domain Decomposition Methods. II. SIAM, Philadelphia, 1989.

[43] T.F. Chan and D.C. Resasco. *Survey of preconditions for domain decomposition.* Research Report Yale U/DCS/RR-414, 1987.

[44] R. Glowinski and M.F. Wheeler. *Domain decomposition and mixed finite element methods for elliptic problems.* In Domain Decomposition Methods for Partial Differential Equations. I.. SIAM, Philadelphia, 1988.

[45] L.D. Marini and A. Quarteroni. *An iterative procedure for domain decomposition methods : a finite element approach.* In Domain Decomposition Methods for Partial Differential Equations. I.. SIAM, Philadelphia, 1988.

[46] D. Funaro, A. Quarteroni and P. Zanolli. *An iterative procedure with interface relaxation for domain decomposition.* To appear in SIAM J. Numer. Anal..

[47] L.D. Marini and A. Quarteroni. *A relaxation procedure for domain decomposition methods using finite elements.* I.A.N.-C.N.R. publication n.577, Pavia, 1987.

[48] A. Quarteroni and A. Valli. *Domain decomposition for a generalized Stokes problem.* To appear in Proc. of the 2nd ECMI Conf., Glasgow, 1988.

[49] A. Quarteroni. *A relaxation procedure for domain decomposition methods for the Stokes equations.* In Domain Decomposition Methods. II. SIAM, Philadelphia, 1989.

[50] A. Quarteroni. *Domain decomposition for the numerical solution of partial differential equations.* I.A.N.-C.N.R. publication n.591, Pavia, 1987.

[51] A. Quarteroni and G. Sacchi Landriani. *Parallel algorithms for the capacitance matrix method in domain decompositions.* I.A.N.-C.N.R. publication n.595, Pavia, 1987.

[52] F. Gastaldi, A. Quarteroni and G. Sacchi Landriani. *Effective methods for the treatment of interfaces separating equations of different character.* To appear in CMEM, 1989.

[**53**] T.F. Chan. *A survey of preconditioners for domain decomposition.* SIAM J. Numer. Anal..

[**54**] P.L. Lions. *On the Schwarz alternating method. II.* In Domain Decomposition Methods, SIAM, Philadalphia, 1989.

[**55**] T.F. Chan, T. Hou and P.L. Lions. *Geometry independent convergence results for domain decomposition algorithms.*

[**56**] J. Bourgat, R. Glowinski, P. Le Tallec and M. Vidrascu. *Variational formulation and algorithm for trace operator in domain decomposition calculations.* In Domain Decomposition Methods, SIAM, Philadelphia, 1989.

[**57**] D. Keyes and W. Gropp. *A comparison of domain decomposition techniques for elliptic partial differential equations.* SIAM J. Sci. Stat. Comp., **8** (1987), 166–202.

[**58**] K. Miller. *Numerical analogs to the Schwarz alternating procedure.* Numerische Mathematik, **7** (1965), 91–103.

[**59**] W.P. Tang. *Schwarz splitting and template operators.* Ph. D. Thesis, Stanford University, 1987.

[**60**] W.P. Tang. *Relief from the pain of overlap. Generalized Schwarz splittings.* Preprint.

[**61**] P. Le Tallec. Personnal communication.

[**62**] M. Fortin. Work in preparation.