CHAPTER 1

Towards a Unified Theory of Domain Decomposition Algorithms for Elliptic Problems*

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Abstract. A distinction is often made between domain decomposition algorithms for elliptic partial differential equations which use overlapping subregions and those which do not. Schwarz's alternating method, the oldest of them all, belongs to the first category. It has recently been discovered that many of the algorithms that belong to the second category can also be regarded as generalizations of the classical Schwarz method or an additive variant thereof. This new approach provides new tools for the analysis and development of domain decomposition algorithms. In this paper, we introduce an abstract additive Schwarz method and develop a simple framework for the analysis of its rate of convergence. We show that it is possible and convenient to specify algorithms in terms of a set of subspaces and related orthogonal projections. This family of algorithms is further expanded by replacing the linear systems of equations, which correspond to the projections, by suitable preconditioners.

We illustrate the usefulness of this approach by considering additive Schwarz methods of a relatively conventional type, iterative substructuring methods, domain decomposition methods defined on the curves or surfaces which subdivide the region, and iterative refinement methods. Throughout we work in a framework of conforming finite elements and self-adjoint problems, but we also mention some new results for more general elliptic finite element problems.

1. Introduction. In this paper, we survey our recent research on domain decomposition and related algorithms for elliptic finite element problems. One of our aims is to develop a set of common and powerful tools for the development and analysis of a great variety of methods. We have recently discovered that many iterative substructuring methods can be viewed as so-called additive Schwarz methods and that the analysis of these methods can be simplified; cf. Dryja and Widlund [28]. Here, we discuss and extend this result and provide further evidence that many domain decomposition methods fit into a framework provided by the Schwarz additive methods.

We note that the bounds given for the condition number of the iteration operators for many domain decomposition algorithms considered in this and other papers are either

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uniform in the number of subdomains and subregions or grow only polynomially in the logarithm of the number of degrees of freedom associated with an individual subregion. The algorithms are therefore, in a certain sense, almost optimal.

The paper is organized as follows. After introducing two elliptic model problems and certain finite element methods in Section 2, we begin Section 3 by reviewing Schwarz's alternating algorithm in its classical setting. Following Sobolev [50] and P. L. Lions [34], we indicate how this algorithm can be expressed in a variational form. Since this formulation is very convenient for the analysis of finite element problems, we work in this Hilbert space setting throughout the paper.

In Section 3, we also introduce the additive variant of Schwarz's method; cf Dryja [24], Dryja and Widlund [26], [28], Matsokin and Nepomnyaschikh [41] and Nepomnyaschikh [44]. We present it in a general form and show how algorithms of this kind can be defined in terms of a set of subspaces and projections. An additive Schwarz method can be viewed as an iterative method for the solution of an auxiliary linear problem that has the same solution as the original finite element problem. We also introduce tools, which make it possible to estimate the condition number $\kappa(P)$ of the operator P of this new problem.

It is often straightforward to obtain an upper bound for the spectrum of P. A lower bound of the eigenvalues is obtained in terms of an upper bound of a Rayleigh quotient which measures the extent by which the subspaces are linearly independent. (If the subspaces are orthogonal, the Schwarz method converges in one step.) We show that preconditioners can be used as approximate solvers for the linear systems, which are related to the individual projections, and that it is easy to obtain bounds for the condition number of the resulting algorithm in terms of the spectrum of the basic operator and any bounds for the individual preconditioned problems that might be available.

Section 3, thus, provides us with a basic framework. In the rest of the paper, we turn to a series of applications.

In Section 4, we summarize our work on additive Schwarz methods of a type directly related to the classical case. For previous discussions of this algorithm, see Dryja [24], Dryja and Widlund [26], [28], with the most details given in [28]. We note that this method has been extended successfully to certain stationary and parabolic convection-diffusion problems in the dissertation of Xiao-Chuan Cai [20],[19], and to mixed finite element methods of the Raviart-Thomas type, cf. [46], in the dissertation of Tarek Mathew [40]. In this paper, we are unable to provide details about their work since it would require the introduction of many notations, etc. We also discuss a result on multilevel algorithms obtained in collaboration with Xuejun Zhang, a Courant Institute graduate student.

In Section 5, we discuss a well-known family of domain decomposition methods known as iterative substructuring methods; cf. e.g. Agoshkov [1], Bjørstad and Hvidsten [6], Bjørstad and Widlund [9], Bramble, Pasciak and Schatz [15], [14], [16], [17], [18], Dryja [23], Dryja, Proskurowski and Widlund [25], Dryja and Widlund [28], Lebedev [32], Marchuk, Kuznetsov and Matsokin [39], Quarteroni [45], Smith and Widlund [49] and Widlund [54]. These algorithms are based on a non-overlapping subdivision $\{\Omega_i\}$ of the region Ω. (Borrowing a term from structural engineering, the subregions are often called substructures.) Surprisingly enough, as shown already in Dryja and Widlund [28], many of these methods also fit well in the Schwarz framework. In this paper, we further extend our analysis and include a more detailed discussion of the three-dimensional case. In this study, it is fruitful to take a substructure by substructure view, estimating the contribution to the strain energy from an individual substructure in terms of the corresponding local contribution to the preconditioner. This approach is similar to that of Jan Mandel, and his coworkers Babuška, Craig and Pitkäranta, cf. [2], [37], [36], [35], who have begun a systematic study of domain

decomposition methods for p–version finite element methods; cf. also Bramble, Pasciak and Schatz [18] for a fundamental study of h–version methods. This approach highlights how preconditioners can be constructed from parts which are strictly local with respect to an individual substructure, parts which involve interaction between pairs of neighboring substructures and a coarse global model with relatively few degrees of freedom. Many of the results which were obtained for two subregions in the early development of the theory can now be recycled to yield useful results for the much more interesting case of many substructures. We note that some of our results have been extended to a class of two-dimensional, stationary and parabolic convection-diffusion problems by Xiao-Chuan Cai [20], [19].

In Section 6, we consider Schwarz methods on the lower dimensional manifolds formed by the curves or surfaces, which partition the domain into substructures. These algorithms can also be viewed as Schwarz methods for problems of potential theory. This idea has previously been discussed by Nepomnyaschikh [44] for the case of a few substructures and without a full development of the theory. We include a discussion and estimates of the rate of convergence of a method developed by Bourgat, Glowinski, Le Tallec and Vidrascu [11] and also discuss some recent work by Barry Smith, a Courant Institute graduate student; cf. [48].

In Section 7, we survey our recent work on iterative refinement methods. These are methods for the solution of the linear systems of algebraic equations, which arise from elliptic finite element problems defined on composite meshes. The study of these methods was pioneered by McCormick and his coworkers Hart and Thomas [31], [42], [43]. An analysis of two-level algorithms is given in Bramble, Ewing, Pasciak and Schatz [13] and in Mandel and McCormick [38]. Proofs of the results discussed in this section are given in Dryja and Widlund [27] and Widlund [56]. The main difficulties in this work are related to an effort of obtaining bounds for the rate of convergence which are independent of the number of refinement levels as well as the number of degrees of freedom. We note that in the dissertation of Tarek Mathew [40], certain results are obtained on iterative refinement methods for Raviart-Thomas finite elements.

In this paper, we primarily consider algorithms that are of additive Schwarz type. We note that a variety of multiplicative algorithms also can be defined systematically. At present, the general multiplicative case appears to be less well understood than the additive, except in the case of two subspaces. For the two subspace case, Bjørstad and Mandel [7] have recently extended their earlier work, cf. [5] and [38], obtaining a detailed comparison of the spectra of the additive and multiplicative algorithms. We note that there are cases for which a satisfactory theory already exists in the case of more than two subspaces. Thus the algorithm of Bank, Dupont and Yserentant [3] is a multiplicative variant of the original, additive algorithm of Yserentant's [58]; it is just as well understood. Similarly, the basic theory for the multilevel iterative refinement methods has advanced to the same level in the two cases; cf. Widlund [56] and Dryja and Widlund [27]. We note, finally, that in the recent thesis of Mathew important progress is reported for the general multiplicative case; cf. Mathew [40] and a remark in Section 3.

2. Model Problems and Finite Elements. In this section, we introduce finite element approximations of a standard Poisson equation and a special second order elliptic problem with variable coefficients.

In the model problems, the continuous and discrete problems are of the form

$$a(u, v) = f(v), \quad \forall\, v \in V\,,$$

and

$$a(u_h, v_h) = f(v_h), \quad \forall\ v_h \in\ V^h\ , \tag{1}$$

respectively. We consider homogeneous Dirichlet problems on bounded Lipschitz regions in two or three dimensions and continuous, piecewise linear finite elements. For the Poisson problem, the bilinear form is defined by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx\ . \tag{2}$$

This form defines a semi-norm $|u|_{H^1(\Omega)} = (a(u,u))^{1/2}$ in the Sobolev space $H^1(\Omega)$. It is a norm of $V = H_0^1(\Omega)$. Here $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ functions with zero trace; all elements of V and its subspace V^h vanish on $\partial\Omega$, the boundary of Ω. We note that if $\tilde{\Omega} \subset \Omega$, then any element of $H_0^1(\tilde{\Omega})$ can be extended by zero to an element in $H_0^1(\Omega)$, and that the extension operator is continuous. We therefore regard $H_0^1(\tilde{\Omega})$ as a subspace of $H_0^1(\Omega)$.

Almost all our results can be extended immediately to general conforming finite element approximations of any self-adjoint elliptic problem, which can be formulated as a minimization problem, and to more general boundary conditions. However, some of the bounds can be very poor if there is a great variation in the values of the coefficients. In this paper, we consider only one generalization,

$$a(u, v) = \int_\Omega \rho(x)\nabla u \cdot \nabla v\, dx\ , \tag{3}$$

where $\rho(x) > 0$ can be discontinuous, but varies slowly in each subregion. There is experimental evidence that some domain decomposition methods of Neuman-Dirichlet type perform quite poorly for certain problems of the form (3); cf. Greenbaum, Li and Chao, [30]. For other algorithms a satisfactory theory already exists with bounds which only depend on the local variation of $\rho(x)$; see Section 5. For still others, this issue is still to be fully addressed.

The triangulation of Ω is introduced in the following way. The region is first divided into non-overlapping substructures Ω_i, $i = 1, \cdots, N$. We assume that the substructures are chosen so that the discontinuities of $\rho(x)$ occur only at substructure boundaries. To simplify the description, we confine most of our study to triangular (simplicial) substructures. In such a case, the original region must, of course, be a polygon (polyhedron). We note that the rate of convergence of the iterative methods, which are considered in this paper, depend only very mildly on the size of the substructures. The size of the subregions can therefore be selected primarily to minimize the cost of a single iteration on a particular computer system.

All the substructures Ω_i are further divided into elements. The common assumption in finite element theory that all elements are shape regular is adopted and the same assumption is made concerning the substructures. On the element level this means that there is a uniform bound on h_K/r_K , which is independent of the number of degrees of freedom. Here h_K is the diameter of the element K and r_K the diameter of the largest inscribed sphere in K.

Since $a(u_h, v_h) = a(u, v_h)$, $\forall\ v_h \in\ V^h$, the finite element solution is the projection of the exact solution onto the finite element space with respect to the inner product defined by the bilinear form. The problems defined on the subregions, from which preconditioners for the entire problem can be assembled, can often similarly be viewed in terms of orthogonal projections onto subspaces directly associated with the subregion in question.

We also need to use matrix representations of the finite element problems since many algorithmic details can best be described using matrix notations. When doing this, it is convenient to consider domain decomposition in light of structural engineering computational practices. The elements of the stiffness matrix K are given by

$$k_{l,m} = a(\varphi_l, \varphi_m) ,$$

where φ_l and φ_m are standard finite element basis functions. Since an integral over Ω can be written as a sum of integrals over the substructures, the stiffness matrix can be assembled from the stiffness matrices $K^{(i)}$, which have the elements

$$k^{(i)}_{l,m} = a_{\Omega_i}(\varphi_l, \varphi_m) .$$

Here the relevant l and m correspond to degrees of freedom associated with the closure of the substructure Ω_i. The bilinear form $a_{\Omega_i}(u_h, v_h)$ represents the contribution to the integral $a_\Omega(u_h, v_h)$ from the substructure Ω_i.

This so-called subassembly process can be summarized in the formula

$$x^T K y = \sum x^{(i)^T} K^{(i)} y^{(i)}, \tag{4}$$

where $x^{(i)}$ is the subvector of parameter values associated with $\bar{\Omega}_i$.

Remark. In practice, the subassembly process is often used recursively, creating larger and larger so-called super elements. This process can be interleaved with the elimination of all the variables which, at the level in question, are coupled only to other variables of the same super element; cf. e.g. [4]. Except for a brief discussion in Section 4, we consider only three levels in this paper: the elements with a characteristic diameter h, the substructures with a diameter on the order of H and the entire region Ω, which, without loss of generality, is assumed to have unit diameter.

If we divide the subvectors $x^{(i)}$ associated with the i-th substructure into two, $x^{(i)}_I$ and $x^{(i)}_B$, corresponding to the variables which are interior to the substructure and those which are shared with other substructures, then the matrix $K^{(i)}$ can be written as

$$\begin{pmatrix} K^{(i)}_{II} & K^{(i)}_{IB} \\ K^{(i)T}_{IB} & K^{(i)}_{BB} \end{pmatrix} .$$

Since the interior variables are associated with only one of the substructures, they can be eliminated locally and in parallel. The reduced matrix is a so-called Schur complement and has the form

$$S^{(i)} = K^{(i)}_{BB} - K^{(i)^T}_{IB} K^{(i)^{-1}}_{II} K^{(i)}_{IB}.$$

It is now easy to show that if the corresponding Schur complement of the global stiffness matrix K is denoted by S, then,

$$x^T_B S y_B = \sum x^{(i)^T}_B S^{(i)} y^{(i)}_B . \tag{5}$$

The elimination of the interior variables from the substructures can be viewed in terms of orthogonal projections, with respect to the bilinear form, of the solution u_h of equation (1) onto the subspaces $H^1_0(\Omega_i) \bigcap V^h$, $i = 1, \cdots, N$. It is easy to show that these subspaces are orthogonal, in the sense of the bilinear form (4), to the so-called piecewise discrete harmonic functions given by

$$K^{(i)}_{II} x^{(i)}_I + K^{(i)}_{IB} x^{(i)}_B = 0, \quad \forall i.$$

If the local problems are solved exactly, what remains is to find a sufficiently accurate approximation of the part of the solution which is piecewise discrete harmonic. This can be done by approximately solving the reduced linear system with the matrix S. A specific iterative substructuring method is obtained by selecting a preconditioner for the matrix S. Once an approximation of the solution has been found on the boundaries of the substructures, the solution can be found everywhere by separately solving local Dirichlet problems on each substructure.

An important family of domain decomposition algorithms can be derived by replacing each contribution $S^{(i)}$ to the Schur complement by a different matrix $\hat{S}^{(i)}$. A preconditioner $\hat{S}$ of S is then created by assembling the local contributions in the same way as in equation (5). When choosing $\hat{S}$, it is important to create a linear system, which is cheaper to solve and, at the same time, to ensure that $\kappa(\hat{S}^{-1}S)$ is small. A natural idea is to eliminate the coupling between the groups of variables associated with different edges (faces) of the individual substructures. Further details are given in Dryja and Widlund [28], where this process is interpreted as a splitting in the sense of Varga [51]. In the analysis, a simple and powerful idea is based on the fact that if

$$c_i \, {x_B^{(i)}}^T \hat{S}^{(i)} x_B^{(i)} \leq {x_B^{(i)}}^T S^{(i)} x_B^{(i)} \leq C_i \, {x_B^{(i)}}^T \hat{S}^{(i)} x_B^{(i)} \ , \ \forall x_B^{(i)}, \tag{7}$$

then,

$$\underline{c} \, x_B^T \hat{S} x_B \leq x_B^T S x_B \leq \bar{C} \, x_B^T \hat{S} x_B \ , \ \forall x_B, \tag{8}$$

where

$$\underline{c} = \min_i c_i \quad \text{and} \quad \bar{C} = \max_i C_i \ . \tag{9}$$

When this technique can be used, it is as easy to derive satisfactory bounds for equations (3) as for (2); cf. Section 5.

3. Multiplicative and Additive Schwarz Methods. In this section, we first review Schwarz's classical alternating method [47], its variational formulation, cf. P. L. Lions [34], and an additive variant of the algorithm, cf. Dryja [24], Dryja and Widlund [26] and Matsokin and Nepomnyaschikh [41]. We then discuss a general approach to the analysis of the rate of convergence of the additive algorithms.

We begin by briefly discussing the classical formulation of Schwarz's method in the case of the continuous Poisson equation. There are two fractional steps corresponding to two overlapping subregions, Ω_1' and Ω_2', the union of which is the region Ω . Let an initial guess $u^0 \in V$ be given. The iterate u^{n+1} is determined from u^n by sequentially updating the approximate solution in the two subregions:

$$\begin{aligned} -\Delta u^{n+1/2} &= f && \text{in } \Omega_1', \\ u^{n+1/2} &= u^n && \text{on } \partial\Omega_1', \end{aligned}$$

and

$$\begin{aligned} -\Delta u^{n+1} &= f && \text{in } \Omega_2', \\ u^{n+1} &= u^{n+1/2} && \text{on } \partial\Omega_2' . \end{aligned}$$

We could just as well have written down the finite element version of the algorithm. From now on, we only consider that case. It is easy and convenient to describe this method

in terms of two projections P_i, $i = 1,2$, onto $V_i^h = H_0^1(\Omega_i') \bigcap V^h$; cf. Lions [34]. The projections are defined by

$$a(P_i v_h, \phi_h) = a(v_h, \phi_h), \ \forall \phi_h \in V_i^h \ . \tag{10}$$

It is also easy to show that the error propagation operator of this multiplicative Schwarz method is

$$(I - P_2)(I - P_1).$$

This algorithm can therefore be viewed as a simple iterative method for solving

$$(P_1 + P_2 - P_2 P_1) u_h = g_h, \tag{11}$$

with an appropriate right-hand side g_h .

This algorithm can be generalized immediately to any number of subspaces. Let

$$V^h = V_1 + V_2 + \ldots + V_N \ .$$

The subspaces V_i can be chosen quite arbitrarily. The projections are defined as in equation (10).

The operator of (11) is a polynomial of degree two and, thus, the method is not ideal for parallel computing, since two sequential steps are involved. If more than two subspaces are used, this effect is further pronounced even if the degree of the polynomial representing the multiplicative algorithm often is lower than maximal. This is so because a product of two projections associated with subspaces that do not intersect nontrivially vanishes; cf. further Widlund [53], [55]. This can best be described in terms of the coloring of an undirected graph in which the nodes represent the subspaces and the edges nontrivial intersections of pairs of subspaces. The problems associated with all subspaces of the same color can be solved in parallel since those subspaces are mutually orthogonal. For the purpose of theory, we can then regard all subspaces of the same color as one subspace; this can greatly improve the upper bound for the eigenvalues of the operator P, which we are about to introduce.

The basic idea behind the additive form of the algorithm is to work with the simplest possible polynomial in the projections. The equation

$$P u_h = (P_1 + P_2 + ... + P_N) u_h = g_h' \ , \tag{12}$$

is solved by an iterative method. (In fact, we could just as well work with $\sum_{i=1}^{N} \alpha_i P_i$, where $\alpha_i > 0$ are suitably chosen constants.) Since we can show that the operator P is symmetric and positive definite, with respect to the bilinear form, the method of choice is the conjugate gradient method. Equation (12) must have the same solution as equation (1), i.e. the correct right-hand side must be found. Since $a(u_h, \phi_h) = f(\phi_h)$, by equation (1), the right-hand side g_h' can be constructed by solving equation (10) for all values of i and adding the results. It is similarly possible to apply the operator P of equation (12) to any element of V^h by applying each projection P_i to the element and adding the results. Most of the work, in particular that which involves the individual projections, can be carried out in parallel. We note that an additive Schwarz algorithm is fully specified by its subspaces. In this survey paper, we will sometimes just specify the subspaces without much discussion of how the corresponding algorithm can be implemented.

It is well known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to $\sqrt{\kappa}$, where κ is the condition number of the relevant operator; see e.g. Golub and Van Loan [29] . We

therefore need to establish that the operator P of equation (12) is not only invertible but also that satisfactory upper and lower bounds on its eigenvalues can be obtained.

An upper bound for the eigenvalues of P is given by N since P is the sum of projections. By combining all subspaces of the same color into one, as indicated above, this bound can be improved by replacing N by the number of colors of any coloring of that graph. For several algorithms this results in an upper bound, which is independent of the number of subspaces. In other instances, a useful technique is provided by strengthened Cauchy inequalities; cf. Mandel and McCormick [38], Widlund [56] and Yserentant [58].

A lower bound can often be obtained conveniently by using a lemma, inspired by Lions [34]; the simple proof is also given in Widlund [56].

LEMMA 1. *Let $u_h = \sum_{i=1}^{N} u_{h,i}$, where $u_{h,i} \in V_i$, be a representation of an element of $V^h = V_1 + ... + V_N$. If the representation can be chosen so that $\sum_{i=1}^{N} a(u_{h,i}, u_{h,i}) \leq C_0^2 a(u_h, u_h), \forall u_h \in V^h$, then $\lambda_{\min}(P) \geq C_0^{-2}$.*

Remark. If we expand individual subspaces, there is a larger choice in selecting $u_{h,i} \in V_i$. The best bound for C_0 can then only improve. If we can expand the subspaces without worsening the upper bound, which is often possible, our estimate of $\kappa(P)$ improves. On the other hand, an enlarged subspace also means that the subproblem has more variables and that it is worse conditioned. For the special case of the classical Schwarz method on two regions, this tradeoff is well understood; for a discussion of precise estimates of the rate of convergence; see Bjørstad and Widlund [10].

Remark. In his recent dissertation, Tarek Mathew [40] has given a bound on the rate of convergence of the multiplicative algorithm in terms of C_0 and the number of colors of the graph. While his bound can probably be improved considerably, it nevertheless shows that if these parameters are bounded independently of the mesh parameters, then the spectral radius of the error propagation operator is uniformly bounded by a constant that is less than 1.

One of the attractive features of the framework introduced in this section is the ease by which variants of the basic algorithm, obtained by replacing the local linear systems by preconditioners, can be analyzed. Let us, for example, consider the additive Schwarz method for a finite element approximation of the problem discussed in the beginning of this section. We can write the projection P_1 in matrix terms. After a suitable permutation of the variables, it corresponds to

$$y = P_1 x = \begin{pmatrix} K_{(1)}^{-1} & 0 \\ 0 & 0 \end{pmatrix} Kx \ . \tag{13}$$

Here $K_{(1)}$ represents the stiffness matrix of the Dirichlet problem on Ω_1'. It is easy to see that the matrix of (13) is symmetric in the K-inner product that corresponds to the bilinear form. If $K_{(1)}^{-1}$ is replaced by $\hat{K}_{(1)}^{-1}$, etc., then it is easy to show by using a Rayleigh quotients argument that the eigenvalues of the resulting operator $\hat{P}$ satisfy

$$\lambda_{\min}(P) \min_i (\lambda_{\min}(K_{(i)}\hat{K}_{(i)}^{-1})) \ \leq \ \lambda_{\min}(\hat{P}), \tag{14}$$

and

$$\lambda_{\max}(\hat{P}) \ \leq \ \lambda_{\max}(P) \max_i (\lambda_{\max}(K_{(i)}\hat{K}_{(i)}^{-1})) \ . \tag{15}$$

An estimate of $\kappa(\hat{P})$ follows immediately,

$$\kappa(\hat{P}) \leq \frac{\max_i \lambda_{\max}(K_{(i)}\hat{K}_{(i)}^{-1})}{\min_i \lambda_{\min}(K_{(i)}\hat{K}_{(i)}^{-1})} \kappa(P). \tag{16}$$

4. Additive Schwarz Methods on Overlapping Subregions. We now describe the additive Schwarz method introduced in Dryja and Widlund [26], [28]; cf. also Dryja [24]. In those papers only Poisson's equation was considered. In the original technical report, good results were given only for problems in two dimensions, but we now need no such restrictions. The most satisfactory proof, so far, is given in [28]. Here, we also discuss problems of the form (3).

We start with the same triangular (simplicial) non-overlapping substructures Ω_i that have been considered before. We extend each substructure to a larger region Ω_i'. We assume that the overlap is generous, assuming that the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by a fixed fraction of H_i, the diameter of Ω_i . We also assume that $\partial\Omega_i'$ does not cut through any element. We make the same construction for the substructures that are next to the boundary except that we cut off the part of Ω_i' that is outside of Ω .

Remark. The analysis of Schwarz methods is more complicated when the overlap is less generous; cf. a discussion in Lions [34]. Such a situation occurs if the region is L-shaped and partitioned into two overlapping rectangles. A similar situation is discussed in Dryja and Widlund [28] in an analysis of iterative substructuring methods; cf. also Section 5.

Our finite element space is represented as the sum of N+1 subspaces

$$V^h = V_0^h + V_1^h + ... + V_N^h .$$

The first subspace V_0^h is equal to V^H, the space of continuous, piecewise linear functions on the coarse mesh defined by the substructures Ω_i . The other subspaces are related to the subdomains in the same manner as in a traditional Schwarz algorithm, i.e. $V_i^h = H_0^1(\Omega_i') \bigcap V^h$.

Remark. There are few differences between the problem related to the first subspace and the others. The right hand side of the particular linear system is generated as weighted averages with weights determined by the coarse mesh basis functions. We also need to interpolate the solutions in V_0^h when combining them with the contributions from the other subspaces. The coarse, global approximation of the elliptic equation is otherwise quite similar to the local problems.

The global coarse problem provides a mechanism for the global transportation of information. As shown in Widlund [54], the rate of convergence of any domain decomposition method, for which the interaction is only through next neighboring subdomains, has a condition number which grows at least as fast as $1/H^2$.

The following result is established in Dryja and Widlund [28].

THEOREM 1. *The operator P of the additive algorithm defined by the spaces V^H and V_i^h, applied to equation (2), satisfies the estimate $\kappa(P) \leq const.$*

In the proof, a quasi-interpolant, $\hat{I}_H$, which is a bounded operator in $H_0^1(\Omega)$ and satifies

$$\|u_h - \hat{I}_H u_h\|_{L_2(\Omega)} \leq const.H|u_h|_{H^1(\Omega)},$$

is used. By examining the proof, it is easy to see that if uniform bounds of the same form hold for the weighted spaces $L_{2,\rho}(\Omega)$ and $H_{0,\rho}^1(\Omega)$, defined by

$$\int_\Omega \rho(x)|u|^2 dx \quad \text{and} \quad \int_\Omega \rho(x)|\nabla u|^2 dx,$$

respectively, then the optimality also holds for the variable coefficient case (3). This is a difficult problem, still not fully understood; cf. Xu [57] for partial results. We note that for two-dimensional problems the proof given in Dryja and Widlund [26] works just as well

for equation (3). The bound of the condition number given in that paper is proportional to $(1 + \log(H/h))$.

We have also considered the use of more than two levels of partitioning of the region and established that for equation (2) $\kappa(P)$ does not grow faster than quadratically in the number of levels. We do not know if this result can be improved. For convex regions, we have shown that the growth is linear in the number of levels. This result cannot be improved. Our work has been carried out jointly with Xuejun Zhang, who has also made a large number of numerical experiments. Rapid convergence can be obtained when using such methods solving only very small linear systems of equations.

As we have already pointed out in the introduction, Xiao-Chuan Cai [20], [19] and Tarek Matew [40] have extended our analysis to certain nonsymmetric and indefinite problems.

5. Iterative Substructuring Methods as Schwarz Methods. In this section, we show how iterative substructuring methods can be derived by using the framework of subspaces and projections developed in Section 3. We first consider problems in the plane and give an estimate of the condition number of the operator P that corresponds to a specific choice of subspaces. For this basic algorithm, the proof of the estimate does not rely on the extension theorems, which have played an important role in previous estimates for iterative substructuring methods; cf. Widlund [52]. We then indicate how results for the special case of two substructures, and an argument about preconditioners for the subspace problems, can be used to derive different algorithms, including one due to Bramble, Pasciak and Schatz [14]. We also consider a second choice of subspaces and show that the resulting algorithm extends to a fast method for the three-dimensional case. The resulting algorithm is similar but not identical to one of the two algorithms considered in Bramble, Pasciak and Schatz [18]; cf. also Dryja [23] for early work on the three-dimensional case. The discussion in this section extends that in Dryja and Widlund [28] and relies in part on ideas from Mandel [36].

We assume that the region is divided into substructures and elements as in Section 2. We first consider the two-dimensional case and introduce an additive Schwarz method. We use the coarse space V^H introduced in Section 4 and let the subregions $\Omega_{ij} = \Omega_i \bigcup \Gamma_{ij} \bigcup \Omega_j$ play the same role as Ω_i' in Section 4. Here Ω_i and Ω_j are adjacent substructures with a common edge Γ_{ij}. The local subspaces V_{ij}^h are thus defined as $V_{ij}^h = H_0^1(\Omega_{ij}) \bigcap V^h$.

Compared with the subspaces used in the previous section, we use less overlap in the sense that only the elements of V^H can differ from zero at the vertices of the substructures. This is reflected in a poorer bound on the condition number when we work with Lemma 1.

THEOREM 2. *In the two-dimensional case, the operator P of the additive algorithm defined by the spaces V^H and V_{ij}^h satisfies the estimate $\kappa(P) \leq const.(1 + log(H/h))^2$ for equations (2) and (3). The constant is independent of h, H and the discontinuities of the coefficient $\rho(x)$.*

For this algorithm, $\lambda_{\max}(P) \leq 4$; cf. Dryja and Widlund [28]. It is easy to see that no more than four colors are needed to color the graph; each interior substructure is covered exactly three times by the subspaces associated with its three edges and, in addition, we have the global space V^H which intersects all other subspaces. In our proof of the lower bound of the spectrum of P, given fully in [28], we use Lemma 1 and the following lemma, which also plays an important role in the more traditional theory of iterative substructuring algorithms.

LEMMA 2. *Let α be any convex combination of values of $u_h(x), x \in \bar{\Omega}_i$. Then*

$$\|u_h - \alpha\|^2_{L^\infty(\Omega_i)} \leq const.\ (1 + \log(H/h))|u_h|^2_{H^1(\Omega_i)}\ .$$

Variations of this result date back at least to 1966 and proofs are given in a number of papers; see e.g. Bramble [12], Bramble, Pasciak and Schatz [14] or Yserentant [58].

Since all the elements of V_{ij}^h vanish at the vertices of the substructures, we must choose the interpolant $I_H u_h$ as the element in V^H when, as required by Lemma 1, we define the representation of u_h . It is easy to show that $|I_H u_h|^2_{H^1(\Omega_i)}$ can be estimated from above by

$$\sum_{k,l \in \mathcal{V}_i} (u_h(x_k) - u_h(x_l))^2 , \tag{17}$$

where the $\mathcal{V}_i$ is the set of vertices of the substructure Ω_i . It then follows from Lemma 2 that

$$|I_H u_h|^2_{H^1(\Omega_i)} \leq const.(1 + \log(H/h))|u_h|^2_{H^1(\Omega_i)} . \tag{18}$$

For a bound on the other terms in the decomposition, required when using Lemma 1, see Dryja and Widlund [28].

Using the method with local preconditioners developed in the end of Section 3, a number of algorithms can be derived from the basic method that we have just introduced. All that is needed is to replace the Dirichlet problem on Ω_{ij} by a preconditioner with a symmetric, positive definite coefficient matrix. A proof of the main result of Bramble, Pasciak and Schatz [14] can be obtained in this way by using (14), (15), Theorem 2 and a bound for the condition number of the two subregion problem defined on Ω_{ij}. Such local bounds are given e.g. in Bjørstad and Widlund [8], Bramble, Pasciak and Schatz [15] and Dryja [21], [22].

In our previous paper, only equation (2) was considered. By using the formulas (7), (8), and (9), the proof can be extended immediately to equation (3); the whole machinery can also be used locally. In this variant of the proof, only four subspaces play a role, namely, the three-dimensional space of linear functions on Ω_i, and the restriction to Ω_i of the spaces V_{ij}^h which correspond to the three edges of that substructure.

In preparation for the three-dimensional case, we develop an alternative basic algorithm for the two-dimensional problems. This algorithm is defined in terms of subspaces related to individual substructures. For any substructure Ω_i, we keep the three local subspaces obtained as above from the spaces V_{ij}^h. For an interior substructure, we replace the three-dimensional space of linear functions, used previously, by the three-dimensional space spanned by

$$\psi_k^{(i)} = \varphi_k - \beta_k^{(i)} \sum_{l \in \mathcal{V}_i} \varphi_l + \beta_k^{(i)}, \ k \in \mathcal{V}_i. \tag{19}$$

Here φ_k is the standard nodal basis function associated with an element of $\mathcal{V}_i$. We note that, just as the regular nodal basis functions, $\psi_k^{(i)}(x_l) = \delta_{kl}, \quad k, l \in \mathcal{V}_i$. The positive constants $\beta_k^{(i)}$ are chosen below, so that

$$\sum_{k \in \mathcal{V}_i} \beta_k^{(i)} = 1.$$

Then,

$$\sum_{k \in \mathcal{V}_i} \psi_k^{(i)} = 1.$$

This subspace, thus, also contains the constants, i.e. the null space of the elliptic operator restricted to Ω_i with a Neumann boundary condition. (When these methods are extended to

elasticity problems, we similarly have to include the whole null space of that operator in the space that accounts for the global transportation of information.) For a substructure that intersects the boundary, we simply use the space spanned by the standard basis functions φ_k corresponding to the vertices of the substructures where a Dirichlet condition is not imposed. For the correct definition of boundary substructures for the case of three dimensions; see below.

A straightforward computation shows that, for an interior substructure, the quadratic form associated with this subspace is given by

$$\sum_{k\in\mathcal{V}_i} k_{kk}^{(i)}(u_h(x_k) - \bar{u}_h^{(i)})^2 . \tag{20}$$

Here, $\bar{u}_h^{(i)}$ is the weighted average of $u_h(x_k)$, $k \in \mathcal{V}_i$, with the weights $\beta_k^{(i)}$. It is easy to see that this quadratic form can be bounded from above and below by the form (17) given in the discussion of the previous algorithm.

Inspired by Mandel [36], we now introduce the quadratic form

$$\mathcal{J}(u_h, y) = \sum_i \sum_{k\in\mathcal{V}_i} k_{kk}^{(i)}(u_h(x_k) - y^{(i)})^2 - 2\sum_i \sum_{k\in\mathcal{V}_i} u_h(x_k) b_k^{(i)}. \tag{21}$$

For substructures which have at least one vertex on $\partial\Omega$, we set $y^{(i)} = 0$. In three dimensions, we adopt the same rule for all substructures which have at least an edge on $\partial\Omega$, but treat those which only touch the boundary at individual points as if they were interior.

The vector $b^{(i)}$ corresponds to the part of the load vector which is associated with Ω_i. Its components are computed by evaluating an appropriate functional at $\psi_k^{(i)}$.

It is easy to show that if

$$\beta_k^{(i)} = \frac{k_{kk}^{(i)}}{\sum_{l\in\mathcal{V}_i} k_{ll}^{(i)}} ,$$

then,

$$\bar{u}_h^{(i)} = \arg\min_{y^{(i)}} \sum_{k\in\mathcal{V}_i} k_{kk}^{(i)}(u_h(x_k) - y^{(i)})^2. \tag{22}$$

Therefore, the solution of the linear system corresponding to these subspaces can be obtained by minimizing $\mathcal{J}$ with respect to u_h and y. By setting the gradient of $\mathcal{J}$ equal to zero, we obtain a linear system of equations, which naturally can be written in a two-by-two block form. The diagonal blocks are diagonal with elements $\sum_i k_{kk}^{(i)}$ and $\sum_{k\in\mathcal{V}_i} k_{kk}^{(i)}$, respectively. The off-diagonal blocks are sparse and are also given by $k_{kk}^{(i)}$.

Since the diagonal blocks are diagonal, it is easy to eliminate all the variables $u_h(x_k)$. This elimination step results in a sparse system, similar to that of a finite difference problem on a coarse mesh that is dual to the one used to define the space V^H. It is often feasible and economical to solve this system by a direct method. Once the values of $\bar{u}_h^{(i)}$ are known, it is easy to determine the values of $u_h(x_k)$, $k \in \mathcal{V}_i$, and the corresponding element of the subspace. The contributions of the other subspaces can now be computed, while observing that we must make the sum of all the contributions continuous across the interfaces between the substructures.

Remark. The method just outlined draws heavily on the work of Bramble, Pasciak and Schatz [16]. However, the current algorithm uses a different average and basis. Our

theory does not require that the linear system, which determines the averages, to have a $M-$matrix. Our method can therefore be extended to the elliptic systems of elasticity; cf. also Mandel [36].

It can be shown, straightforwardly, that there is no longer a need to insist on triangular substructures. In the general case, we can simply redefine $\mathcal{V}_i$ as the set of nodal points which belong to the boundaries of at least three substructures.

Lemma 2 holds only in two dimensions; it resembles a Sobolev inequality which is far from valid in three dimensions. However, for finite element functions, we can find a useful bound of the L_2 – norm over an interval in terms of the strain energy. Let $\mathcal{W}_i$ denote the wire basket of the substructure Ω_i. This is the union of the edges of the tetrahedral substructure; more generally, it is defined as the set of nodal points of Ω_i, which belong to the boundaries of at least three substructures. The following lemma is essentially a corollary of Lemma 2.

LEMMA 3. *Let α be any convex combination of values of $u_h(x), x \in \mathcal{W}_i$. Then*

$$h\|u_h - \alpha\|^2_{l^2(\mathcal{W}_i)} \leq const.\ (1 + \log(H/h))|u_h|^2_{H^1(\Omega_i)}\,.$$

An algorithm for the three-dimensional case can now be defined in terms of the subspaces V_{ij}^h, which are defined just as in the two-dimensional case, and a special subspace. This subspace is given in terms of a basis defined as in (19), where we replace the set $\mathcal{V}_i$ by $\mathcal{W}_i$. The study of the linear algebra problem associated with this basis proceeds very much as in the two-dimensional case. We only note that when we compute the expression corresponding to (20), we also get off-diagonal terms corresponding to nodes on $\mathcal{W}_i$, which are next neighbors. It is however easy to show that these contributions can be disregarded. The rest of the analysis and the development now follows just as in the two-dimensional case. We note that the number of elements in $\mathcal{W}_i$ grows linearly with H/h. The nontrivial part of the sparse linear system of the special subspace has a dimension, which is equal to the number of interior substructures.

We have obtained the following result.

THEOREM 3. *The operator P of the additive algorithm defined by the spaces V_{ij}^h and that defined by the basis functions given by (19), or the corresponding formula for the three-dimensional case, satisfies the estimate $\kappa(P) \leq const.(1+log(H/h))^2$ for equations (2) and (3). The constant is independent of h, H and the discontinuities of the coefficient $\rho(x)$.*

6. Schwarz Methods for Problems on the Interfaces. We have already seen that the original linear systems of equations can be reduced to smaller systems, where the remaining variables are associated with the boundaries $\partial\Omega_i$ of the substructures. We have also shown that this reduced set of variables are associated with the piecewise discrete harmonic functions. This is quite similar to potential theory for elliptic problems where there is also a reduction in dimension.

Here, we only consider the first model case (2), since at this time, we do not know if bounds independent of the variations of the coefficients of equation (3) can be obtained for the methods that we are going to introduce. We denote by $V_{harm}^h(\Gamma)$ the space of piecewise discrete harmonic functions defined by the values of the finite element functions on the set

$$\Gamma = \bigcup \partial\Omega_i \setminus \partial\Omega. \tag{23}$$

The space of traces $H^{1/2}(\Gamma)$ are the restrictions of $H^1(\Omega)$ to Γ. As is well known, the norm of this space can be defined by

$$|u|^2_{H^{1/2}(\Gamma)} = \sum_i |u|^2_{H^{1/2}(\partial\Omega_i)}, \tag{24}$$

where

$$|u|^2_{H^{1/2}(\partial\Omega_i)} = \int_{\partial\Omega_i}\int_{\partial\Omega_i} (|u(x(s)) - u(x(t))|^2/|x(s) - x(t)|^d)dsdt. \tag{25}$$

Here, $d = 2, 3$, for problems with $\Omega \subset R^d$.

We consider several different additive Schwarz methods. The first is defined by the partitioning

$$V^h_{harm}(\Gamma) = \sum_i V^h_{harm,i}, \tag{26}$$

where $V^h_{harm,i}$ is the subspace of $V^h_{harm}(\Gamma)$ with zero values on Γ outside $\partial\Omega_i$. By using the same technique as in the previous three sections, it is easy to see that the operator P, which correspond to this set of subspaces, is bounded uniformly from above. We can use the techniques of Widlund [54] and Dryja [23] for the two- and three-dimensional cases, respectively and Lemma 1, to obtain the bound $C_0^2 \leq const.(1 + \log(H/h))^2$. It is not possible to obtain a uniform bound, since then the spaces $H^{1/2}_{00}$ and $H^{1/2}$ would have to be the same; cf. Lions and Magenes [33].

The computations associated with the space $V^h_{harm,i}$ involve piecewise discrete harmonic functions, which differ from zero in all the substructures which are next neighbors of Ω_i. The coefficient matrix of the linear system, which is solved when computing the projection of V^h_{harm} onto $V^h_{harm,i}$ is the principal minor of S associated with the nodes on $\partial\Omega_i$. It also involves contributions from the matrices $S^{(j)}$ of the neighbors of Ω_i . It is natural to replace this matrix by $S^{(i)}$, since the use of this preconditioner only involves a problem on one substructure.

What we have just given is an alternative derivation of an algorithm introduced by Bourgat, Glowinski, Le Tallec and Vidrascu [11]. We note that there is a minor technical issue related to the fact that $S^{(i)}$ is singular for interior substructures, but that it is easy to solve that problem. Using the formalism developed in Section 2, we can write down the following formula for the preconditioner which corresponds to this algorithm,

$$x_B^T \hat{S}^{-1} y_B = \sum x_B^{(i)T} S^{(i)-1} y_B^{(i)}. \tag{27}$$

Using Lemma 3.2 in Widlund [54], the corresponding estimates developed for the three-dimensional case in Dryja [23] and the estimate (16), we can conclude the proof of the following result.

THEOREM 4. *The operator $\hat{P}$ of the additive algorithm defined by the spaces $V_{harm,i}$, applied to equation (2), and the local preconditioner defined by $S^{(i)-1}$ satisfies the estimate $\kappa(P) \leq const.(1 + \log(H/h))^4$. The constant in the estimate is independent of h but not of H.*

By using an elementary argument given in Widlund [54], it is easy to see that $\kappa(P)$ must grow at least as fast as $1/H^2$, since in each iteration, information is exchanged only with neighboring substructures. The performance of the algorithm, for the case of many substructures, can be improved by introducing the same spaces of modest dimension, which provide some global transportation of information in the methods considered in Section 5. After this modification, the constant in the theorem can be shown to be independent of H as well.

We note that in each iteration of this algorithm, both a Neumann and a Dirichlet problem have to be solved on each substructure. This results in twice as much work per iteration as for many other domain decomposition methods.

If there is a red-black ordering of the substructures, there is another way of analyzing this algorithm. We first consider the two subdomain case. We note that the inverse of the preconditioner corresponding to the Neumann-Dirichlet algorithm is $S^{(1)^{-1}}$ or $S^{(2)^{-1}}$ depending on the roles assigned to the substructures; cf. Bjørstad and Widlund [8]. We can therefore view the preconditioner introduced in this section as an average of Neumann-Dirichlet operators. In this simple case, the condition number remains uniformly bounded since there are no so-called cross points. Similarly, we can use a result by Dryja [23] to prove that the algorithm for three dimensions, which incorporates the global subspace introduced in the previous section, has a condition number which is bounded by $const.(1+\log(H/h))^3$. This again follows from the observation that this new preconditioner can be viewed as an average of two instances of the operator which arises in the study of the version of the Neumann-Dirichlet algorithm considered in Dryja [23].

We conclude this section by a brief discussion of some recent work by Barry Smith, a Courant Institute graduate student. His algorithm, described in Smith [48], is an additive Schwarz method, which uses the subspace V^H both in two and three dimensions. We give some details only for the three-dimensional case. In addition to V^H, there are subspaces associated with each face, edge and vertex of the substructures. The subspace associated with a face is $H_{00}^{1/2}(\Gamma_{ij}) \bigcap V_{harm}^h$ and therefore closely related to the space V_{ij}^h introduced in Section 5. Similarly, for an edge (vertex), Smith uses a space associated with a neighborhood in Γ of the edge (vertex) which extends a distance on the order of the diameter of Ω_i. The success of this approach can be explained informally by noting that the overlap is relatively generous. The numerical results, so far confined to two dimensions and simple substructures with benign aspect ratios, show great promise. In addition to Poisson's equation, membrane and shell models have been tested. So far, the condition numbers have always been less than 4.

The following theoretical result has been established for both two and three dimensions. It also holds for elliptic systems such as those of linear elasticity; cf. Smith [48].

THEOREM 5. *The operator P of the additive algorithm developed by Smith satisfies the estimate $\kappa(P) \leq const.$ The constant in the estimate is independent of h as well as H.*

7. Optimal Iterative Refinement Methods. In our study of iterative refinement methods, we consider problems on a special kind of composite finite element triangulations. We begin by introducing a relatively coarse triangulation of $\Omega^{(1)} = \Omega$, and denote the corresponding space of finite element functions by V^{h_1}. We can think of this space as having a uniform (or relatively uniform) mesh size h_1. Let $\Omega^{(2)}$ be a subregion where we wish to increase the resolution. We do so by subdividing the elements and introducing an additional finite element space V^{h_2}. This space is constructed quite similarly to the previous one and it contains $V^{h_1} \bigcap H_0^1(\Omega_2)$ as a subspace. We assure that the resulting composite space $V^{h_1} + V^{h_2}$ is conforming by having the functions of V^{h_2} vanish on $\partial\Omega^{(2)}$. We repeat this process by selecting a subregion $\Omega^{(3)}$ of $\Omega^{(2)}$ and introducing a further refinement of the mesh and the finite element space, etc. We denote the resulting nested subregions and subspaces by $\Omega^{(i)}$ and V^{h_i}, respectively. Throughout, we assume that

$$\Omega^{(i)} \subset \Omega^{(i-1)} ,$$

and that

$$V^{h_{i-1}} \cap H_0^1(\Omega^{(i)}) \subset V^{h_i} \subset H_0^1(\Omega^{(i)}), \quad i = 2, \ldots, k.$$

The composite finite element space, on the repeatedly refined mesh, is

$$V^h = V^{h_1} + V^{h_2} + \ldots + V^{h_k} .$$

The finite element models on composite meshes is thus systematically constructed by introducing a basic finite element approximation on the entire region and then selecting subregions, and subregions of subregions etc., where the finite element model is further refined in order to gain higher accuracy. In each iteration of the iterative refinement methods, problems representing finite element models on the original region and the subregions, prior to further refinement, are solved. As always, an additive algorithm is defined by specifying the subspaces V_i, or alternatively, the projections P_i.

In order to prove the results given at the end of this section, we need some additional technical assumptions. To make our proofs work, we cannot allow the sets $\Omega_{i-1} \setminus \Omega_i$ to become arbitrarily thin in comparison with the diameter of Ω_{i-1}. We also assume that the area of any triangle on level i can be bounded by *const.* q^{i-j} times the area of the triangle on level j of which it is a part. Here q is a constant < 1 and $j < i$.

The fundamental building blocks of our algorithms are the projections P_i^j, $j = i-1, i$, onto the spaces $V^{h_j} \cap H_0^1(\Omega_i)$. We note that if $j = i-1$, we solve a problem on Ω_i with a coarser mesh than if V^{h_i} were used. The projection P_i^j is defined in terms of the unique element of $V^{h_j} \cap H_0^1(\Omega_i)$, which satisfies

$$a(P_i^j v_h, \phi_h) = a(v_h, \phi_h), \ \forall \phi_h \in V^{h_j} \cap H_0^1(\Omega_i). \tag{28}$$

We consider two different algorithms and distinguish between them by using superscripts. The perhaps most natural algorithm uses the projections $P_i^{(1)} = P_i^i$. The condition number of this algorithm can grow as fast as linearly with k. By the standard argument, it is easy to show that the eigenvalues of $P^{(1)}$ are bounded from above by k. This bound is attained if $V^{h_1} \cap H_0^1(\Omega_k)$ is not empty, i.e. when the mesh size h_1 is fine enough. Any function in this space belongs to V^{h_i}, $i = 1, 2, \ldots, k$, and is exactly reproduced by each of the projection operators. It is therefore an eigenfunction of $P^{(1)}$ with the eigenvalue k. Similarly, any function which belongs to $V^{h_1} \cap H_0^1(\Omega_1 \setminus \Omega_2)$ is an eigenfunction with eigenvalue 1. We have shown in [56], that the eigenvalues of $P^{(1)}$ are bounded from below by a constant. The condition number of $P^{(1)}$ is therefore of order k.

We are principally interested in the additive algorithm defined by the projections $P_i^{(2)} = P_i^i - P_{i+1}^i$, $i \leq k-1$, and $P_k^{(2)} = P_k^k$. It easy to show that these operators are projections and that the composite finite element space V^h is the direct sum of the corresponding subspaces $V_i^{(2)}$. The following results have recently been demonstrated in Dryja and Widlund [27] and Widlund [56]. The latter paper also contains a similar result for a multiplicative algorithm.

THEOREM 6. *The condition number of $P^{(2)}$ is uniformly bounded by a constant. The condition number of $P^{(1)}$ grows at most linearly with the number of refinement levels.*

We note, that Tarek Mathew [40] has developed iterative refinement methods for Raviart-Thomas finite element methods and obtained bounds for their rate of convergence.

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