CHAPTER 16

A Fourier Analysis of the Two-Level Hierachical Basis Multigrid Method for Convection-Diffusion Equations

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Abstract. We consider the solution of a simple one space dimensional convection-diffusion equation by the hierarchical basis multigrid method. The simplicity of this problem allows an exact analysis by standard Fourier techniques. Out results explicitly show the role of upwind discretizations in promoting the convergence of the method.

1. A One Dimensional Model Problem. In this note, we consider the one dimensional convection-diffusion equation

$$-u'' + \beta u' = f \tag{1}$$

in $\Omega = (0,1)$ with periodic boundary conditions

$$\begin{aligned} u(0) &= u(1) \\ u'(0) &= u'(1) \end{aligned} \tag{2}$$

by the hierarchical basis multigrid method [2], [8], [11]. Here β is assumed to be a positive constant. While there is no practical interest in actually solving this problem using the hierarchical basis multigrid method, its simplicity allows one to obtain exact estimates for the two level iteration via Fourier analysis, for the case of uniform mesh spacing. This can then serve a guide and motivation for our subsequent analysis of the more complicated two space dimensional problems with nonuniform meshes. We will consider the two dimensional case elsewhere [1]. The use of multigrid methods for such problems has been studied theoretically and empirically in [4], [5], [10], [9]; the multigrid bibliography in [9] contains many additional references. Our approach here is most closely related to that of Hackbusch [4].

We assume that there exists a solution u to the problem (1)-(2). For this to be true, f must satisfy the consistency condition

$$\int_0^1 f dx = 0$$

and then the solution is determined up to an arbitrary additive constant. We can make the solution unique, for example, by imposing the condition

$$\int_0^1 u dx = 0$$

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A proof of the convergence of the standard multigrid method for both the Dirichlet problem and the periodic boundary value problem in one dimension has been given by Hackbusch in [4]. In that analysis, a finite difference discretization using both centered and upwinded differences were considered.

Here, we will use a streamline diffusion Petrov-Galerkin finite element discretization on a uniform mesh of size h [6], [7]. Let $n > 2$ be an integer and set $h = 1/(2n)$, and $x_k = kh, 0 \leq k \leq 2n-1$. The coarse mesh will have n mesh points $x_{2k}, 0 \leq k \leq n-1$, and $h' = 2h$. Grid functions defined on these meshes are assumed to be periodically extended in the usual fashion. We will refer to the set of coarse grid points as *level 1 vertices*, and $x_{2k+1}, 0 \leq k \leq n-1$, as *level 2 vertices*.

We define the bilinear form $B(\cdot,\cdot)$ by:

$$B(u,v) = \int_0^1 (1 + ch\beta)u'v' + \beta u'v dx$$

Here $c \geq 0$ is a fixed constant. Let $\mathcal{S}_h$ be the space of continuous, periodic, piecewise linear functions associated with the fine mesh $\mathcal{T}_h$. $\mathcal{S}_h$ has dimension $2n$. The n dimensional coarse grid space $\mathcal{S}_{h'} \subset \mathcal{S}_h$, associated with the coarse grid $\mathcal{T}_{h'}$, is similarly defined. The discrete system of equations to be solved is: find $u_h \in \mathcal{S}_h$ such that

$$B(u_h, v) = (f, v + chv') = \int_0^1 f\{v + chv'\}\,dx \tag{3}$$

for all $v \in \mathcal{S}_h$. As with the continuous problem, the solution will be unique up to an additive constant. See [7] for some *a priori* error estimates for $u - u_h$.

Our results show that the rapid convergence of the two level scheme is critically linked to the strength of the upwinding. This is perhaps not surprising, since upwinding results in the addition of a symmetric, positive semidefinite matrix to the stiffness matrix corresponding to the standard Galerkin discretization, and this is usually helpful in speeding convergence of iterative methods. In particular, the rate of convergence depends on the quantity βh. By adding sufficient upwinding, the rate can be made independent of βh; otherwise, one can guarantee convergence only for βh sufficiently small. In some sense, the success or failure of the multigid method is directly linked to the underlying stability of the discretization. Roughly speaking, when the upwinding becomes strong enough to stabilize the discretization, then one can expect a good rate of convergence of the hierarchical basis multigrid method.

For more than two levels, and more than one space dimension, the dependence on βh becomes more complicated, and, as one might expect, imposes additional constaints on the fineness of the coarse grid. Nevertheless, the principle that the stability of the discretization is directly connected with the success of the hierarchical basis multigid method is still quite apparent. We will examine the two dimensional case in [1].

The remainder of this paper is organized as follows. In section 2, we define the hierarchical basis and the corresponding iteration. In section 3, we analyze the convergence via Fourier analysis.

2. Two level scheme. We begin by introducing the hierarchical basis for the space $\mathcal{S}_h$. Let $\hat{\psi}_k$, $0 \leq k \leq 2n-1$, denote the usual *nodal* basis for the space $\mathcal{S}_h$

$$\hat{\psi}_k(x_j) = \delta_{jk}$$

for $0 \leq j \leq 2n-1$. (Because of the periodic boundary conditions, the basis function $\hat{\psi}_0$ has support in the interval (x_{2n-1}, x_{2n}) as well as (x_0, x_1).)

The *hierarchical* basis for $\mathcal{S}_h$ consists of the union of the nodal basis functions for the coarse grid space $\mathcal{S}_{h'}$ and the nodal basis functions for the level 2 nodes, $\hat{\psi}_{2k+1}$, $0 \leq k \leq n-1$. This basis will be denoted ψ_k, $0 \leq k \leq 2n-1$.

The hierarchical basis introduces a natural splitting of the space $\mathcal{S}_h$. If $u \in \mathcal{S}_h$, then we have the unique decomposition $u = v + w$, where $v \in \mathcal{S}_{h'} \equiv \mathcal{V}$ and $w \in span\{\hat{\psi}_{2k+1}\}_{k=0}^{n-1} \equiv \mathcal{W}$.

The matrix formulation of (3) with respect to the hierarchical basis is

$$A_H U_H = F_H \tag{4}$$

where

$$A_H = \begin{bmatrix} a_H+b_H & c_H & -b_H & & & & & -a_H & -c_H \\ -c_H & d_H & c_H & 0 & & & & & 0 \\ -a_H & -c_H & a_H+b_H & c_H & -b_H & & & & \\ & 0 & -c_H & d_H & c_H & 0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & & 0 & -c_H & d_H & & c_H & 0 \\ -b_H & & & & -a_H & -c_H & & a_H+b_H & c_H \\ c_H & 0 & & & & 0 & & -c_H & d_H \end{bmatrix} \tag{5}$$

and

$$\begin{aligned} a_H &= \frac{1+(c+1)h\beta}{2h} \\ b_H &= \frac{1+(c-1)h\beta}{2h} \\ c_H &= \frac{\beta}{2} \\ d_H &= \frac{2+2ch\beta}{h} \end{aligned}$$

The matrix A_H can be transformed in two ways that are of interest to us here. The first transformation is a simple permutation, in which the basis functions associated with the subspace $\mathcal{V}$ are ordered first, and those associated with $\mathcal{W}$ are ordered last. If we denote the relevant permutation matrix by P, the permuted matrix is block 2×2, with $n \times n$ blocks, of the form

$$A'_H \equiv PA_HP^t = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{6}$$

with

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_H+b_H & -b_H & & & -a_H \\ -a_H & a_H+b_H & -b_H & & \\ & \ddots & \ddots & \ddots & \\ & & -a_H & a_H+b_H & -b_H \\ -b_H & & & -a_H & a_H+b_H \end{bmatrix} \\ A_{12} &= -A_{21}^t = c_H \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix} \\ A_{22} &= d_H I_{n\times n} \end{aligned}$$

Let A_N denote the stiffness matrix with respect to the standard nodal basis functions. This matrix is of the form

$$A_N = \begin{bmatrix} a_N+b_N & -b_N & & & -a_N \\ -a_N & a_N+b_N & -b_N & & \\ & \ddots & \ddots & \ddots & \\ & & -a_N & a_N+b_N & -b_N \\ -b_N & & & -a_N & a_N+b_N \end{bmatrix},$$

with

$$\begin{aligned} a_N &= \frac{2+(2c+1)h\beta}{2h} \\ b_N &= \frac{2+(2c-1)h\beta}{2h} \end{aligned}$$

Then the matrix A_N can be transformed to A_H by

$$A_H' = S^t P A_N P^t S \tag{7}$$

Here S is a block 2×2 matrix of the form

$$S = \begin{bmatrix} I & 0 \\ R & I \end{bmatrix}$$

with

$$R = \frac{1}{2} \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}$$

The identity (7) is important for the practical realization of the hierarchical basis multigrid method, since in two space dimensions, the stiffness matrix with respect to the hierarchical basis is much more dense than that for the nodal basis. Explicit computation of the hierarchical basis stiffness matrix should be avoided in implementing the method [3]. However, for theoretical purposes, it is most convenient to formulate the method using the matrix A_H', since this simplifies the mathematical analysis and description.

Within this framework, the 2-level hierarchical basis method is just the block symmetric Gauss-Seidel iteration applied to the linear system

$$A_H' U_H' = F_H' \tag{8}$$

This can be formulated as the two-step process

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} (U_{k+1/2} - U_k) = F_H' - A_H' U_k \tag{9}$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} (U_{k+1} - U_{k+1/2}) = F_H' - A_H' U_{k+1/2} \tag{10}$$

We note that the matrix A_H' is singular with a one-dimensional kernel corresponding to the constant function. The corresponding vector in $\mathcal{R}^{2n}$ has the block form $e^t = (e_1^t \, e_2^t)$, with $e_1^t = (1\,1 \ldots 1)$ and $e_2 = 0$. Since $A_{11} e_1 = A_{21} e_1 = 0$, both parts of the iteration (9)-(10) are also singular, but with the same kernel. If we interpret (8)-(10) restricted to the orthogonal complement subspace for e, then everything is consistent and well defined, including the (generalized) inverses of all the relevant operators.

Standard algebraic manipulations of (9)-(10) lead to a compact definition of the the symmetric block Gauss-Seidel iteration in terms of the preconditioner

$$M' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}^+ \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{11}$$

This can be written as

$$M' = \begin{bmatrix} A_{11} - B_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{12}$$

where

$$B_{11} = \frac{c_H^2}{d_h} \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix} \tag{13}$$

Setting $M' = PMP^t$, we can write M as

$$
(14) \qquad M = \begin{bmatrix}
\hat{a}_H+\hat{b}_H & c_H & -\hat{b}_H & & & & -\hat{a}_H & -c_H \\
-c_H & d_H & c_H & 0 & & & & 0 \\
-\hat{a}_H & -c_H & \hat{a}_H+\hat{b}_H & c_H & -\hat{b}_H & & & \\
 & 0 & -c_H & d_H & c_H & 0 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 0 & -c_H & d_H & c_H & 0 \\
-\hat{b}_H & & & & -\hat{a}_H & -c_H & \hat{a}_H+\hat{b}_H & c_H \\
c_H & 0 & & & & 0 & -c_H & d_H
\end{bmatrix}
$$

with

$$
\begin{aligned}
\hat{a}_H &= a_H - \frac{c_H^2}{d_H} \\
\hat{b}_H &= b_H - \frac{c_H^2}{d_H}
\end{aligned}
$$

To analyze the rate of convergence for (9)-(10), it is sufficient to study the eigenvalues of the matrix $I - M^+A_H$, restricted to the orthogonal complement subspace for e, which can be done exactly using Fourier analysis as we shall see in the next section.

3. Fourier analysis. Let B be any matrix of the form

$$
(15) \qquad B = \begin{bmatrix}
a+b & c & -b & & & & -a & -c \\
-c & d & c & 0 & & & & 0 \\
-a & -c & a+b & c & -b & & & \\
 & 0 & -c & d & c & 0 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 0 & -c & d & c & 0 \\
-b & & & & -a & -c & a+b & c \\
c & 0 & & & & 0 & -c & d
\end{bmatrix}
$$

We define the discrete Fourier vectors $\{q_k\}_{k=0}^{2n-1}$ by

$$
(16) \qquad q_k^t = (1, e^{ik\theta}, e^{i2k\theta}, \ldots, e^{i(2n-1)k\theta})
$$

with $\theta = 2\pi h = \pi/n$. It is straightforward to check that

$$
(17) \qquad
\begin{aligned}
Bq_k &= \alpha_k q_k + \gamma_k q_{k+n} \\
Bq_{k+n} &= \alpha_{k+n} q_{k+n} + \gamma_{k+n} q_k
\end{aligned}
$$

$$
\begin{aligned}
\alpha_k &= \frac{a+b}{2}\{1-\cos(2k\theta)\} + \frac{d}{2} + i\frac{a-b}{2}\sin(2k\theta) + i2c\sin(k\theta) \\
\gamma_k &= \frac{a+b}{2}\{1-\cos(2k\theta)\} - \frac{d}{2} + i\frac{a-b}{2}\sin(2k\theta)
\end{aligned}
$$

Thus the two dimensional subspaces $span\{q_k, q_{k+n}\}$, $0 \le k \le n-1$, are invariant under application of B. Since both A_H and M are of this form, the problem of analyzing the eigenvalues for the $2n \times 2n$ matrix $I - M^+A_H$ is reduced to the analysis of n eigenvalue problems of dimension 2.

To find the eigenvalues of the iteration matrix, we can set

$$
det\{A_H - M - \lambda M\} = 0
$$

Since both A_H and M are of the form (15), Fourier analysis will reduce this $2n \times 2n$ eigenvalue problem to n eigenvalue problems of size 2×2. To simplify notation in this calculation, we will set, for $0 \le k \le n-1$,

$$
\begin{aligned}
\rho &= \beta h \\
\eta &= (1 + c\beta h) \\
&= (1 + c\rho) \\
p_k &= \frac{\hat{a}_H + \hat{b}_H}{2}\{1 - \cos(2k\theta)\} + i\frac{\hat{a}_H - \hat{b}_H}{2}\sin(2k\theta) \\
&= \frac{\sin(k\theta)}{8h\eta}\{(4\eta^2 - \rho^2)\sin(k\theta) + i\rho\eta\cos(k\theta)\} \\
q_k &= \frac{d_H}{2} \\
&= \frac{\eta}{h} \\
r_k &= 2c_H \sin(k\theta) \\
&= \frac{\rho\sin(k\theta)}{h} \\
s_k &= \frac{c_H^2}{d_h}\{1 - \cos(2k\theta)\} \\
&= \frac{\rho^2 \sin^2(k\theta)}{4h\eta}
\end{aligned}
$$

Then a typical 2×2 determinant has the form

$$
det\left\{ s_k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} p_k + q_k + ir_k & p_k - q_k \\ p_k - q_k & p_k + q_k - ir_k \end{bmatrix} \right\} = 0
$$

The straightforward calculation of this determinant shows that

$$
(4p_k q_k + r_k^2)\lambda^2 - 4\lambda s_k p_k = 0
$$

giving $\lambda = 0$ and

$$
\begin{aligned}
\lambda &= \frac{s_k}{q_k + \frac{r_k^2}{4p_k}} \\
&= \frac{\rho^2 \sin^2(k\theta)}{4\eta^2}\left\{\frac{(4\eta^2 - \rho^2)\sin(k\theta) + i\rho\eta\cos(k\theta)}{(4\eta^2 + \rho^2)\sin(k\theta) + i\rho\eta\cos(k\theta)}\right\}
\end{aligned}
$$

from which the estimate

$$
\begin{aligned}
|\lambda| &\le \frac{\rho^2 \sin^2(k\theta)}{4\eta^2} \\
&\le \frac{\rho^2}{4\eta^2} \\
&= \frac{(\beta h)^2}{4(1 + c\beta h)^2}
\end{aligned}
$$

follows. A special case occurs when $k = 0$, and $p_k = r_k = s_k = 0$. This 2×2 block has one Fourier component corresponding to the constant function, which should be ignored. The remaining eigenvalue for this block is $\lambda = 0$.

This analysis shows that the iteration will converge ($|\lambda| < 1$) for all

$$
c \ge \max\left(\frac{\beta h - 2}{2\beta h}, 0\right)
$$

This set of conditions is similar to that required for stability of the underlying streamline diffusion discretization. If the velocity β or the mesh spacing h is small enough ($\beta h < 2$), then no upwinding is required, and we have convergence for any $c \geq 0$. However, for larger βh, $c > 0$ is required to achieve stability of the discretization, and also convergence of the 2-level iteration. We note that choosing $c > 1/2$ will always guarantee convergence of the 2-level iteration, independent of the size of βh.

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