An Analysis of Convergence of the Multigrid Method for Stiff Problems

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Abstract. If on tetrahedrons ω_i with vertices at 'coarse' nodes the variations of values of coefficients of the elliptic operator are uniformly bounded by quantity S, the convergence rate of the two-grid method suggested in the paper is dependent only on the quantity S, i.e. independent of the value of a coefficient jump at the boundary of 'coarse' tetrahedrons ω_i.

1. Introduction. Let Ω be a bounded domain in R^3 with the piecewise-differentiable boundary.

Let then functions a_{ij} be piecewise-continuous and for the quadratic form $a(\xi,x) = \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j$ a piecewise-continuous function $\lambda(x) > 0$ exists such that

$$\inf_{x\in\Omega,|\xi|=1} a(\xi,x)/\lambda(x) > 0; \qquad \sup_{x\in\Omega,|\xi|=1} a(\xi,x)/\lambda(x) < \infty. \tag{1.1}$$

Let then $b(x) \geqslant 0$, $b \in L_1(\partial\Omega)$ on a set of positive measure. Introduce on $W_2^1(\Omega)$ a scalar product

$$[u,v] = \int_\Omega \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx + \int_{\partial\Omega} buv\, d\Sigma. \tag{1.2}$$

For $f \in L_2(\Omega)$, $\psi \in L_2(\partial\Omega)$ consider the functional minimum problem

$$\Phi(u) = [u,u] - 2\int_\Omega uf\, dx - 2\int_{\partial\Omega} u\psi\, d\Sigma. \tag{1.3}$$

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In this paper we will analyse the multigrid iterative method of computation of the minimum of the functional $\Phi(u)$ on finite element spaces of functions.

Let Ω_0 be a partition of the domain Ω into tetrahedrons Δ_m^0. Denote by ω_0 a set of vertices of tetrahedrons of the partition Ω_0. Assume that the nodes of ω_0 which are not internal points of the domain Ω belong to its boundary $\partial\Omega$. Then again partition the tetrahedrons Δ_m^0 into finer tetrahedrons Δ_n^1. Denote by Ω_1 a set of tetrahedrons of fine partition and by ω_1 a set of their vertices. By $|\omega_k|$, $k = 0,1$, denote the number of nodes in ω_k.

Let R_k, $k = 0,1$, be linear spaces of grid functions on the nodes ω_k, and let x_k^m be a node from ω_k. Denote by $st(x_k^m)$ the star of the node x_k^m, i.e. the set of tetrahedrons of the partition Ω_k which have the node x_k^m as their vertex. Then introduce the basis function $\varphi_k^m(x)$ equal to unity at the node x_k^m,to zero at the other nodes of the star $st(x_k^m)$ and linear on each tetrahedron of the star. Outside $st(x_k^m)$ the function φ_k^m is assumed to be equal to zero. Then functions φ_0^m are said to be 'coarse-basis', and φ_1^n are said to be 'fine-basis'. Now construct finite element spaces H_k, $k = 0,1$:

$$H_k:\ v = \sum_{x_k^m \in \omega_k} v(x_k^m)\varphi_k^m(x). \tag{1.4}$$

This formula defines isomorphism $J_k: R_k \to H_k$. It is obvious that $H_0 \subset H_1$. Denote this inclusion by I and define the inclusion i_1 of R_0 into R_1:

$$i_1 = J_1^{-1} I J_0. \tag{1.5}$$

It is not difficult to verify that $(i_1 u)(x_1^m) = (IJ_0 u)(x_1^m)$. Obviously, i_1 extends definition of grid functions from R_0 given at coarse nodes of ω_0 to fine nodes of $\omega_1 \backslash \omega_0$ by interpolation defined by the mapping J_0.

Let us introduce on R_k the scalar products

$$(\varphi,\psi) = \int_\Omega (J_k\varphi)\cdot\overline{(J_k\psi)}\,dx, \qquad [\varphi,\psi] \stackrel{\text{def}}{=} [J_k\varphi, J_k\psi]$$
$$<\varphi,\psi>_k = \sum_{x_k^m \in \omega_k} \varphi(x_k^m)\psi(x_k^m). \tag{1.6}$$

2. Projector P of space R_1 onto i_1R_0. The finite element approximation of the elliptic operator L corresponding to problem (1.4) is the linear operator L_k on the space R_k satisfying the relation $<L_k u,v>_k = [u,v]$ $\forall u,v \in R_k$. As well known, minimization problem (1.3) on the space H_k can be reduced to the solution of the linear equation

$$L_k u_k = f_k, \qquad u_k \in R_k \tag{2.1}$$

where the right-hand side f_k can be computed by standard procedures using f and ψ [1].

In the multigrid iterative process contained in [2,3] for solving the equation $L_1 u_1 = f_1$ the fundamental part is played by the projector P of orthogonal projection with respect to the scalar product [,] of the space R_1 onto $i_1 R_0$.

The function $(1-P)u$ is obviously orthogonal to $i_1 R_0$. To compute this function, it is convenient to pass to the space H_1. Then,

$$J_1(1-P)u = J_1 u + \sum_{m=1}^{|\omega_0|} v_m \varphi_0^m$$

and the condition $[J_1(1-P)u, \varphi_0^n] = 0$, $n = 1,\ldots,|\omega_0|$, leads to the system of equations

$$\sum_{m=1}^{|\omega_0|} v_m [\varphi_0^m, \varphi_0^n] = -[J_1 u, \varphi_0^n]. \tag{2.2}$$

As known, the matrix $\|[\varphi_0^m, \varphi_0^n]\|$ coincides with the matrix L_0. To compute the function $(1-P)u$, it is therefore necessary to solve equation (2.2) on the coarse grid ω_0.

3. Iterative process. For the case where the estimating function $\lambda(x)$ from condition (1.1) is continuous on tetrahedrons of the coarse partition Ω_0, in [3] we suggested an iterative process whose convergence rate is independent of the value of $\lambda(x)$.

Thus, denote by D_1 a diagonal matrix equal to the diagonal of the operator L_1, and let u_0 be the solution to equation (2.1) on the coarse grid, i.e. for $k = 0$. Let us consider the iterative process

$$Y^n = (1-P)X^n + i_1 u_0, \quad X^0 = 0, \quad n \geqslant 0 \tag{3.1}$$

$$X^{n+1} = Y^n - \tau_n D_1^{-1}(L_1 Y^n - f_1), \quad n \geqslant 0. \tag{3.2}$$

Here, the step size τ_n is determined by the gradient descent method for the operator $D_1^{-1} L_1 (1-P)$:

$$\tau_n = \langle D_1 \xi_n, \xi_n \rangle_1 / \langle L_1 (1-P)\xi_n, \xi_n) \rangle_1$$
$$\xi_n \stackrel{\text{def}}{=} D_1^{-1}(L_1 Y^n - f_1). \tag{3.3}$$

Thus, to start iterative process (3.1), (3.2), it is first necessary to solve equation (2.1) on the coarse grid and then to successively perform computations by formulae (3.1), (3.2). It was proved in [2] that

$$P u_1 = i_1 u_0. \tag{3.4}$$

Hence, iterative process (3.1), (3.2) fixes the projection of Y^n onto $i_1 R_0$.

The modifications of iterative process (3.1), (3.2) without justification of its convergence for stiff problems can be found in publications by various authors (see, for example, [4]). Another approach to the multigrid method in case of the coefficients being continuous on the tetrahedrons Ω_0 was suggested in [5].

Below, we impose constraints on the function $\lambda(x)$ under which in the estimate

$$<L_1 r_{n+1}, r_{n+1}>_1 \leqslant (1-q)<L_1 r_n, r_n>, \qquad r_n = Y^n - u_1 \tag{3.5}$$

the contraction coefficient q is independent of the step size of grid nodes and the spread in values of $\lambda(x)$, i.e. of the 'stiffness' of problem.

To prove it, we will suggest another technique making it possible to avoid the use of the second main inequality for elliptic equations.

Let us analyse the algebraic structure of the formulae of process (3.1), (3.2). Denote by P^* an operator adjoint to P with respect to the scalar product $< , >_1$, and let D_1 be an arbitrary symmetric, positive definite operator in R_1. Introduce the linear space $R_D \overset{\text{def}}{=} D_1^{-1}(1-P^*)R_1$ and set

$$\begin{aligned} m(L) &= \inf_{v \in R_D} <L_1(1-P)v,v>_1/<D_1 v,v>_1 \\ M(L) &= \sup_{v \in R_D} <L_1(1-P)v,v>_1/<D_1 v,v>_1 \\ q(L) &= [M(L)-m(L)]/[M(L)+m(L)]. \end{aligned} \tag{3.6}$$

Lemma 3.1. Iterative process (3.1), (3.2) for $n \geqslant 0$ satisfies estimate (3.5) wit $q = q(L)$.

Proof. The space R_1 can be decomposed into the direct sum $R_1 = R_D \oplus PR_1$ orthogonal to $<D_1 \cdot, \cdot>_1$. Indeed, assume that $<D_1 \cdot D_1^{-1}(1-P^*)v,w>_1 = 0$ $\forall v \in R_1$. It means that $(1-P)w = 0$, i.e. $w \in PR_1$. Denote by Q the projector R_1 onto R_D orthogonal to $<D_1 \cdot, \cdot>$.

Then the equality $L_1 P = P^* L_1$ proved in [2] implies the equality

$$D_1^{-1}L_1(1-P) = D_1^{-1}(1-P^*)L_1. \tag{3.7}$$

Hence, we have $D_1^{-1}L_1(1-P)R_1 \subset R_D$ and the operator $D_1^{-1}L_1(1-P)$ is self-adjoint to $<D_1 \cdot, \cdot>_1$.

Formula (3.1) implies the equality

$$Y^n - u_1 = (1-P)(X^n - u_1). \tag{3.8}$$

Since for the introduced projector Q we have $(1-P) = (1-P)Q$, then by virtue of (3.7) we obtain

$$\begin{aligned} <L_1(Y^{n+1}-u_1),(Y^{n+1}-u_1)>_1 &= <L_1(1-P)(X^{n+1}-u_1),(1-P)(X^{n+1}-u_1)>_1 \\ &= <L_1(1-P)Q(X^{n+1}-u_1),(1-P)Q(X^{n+1}-u_1)>_1. \end{aligned} \tag{3.9}$$

Then determine $Q(X^{n+1} - u_1)$ from (3.2) and substitute it into (3.9). Equality (3.8) implies that $PY^n = Pu_1$. Therefore,

$$L_1 Y^n - f_1 = L_1(Y^n - u_1) = L_1(1-P)(Y^n - u_1).$$

Hence, by virtue of (3.2) we have

$$\begin{aligned} X^{n+1} - u_1 &= Y^n - u_1 - \tau_n D_1^{-1} L_1(Y^n - u_1) \\ &= Y^n - u_1 - \tau_n D_1^{-1} L_1(1-P)(Y^n - u_1). \end{aligned} \tag{3.10}$$

Formula (3.7) implies that the second term in the right-hand side of (3.10) belongs to R_D. Make the operator Q act on equality (3.10). Then taking into account that $1 - P = (1-P)Q$ we obtain

$$Q(X^{n+1} - u_1) = Q(Y^n - u_1) - \tau_n D_1^{-1} L_1(1-P)Q(Y^n - u_1). \tag{3.11}$$

The right-hand side in (3.11) defines the iterative process on R_D for the self-adjoint operator $D_1^{-1}L_1(1-P)$ whose step τ_n in the gradient descent method is computed by formula (3.3). Substitute (3.11) into (3.9). Taking into account the convergence rate estimates for the steepest descent method [6] and the estimate

$$<L_1(1-P)v,(1-P)v>_1 = [(1-P)v,v] \leqslant [v,v] = <L_1 v,v>_1$$

we obtain

$$\begin{aligned} &<L_1(1-P)Q(X^{n+1} - u_1),Q(X^{n+1} - u_1)>_1 \\ &\leqslant (1 - q(L))<L_1(1-P)Q(Y^n - u_1),(1-P)Q(Y^n - u_1)>_1 \\ &= (1 - q(L))<L_1(1-P)(Y^n - u_1),(1-P)(Y^n - u_1)>_1. \end{aligned} \tag{3.12}$$

Equality (3.8) implies that $Y^n - u_1 = (1-P)(Y^n - u_1)$. Hence, (3.9), (3.12) imply estimate (3.5) for $q = q(L)$. This completes the proof of the lemma.

4. Estimates for $m(L)$, $M(L)$. Let us introduce the value δ_λ:

$$\delta_\lambda = \max_m \left(\operatorname{Vrai\,max}_{\Delta_m^0} \lambda \,/\, \operatorname{Vrai\,min}_{\Delta_m^0} \lambda\right) \tag{4.1}$$

and similarly define δ_b [the function $b(x)$ is contained in (1.2)]. Let it be known that in the partitions Ω_k on each of the tetrahedrons Δ_m^k the ratio of the length of the maximal edge to that of the minimal one does not exceed γ, and each of the tetrahedrons Δ_m^0 contains, at most, ν fine tetrahedrons Δ_n^1. The following theorem is valid.

Theorem 4.1. A constant C dependent on γ, ν and the boundary $\partial\Omega$ exists such that

$$m(L) \geqslant C/[\max(\delta_\lambda,\delta_b)]^2. \tag{4.2}$$

For arbitrary integrated values a_{ij} and b we have

$$M(L) \leqslant 4. \tag{4.3}$$

Auxiliary manipulations. Introduce the notation

$$[u,v](\Delta) \overset{\text{def}}{=} \int_{\Omega\cap\Delta} \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \mathrm{d}x + \int_{(\partial\Omega)\cap\Delta} buv \,\mathrm{d}\Sigma. \tag{4.4}$$

Let $\omega(\Delta_m^i)$ be a set of numbers of vertices of the tetrahedron Δ_m^i. Set

$$\begin{aligned} <D_1u,v>_1(\Delta_m^1) &= \sum_{n\in\omega(\Delta_m^1)} [\varphi_1^n,\varphi_1^n](\Delta_m^1)u_n v_n \\ <D_1u,v>_1(\Delta_k^0) &= \sum_{\Delta_m^1\in\Delta_k^0} <D_1u,v>_1(\Delta_m^1). \end{aligned} \tag{4.5}$$

Then for each function $v \in R_1$ construct a coarse-basis function φ coinciding with v at coarse nodes of ω_0:

$$\varphi = \Pi v \overset{\text{def}}{=} \sum_{m=1}^{|\omega_0|} v(x_0^m)\varphi_0^m(x). \tag{4.6}$$

It is obvious that $\Pi^2 = \Pi$. Denote the linear space $(1-\Pi)R_1$ by R_1/R_0.
To estimate $m(L)$ from below, we will make use of the values

$$C_{10} \overset{\text{def}}{=} \sup_{v\in R_1} [\Pi v,\Pi v]/[v,v] \tag{4.7}$$

$$m_{10} \overset{\text{def}}{=} \inf_{\Delta_m^0} \inf_{\psi\in R_1/R_0} [\psi,\psi](\Delta_m^0)/<D_1\psi,\psi>_1(\Delta_m^0). \tag{4.8}$$

The proof of the estimate from below.

Lemma 4.1. The following estimate is valid:

$$m(L) \geqslant m_{10}/2(1+C_{10}). \tag{4.9}$$

Proof. Set $\varphi = \Pi(u - Pu)$. By virtue of orthogonality of the subspaces R_D and PR_1 to $<D_1\cdot,\cdot>$ we have

$$\begin{aligned} <D_1u,u>_1 &\leqslant <D_1(u-Pu-\varphi),(u-Pu-\varphi)>_1 \\ &= <D_1u,u>_1 + <D_1(Pu+\varphi),(Pu+\varphi)>_1. \end{aligned} \tag{4.10}$$

Then (4.7) implies the inequality

$$<L_1(u - Pu - \varphi),(u - Pu - \varphi)>_1 \leqslant 2(1 + C_{10})<L_1(1 - P)u,u>_1. \quad (4.11)$$

Inequalities (4.10) and (4.11) on R_D imply the estimate

$$\frac{<L_1(u - Pu - \varphi),(u - Pu - \varphi)>_1}{<D_1(u - Pu - \varphi),(u - Pu - \varphi)>_1} \leqslant 2(1 + C_{10})\frac{<L_1(1 - P)u,u>_1}{<D_1u,u>_1}. \quad (4.12)$$

Set $\psi = u - Pu - \varphi$. It is obvious that ψ belongs to R_1/R_0.

Below we will show that

$$<D_1u,v>_1 = \sum_{\Delta_k^0 \in \Omega_0} <D_1u,v>_1(\Delta_k^0). \quad (4.13)$$

Therefore, the left-hand side of (4.12) can be presented in the form of fraction

$$\left[\sum_{\Delta_m^0 \in \Omega_0} [\psi,\psi](\Delta_m^0)\right] / \left[\sum_{\Delta_m^0 \in \Omega_0} <D_1\psi,\psi>_1(\Delta_m^0)\right]. \quad (4.14)$$

Now make use of the elementary inequality for positive values a_i and b_i:

$$\min_i (a_i/b_i) \leqslant (\Sigma a_i)/(\Sigma b_i) \leqslant \max_i (a_i/b_i). \quad (4.15)$$

Hence, by virtue of (4.15) and the definition of m_{10} from (4.12), (4.14) we obtain estimate (4.9).

The proof of relation (4.13). Let $st(x_1^n)$ be stars of nodes x_1^n, $n = 1,...,|\omega_1|$. Obviously, we have

$$<D_1v,v>_1 = \sum_{n=1}^{|\omega_1|} [\varphi_1^n,\varphi_1^n]v_n^2 = \sum_{n=1}^{|\omega_1|} v_n^2 \sum_{\Delta_m^1 \in st(x_1^n)} [\varphi_1^n,\varphi_1^n](\Delta_m^1). \quad (4.16)$$

Rearrange the terms in sum (4.16). Note that the tetrahedron Δ_m^1 is contained in those stars $st(x_1^n)$ whose nodes x_1^n are its vertices: $x_1^n \in \omega(\Delta_m^1)$. Hence, if we group all summands relating to simplexes Δ_m^1, $m = 1,...,|\omega_1|$, we obtain formula (4.13). This completes the proof of the lemma.

The new technique suggested for estimating $m(L)$ consists in exploiting values (4.7) and (4.8) and in estimating them via δ_λ, δ_b, ν and γ. In this way we manage to avoid using the second main inequality for the elliptic equation.

The proof of Theorem 4.1. Estimate from above (4.3). The following string of equalitites is valid

$$<L_1v,v>_1 = [J_1v,J_1v] = \sum_{n=1}^{|\omega_1|} [J_1v,J_1v](\Delta_m^1) = \sum_{\Delta_m^1 \in \Omega_1} \sum_{l,n \in \omega(\Delta_m^1)} [\varphi_1^l,\varphi_1^n](\Delta_m^1)v_lv_n. (4.17)$$

Then exploiting the estimate

$$|[\varphi_1^l,\varphi_1^n](\Delta_m^1)v_l v_n| \leq 2^{-1}[\varphi_1^l,\varphi_1^l](\Delta_m^1)v_l^2 + 2^{-1}[\varphi_1^n,\varphi_1^n](\Delta_m^1)v_n^2$$

by virtue of (4.17) we find

$$<L_1 v,v>_1 \leq 4 \sum_{\Delta_m^1 \in \Omega_1} <D_1 v,v>_1(\Delta_m^1). \qquad (4.18)$$

Inequality (4.3) is implied by relations (4.13), (4.5) and inequality (4.18).

Estimate from below. Estimate first C_{10}. Show that if

$$[v,v](\Delta_m^0) = 0, \quad \text{then} \quad [\Pi v,\Pi v](\Delta_m^0) = 0. \qquad (4.19)$$

Formula (1.2) implies that if $[v,v](\Delta_m^0) = 0$, then almost everywhere on $\partial\Omega \cap \Delta_m^0$ we have $b(x) = 0$. Then the expression $[v,v](\Delta_m^0)$ is a quadratic form of values of the function v at nodes of $\bigcup_{\Delta_n^1 \in \Delta_m^0} \omega(\Delta_n^1)$, and the expression $[\Pi v,\Pi v](\Delta_m^0)$ is a quadratic form of values of the function v at nodes of $\omega(\Delta_m^0)$. The form $[\Pi v,\Pi v](\Delta_m^0)$ vanishes in the linear subspace

$$v(x_0^{j_1}) = \ldots = v(x_0^{j_4}), \qquad \{x_0^{j_1},\ldots,x_0^{j_4}\} = \omega(\Delta_m^0) \qquad (4.20)$$

and the form $[v,v](\Delta_m^0)$ vanishes on the linear subspace

$$v(x_1^{i_1}) = \ldots = v(x_1^{i_v}), \qquad \{x_1^{i_1},\ldots,x_1^{i_v}\} = \bigcup_{\Delta_n^1 \in \Delta_m^0} \omega(\Delta_n^1). \qquad (4.21)$$

Since $\omega(\Delta_m^0) \subset \bigcup_{\Delta_n^1 \in \Delta_m^0} \omega(\Delta_n^1)$, null space (4.21) is contained in null space (4.20) and, hence, relation (4.19) is valid. Thus,

$$\sup_{v \in R_1} [\Pi v,\Pi v]/[v,v] \leq \max_m{}' \sup_{v \in R_1} [\Pi v,\Pi v](\Delta_m^0)/[v,v](\Delta_m^0). \qquad (4.22)$$

Here, $\max'$ is taken for those m for which we have $[v,v](\Delta_m^0) \neq 0$. For such m we have

$$[\Pi v,\Pi v]/[v,v] \leq \max(\delta_\lambda,\delta_b) \max_m \frac{\int_{\Delta_m^0} |\nabla(J_0 \Pi v)|^2 \mathrm{d}x + \int_{\Delta_m^0 \cap \partial\Omega} (J_0 \Pi v)^2 \mathrm{d}\Sigma}{\int_{\Delta_m^0} |\nabla(J_1 v)|^2 \mathrm{d}x + \int_{\Delta_m^0 \cap \partial\Omega} (J_1 v)^2 \mathrm{d}\Sigma}. \qquad (4.23)$$

The quadratic forms

$$\int_{\Delta_m^0} |\nabla(J_0 \Pi v)|^2 \mathrm{d}x, \qquad \int_{\Delta_m^0} |\nabla(J_1 v)|^2 \mathrm{d}x$$

have null spaces (4.20) and (4.21), respectively. Consider their relation on the subspace orthogonal to null space (4.21) and using standard arguments we obtain

$$\int_{\Delta_m^0} |\nabla(J_0 \Pi v)|^2 \mathrm{d}x \leq C_1(\gamma,\nu,\partial\Omega) \int_{\Delta_m^0} |\nabla(J_1 v)|^2 \mathrm{d}x. \tag{4.24}$$

Likewise, we find

$$\int_{\Delta_m^0 \cap \partial\Omega} (J_0 \Pi v)^2 \mathrm{d}\Sigma \leq C_1(\gamma,\nu,\partial\Omega) \int_{\Delta_m^0 \cap \partial\Omega} (J_1 v)^2 \mathrm{d}\Sigma. \tag{4.25}$$

Estimates (4.24) and (4.25) imply the estimate for C_{10}.

By virtue of (4.20) the estimate for m_{10} is obtained similarly to that for C_{10}. This completes the proof of the theorem.

Application to computing real problems. If on the tetrahedrons of the coarse partition Ω_0 the variations of values of the functions λ and b are great, it is necessary to modify the coarse partition Ω_0 by adding to it, instead of some coarse tetrahedrons with large values δ_λ and δ_b, tetrahedrons of fine partition which make part of coarse tetrahedrons. For new 'coarse nodes' $\tilde{x}_0^m$ which are vertices of the newly included (formerly fine) tetrahedrons we take old fine-basis functions for new 'coarse-basis' functions. The partition Ω_1 and the basis functions corresponding to it are taken without any modifications.

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