CHAPTER 11

Analysis and Test of a Local Domain-Decomposition Preconditioner

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Abstract

We study in this paper a local domain decomposition preconditioner which has been described by Glowinski et al. (1987) in the framework of unstructured meshes and arbitrary geometries. We derive a theoretical upper bound on the rate of convergence of the corresponding method as a function of the discretization step and of the size of the subdomains. We also describe practical results which were obtained on parallel computers with shared and distributed memory architecture and which confirm the theoretical estimates.

1 Introduction:

Among the various Domain-Decomposition methods which can solve large scale finite-element problems, the one that processes the Schur complement at the interface by a Preconditioned Conjugate Gradient is particularly interesting because it can handle unstructured meshes and arbitrary geometries. Because of the large scale and of the ill-conditioning of the original problem, it is then advantageous to keep the implicit character of the interface problem and to build a local preconditioner for its iterative solution. In this framework, we have selected the approach proposed by Glowinski et al. [5]. This method is recalled in §2 and its convergence is analysed in §3. It is finally tested in the three-dimensional calculation of a composite material beam made of anisotropic strongly heterogeneous linearly elastic materials. This last calculation was run on parallel computers with shared and distributed memory architectures.

2 Schur Complement and Neumann Preconditioner:

2.1 Notation:

We consider the partition of a domain Ω into non-overlapping subdomains Ω_i.
Let us introduce the boundaries (see Figure 1)

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$$\begin{array}{rcll} \partial\Omega &=& \partial\Omega_N \cup \partial\Omega_D\ , & \text{external Dirichlet and Neumann boundaries,} \\ \partial\Omega_{Di} &=& \partial\Omega_D \cap \partial\Omega_i\ , & \text{local Dirichlet boundary,} \\ \Gamma_i &=& \partial\Omega_i \backslash \partial\Omega\ , & \text{local interface,} \end{array}$$

together with the spaces

$$\begin{array}{rcl} V &=& \{v \in H^1(\Omega; R^p)\ ,\ v = 0 \ \text{ on } \ \partial\Omega_D\}\ , \\ V_i &=& \{v \in H^1(\Omega_i; R^p)\ ,\ v = 0 \ \text{ on } \ \partial\Omega_{Di}\}\ , \\ V_{0i} &=& \{v \in H^1(\Omega_i; R^p)\ ,\ v = 0 \ \text{ on } \ \partial\Omega_{Di} \cup \Gamma_i\}\ . \end{array}$$

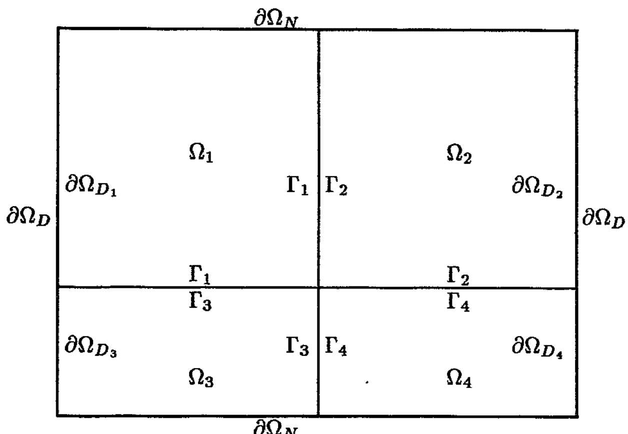


Figure 1: *Definition of the subdomains and of the boundaries.*

Moreover, we define the space $Y = \sum_i \mathrm{Tr}V_i$. It is equal to the space of traces on Γ of functions of V only if there are no cross-points in the interface. In addition, $\mathrm{Tr}_i^{-1}(\lambda)$ will represent any element z of V_i whose trace on Γ_i is equal to λ. Finally, we introduce the elliptic form

$$a_i(u,v) \ = \ \int_{\Omega_i} A_{mnkl}(x)\, \frac{\partial u_m}{\partial x_n}\, \frac{\partial v_k}{\partial x_l}\ ,$$

with $A \in L^\infty(\Omega)$ symmetric and satisfying the strong ellipticity condition

$$A_{mnkl}(x) F_{mn} F_{kl} \geq c_0 |F|^2, \forall F \in R^{p\times N}.$$

Such an assumption is typically satisfied in linear elasticity problems where we have

$$a_i(u,v) \ = \ \int_{\Omega_i} A\, \varepsilon(u) : \varepsilon(v)\ ,$$

with A the elasticity tensor and $\varepsilon(u)$ the linearized strain tensor

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right).$$

Under this notation, the problem to solve becomes

$$\begin{array}{l} \text{Find } u \in V \text{ such that} \\ \displaystyle\sum_i a_i(u,v) \ = \ < f, v > \quad , \quad \forall\, v \in V\ . \end{array} \tag{1}$$

On the discretized problem, we introduce the following notation:
-global unknown:

$$U = \begin{pmatrix} \dot{U} \\ \bar{U} \end{pmatrix} \quad \text{where} \quad \begin{cases} \dot{U} & \text{stands for the unknowns of} \quad (\Omega \cup \partial\Omega_N)\backslash\Gamma \\ \bar{U} & \text{stands for the unknowns of} \quad \Gamma \end{cases}$$

-local unknown:

$$U_i = \begin{pmatrix} \dot{U}_i \\ \bar{U}_i \end{pmatrix} \quad \text{where} \quad \begin{cases} \dot{U}_i & \text{stands for the unknowns of} \quad (\Omega_i \bigcup \partial\Omega_{i,N}) \backslash \Gamma_i \\ \bar{U}_i & \text{stands for the unknowns of} \quad \Gamma_i \; . \end{cases}$$

Using the same convention, we split the local matrices into

$$A_i = \begin{pmatrix} \dot{A}_i & B_i^t \\ B_i & \bar{A}_i \end{pmatrix}$$

and the global gathered matrix into

$$A = \begin{pmatrix} \dot{A} & B^t \\ B & \bar{A} \end{pmatrix}$$

2.2 Steklov-Poincaré operator and Schur complement matrix:

With $\lambda \in Y$ we now associate $z_i(\lambda, f)$ as the solution of

$$\begin{aligned} a_i(z_i, v) &= < f, v > \quad , \quad \forall\, v \in V_{0i}, \\ z_i \in V_i \quad , \quad z_i &= \lambda \quad \text{on } S_i \; , \end{aligned}$$

the Steklov-Poincaré operator S_i given by

$$< S_i\lambda, \mu > = \; a_i(z_i(\lambda, 0)\, , \mathrm{Tr}_i^{-1}\mu) \quad , \quad \forall\, \mu \in Y \; ,$$

and the right-hand side L given by

$$< L, \mu > = \; -\sum_i a_i(z_i(0, f)\, , \mathrm{Tr}_i^{-1}\mu) \quad , \quad \forall\, \mu \in Y \; .$$

Observe that in computing $a_i(z_i(\lambda, 0)\, , \mathrm{Tr}_i^{-1}\mu)$, the choice of the representative element of Tr_i^{-1} is of no importance since, by construction, $z_i(\lambda, 0)$ is orthogonal to any component of this element in Ker (Tr_i).
With this new notation, our initial variational problem classically reduces to the interface problem

$$\left(\sum_j S_j\right)\lambda \; = \; L \qquad \text{in} \quad Y^* \; . \tag{2}$$

Once discretized, this reduction is equivalent to a Gauss block-elimination, which transforms the linear system

$$\begin{pmatrix} \dot{A} & B^t \\ B & \bar{A} \end{pmatrix} \begin{pmatrix} \dot{U} \\ \bar{U} \end{pmatrix} \; = \; \begin{pmatrix} \dot{F} \\ \bar{F} \end{pmatrix}$$

into

$$\begin{pmatrix} \dot{A} & B^t \\ 0 & \bar{A} - B\dot{A}^{-1}B^t \end{pmatrix} \begin{pmatrix} \dot{U} \\ \bar{U} \end{pmatrix} \; = \; \begin{pmatrix} \dot{F} \\ \bar{F} - B\dot{A}^{-1}\dot{F} \end{pmatrix} \; .$$

This system is associated with the Schur complement matrix:

$$S = \bar{A} - B\dot{A}^{-1}B^t = \sum_i S_i = \sum_i \bar{A}_i - B_i\dot{A}_i^{-1}B_i^t \; .$$

The main interest of this method comes from this last property of decomposition into local matrices, allowing the parallelisation of the computations within an iterative method that only requires computation of $S\bar{U}$. A Conjugate Gradient method is well suited for this positive definite matrix. In most cases, even an explicit computation of the S_i would be too expensive. Then, a practical solver for the local Dirichlet Probem consists in factorizing only $\dot{A}_i$.

Remark: In the presence of cross-points on the interface, the functions of $Y = \sum_i \mathrm{Tr}V_i$ might be singular at these cross-points, which means that Y is not included in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. As a consequence, the function $z_i(\lambda, f)$ is not necessarily in $\mathrm{H}^1(\Omega_i)$. This difficulty disappears at the discrete level but explains the dependence on d and h of our theoretical estimates.

2.3 'Neumann' Preconditioner:

We first define a trace operator $\alpha_i Tr$ from V_i into Y satisfying

$$\sum_i \alpha_i Tr(v) = \mathrm{Tr}(v) \quad , \quad \forall\, v \in V \,. \tag{3}$$

For example, at the continuous level, we can often set $\alpha_i = 1/2$. A different definition might be used at the finite element level in order that condition (3) is still satisfied after discretization. Following Morice [14], Agoskov [10], Glowinski and Wheeler [4], and Bourgat, Glowinski, Le Tallec and Vidrascu [5], we now propose as preconditioner the operator

$$M = \sum_i (\alpha_i Tr)\, S_i^{-1}\, (\alpha_i Tr)^t \,.$$

By definition of S_i, the action of the preconditioner M on L is then given by

$$ML = \sum_i \alpha_i Tr(\psi_i)$$

with ψ_i the solution of

$$a_i(\psi_i, v) = L(\alpha_i \,\mathrm{Tr}\, v) \qquad \forall\, v \in V_i \,,\; \psi_i \in V_i \,. \tag{4}$$

At the discretized level, this preconditioner can be written:

$$M = \sum_i D_i S_i^{-1} D_i^t$$

where D_i are local weighting matrices, such that: $D_i : \Gamma_i \to \Gamma_i$ and $\sum_i D_i = I_{|\Gamma}$.
In the case of two domains with $S_1 = S_2$, then $D_i = \frac{1}{2} I_{|\Gamma_i}$ gives the perfect preconditioner, because $M = \frac{1}{4}(S_1^{-1} + S_2^{-1}) = S^{-1}$
The actual matrix-vector product by S_i^{-1} is also performed implicitly, since it only requires the factorization of the complete local stiffness matrix A_i. Indeed,

$$S_i^{-1}\bar{U} = \begin{pmatrix} 0 & Id \end{pmatrix} \begin{pmatrix} \mathring{A}_i & B_i^t \\ B_i & \bar{A}_i \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ Id \end{pmatrix} \bar{U} \,.$$

Remark 1: In the absence of Dirichlet boundary conditions in the definition of V_i, problem (4) is not well-posed. In such situations, we replace a_i in (4) by an equivalent symmetric bilinear form $\tilde{a}_i$ which we take as positive definite on V_i and such that

$$\sum_i \tilde{a}_i(z,z) \geq \sum_i a_i(z,z) \geq c_1 \sum_i \tilde{a}_i(z,z),\; \forall\, z \in V.$$

For example, on the discrete level, the bilinear form $\tilde{a}_i$ is simply obtained by replacing, in the factorization of the finite-element matrix of a_i, all the singular pivots by an averaged strictly positive pivot.

Remark 2: The performance of this preconditioner is not optimal, and the convergence rate that we will show in the next section is not as good as the one proposed by Smith [18]. However, it is much cheaper to implement, and it leads to a simpler data structure, especially in three dimensions. Moreover, let us recall that our algorithm only requires the factorization of the local stiffness matrices, and not the construction of any submatrix of the Schur complement (constructing a $p \times p$ submatrix costs the equivalent of $p/2$ iterations of our PCG). Thus, the initialisation step, which is included in our numerical results, remains reasonably short.

3 Convergence Analysis:

3.1 At the continuous level:

To study the convergence of the conjugate gradient algorithm, applied to the above preconditioned problem, we have to prove the spectral boundedness of $M\sum_i S_i$, that is to exhibit two positive constants k and K such that

$$k((\lambda,\lambda)) \le ((MS\lambda,\lambda)) \le K((\lambda,\lambda)) \ .$$

with $((\cdot,\cdot))$ a given scalar product on Y and $S=\sum_i S_i$. Indeed, let α_1 be the smallest eigenvalue of MS, and λ_1 the corresponding eigenvector. The same for α_n, the largest eigenvalue. Then:

$$\begin{array}{rclcl} ((MS\lambda_1,\lambda_1)) &=& \alpha_1((\lambda_1,\lambda_1)) &\ge& k((\lambda_1,\lambda_1)) \\ ((MS\lambda_n,\lambda_n)) &=& \alpha_n((\lambda_n,\lambda_n)) &\le& K((\lambda_n,\lambda_n)) \end{array}$$

Thus

$$\frac{\alpha_n}{\alpha_1} \le \frac{K}{k} \Rightarrow \mathrm{cond}(MS) \le \frac{K}{k} \ .$$

To prove this spectral boundedness, we endow Y with the scalar product

$$((\lambda,\lambda')) = < S\lambda,\lambda' > \ .$$

This is by construction a scalar product. Indeed,

$$\begin{aligned} < S\lambda,\lambda' > &= \sum_j a_j\left(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\lambda')\right) \\ &= \sum_j a_j(z_j(\lambda,0)\,,\,z_j(\lambda',0))\,. \end{aligned}$$

With this scalar product, we have, using the previous notation,

$$\begin{aligned} ((MS\lambda,\lambda)) &= < SMS\lambda,\lambda > \\ &= < MS\lambda, S\lambda > \\ &= \sum_j a_j(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(MS\lambda)) \\ &= \sum_j a_j(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\sum_i \alpha_i \,\mathrm{Tr}\,\psi_i)) \\ &= \sum_{ij} a_j(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\alpha_i \,\mathrm{Tr}\,\psi_i)) \\ &= \sum_i L(\alpha_i \,\mathrm{Tr}\,\psi_i) \\ &= \sum_i \tilde{a}_i(\psi_i,\psi_i) \ . \end{aligned}$$

Therefore, the whole convergence analysis reduces to the verification of the inequality

$$k\|z(\lambda,0)\|^2 \le \|\psi\|^2 \le K\|z(\lambda,0)\|^2,$$

under the notation

$$\|\varphi\| = \Big(\sum_i \tilde{a}_i(\varphi_i,\varphi_i)\Big)^{1/2} .$$

Theorem 1 : *We have*

$$c_1^2\|z(\lambda,0)\|^2 \le \|\psi\|^2 \le C_2^2\|z(\lambda,0)\|^2,$$

with

$$C_2 = \sup_{\varphi\in\Pi V_i} \frac{\|z\,(\sum \alpha_i\ Tr\varphi_i\,,0)\|}{\|\varphi\|}\,.$$

Remark: In the case of no internal cross-points and if we choose smooth weights α_i, the constant C_2 is well defined. Furthermore, in this case it does not depend on the discretization step h when the spaces V_i are replaced by conforming finite-element spaces V_{ih} provided that we have

$$\mathrm{Tr}\, V_{ih_{|\partial\Omega_i\cap\partial\Omega_j}} = \mathrm{Tr}\, V_{jh_{|\partial\Omega_i\cap\partial\Omega_j}}\ .$$

Indeed, in this case z is the harmonic extension of the function μ defined in $\prod_{ij} \mathrm{Tr}V_i\cap\mathrm{Tr}V_j$ by

$$\mu_{|\partial\Omega_i\cap\partial\Omega_j} = \alpha_i \mathrm{Tr}\,\psi_i + \alpha_j \mathrm{Tr}\,\psi_j\ ,$$

and this obviously depends continuously on ψ_i and ψ_j.
In the case of internal cross points, μ does not belong any more to $\mathrm{H}^{-\frac{1}{2}}(\Gamma)$. Theorem 2 will then give the dependency of C_2 on h and d.

Proof of the theorem: From the Cauchy-Schwarz inequality, we first have by construction,

$$\begin{aligned}
\|\psi\|\,\|z(\lambda,0)\| &= \Big(\sum_i \tilde a_i(\psi_i,\psi_i)\Big)^{1/2}\Big(\sum_j \tilde a_j(z_j(\lambda,0),z_j(\lambda,0))\Big)^{1/2}\\
&\ge \sum_i \tilde a_i(\psi_i, z_i(\lambda,0))\\
&= \sum_i L(\alpha_i\,\mathrm{Tr}\,z_i(\lambda,0))\\
&\ge \sum_i\sum_j a_j\big(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\alpha_i\,\mathrm{Tr}\,z_i(\lambda,0))\big)\\
&\ge \sum_j a_j(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\sum_i \alpha_i\,\mathrm{Tr}\,z_i(\lambda,0)))\\
&\ge \sum_j a_j\big(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\lambda)\big)\\
&= \sum_j a_j(z_j(\lambda,0),z_j(\lambda,0))\\
&\ge c_1\|z(\lambda,0)\|^2\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_i \tilde a_i(\psi_i,\psi_i) &= \sum_i L(\alpha_i\,\mathrm{Tr}\,\psi_i)\\
&= \sum_i\sum_j a_j\big(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\alpha_i\,\mathrm{Tr}\,\psi_i)\big)\\
&= \sum_j a_j(z_j(\lambda,0)\,,\,\mathrm{Tr}_j^{-1}(\sum_i \alpha_i\,\mathrm{Tr}\,\psi_i))\\
&= \sum_j a_j(z_j(\lambda,0)\,,\,z_j(\sum_i \alpha_i\,\mathrm{Tr}\,\psi_i,0))\\
&\le \|z(\lambda,0)\|\,\Big\|(z(\sum_i \alpha_i\,\mathrm{Tr}\,\psi_i\,,0)\Big\|\\
&\le C_2\|z(\lambda,0)\|\,\|\psi\|.
\end{aligned}$$

By combining the two inequalities, we finally obtain

$$c_1^2\|z(\lambda,0)\|^2 \leq \|\psi\|^2 \leq C_2^2\|z(\lambda,0)\|^2$$

Corollary of Theorem 1:
Our preconditioned conjugate gradient algorithm converges at least linearly with asymptotic constant

$$\frac{C_2/c_1 - 1}{C_2/c_1 + 1}.$$

Proof: From Theorem 1, we have seen that the spectrum of the operator MS was bounded above by C_2^2 and bounded below by c_1^2. Hence, the condition number of MS is bounded above by $(C_2/c_1)^2$ which ensures that the associated preconditioned conjugate gradient algorithm converges as announced (see Golub and Van Loan [11] for more details). ∎

Actually, this convergence result is rather conservative. Indeed, our numerical results show that the larger eigenvalues of MS are well separated, which accelerates the convergence of the conjugate gradient algorithm.

3.2 Within the Finite Element Discretisation:

Theorem 2 *: As a function of the discretization step h and of the diameter d of the subdomains, and for conforming finite elements of* 1^{st} *order (i.e. continuous piecewise linear), one has:*

$$C_2 = \sup_{\varphi \in \Pi V_{ih}} \frac{\|z(\sum_i \alpha_i \, Tr\varphi_i, 0)\|_{1,2,\Omega}}{\|\varphi\|_{1,2,\Omega}} \leq \frac{C}{d}(1 + \ln\frac{d}{h}) \ . \tag{5}$$

Remark: Thus, the condition number of MS grows like $\frac{1}{d^2}$. As Widlund [20] stated, this is a classical dependency for elliptic operators, when preconditioned by local operators (at scale d).

Proof of the theorem: Recall that $z(\lambda, 0)$ is the discrete harmonic extension of λ given at the internal interface Γ. We first prove in Lemma 3 that any extension u_{0h} of λ in V_{ih} has an H^1 norm that bounds that of z,

$$\|z(\lambda,0)\|_{1,2,\Omega} \leq C\|u_{0h}\|_{1,2,\Omega} \ . \tag{6}$$

We then construct a function u_{0h} whose trace is equal to $\lambda^\varphi = \sum_i \alpha_i \mathrm{Tr}\varphi_i$ on the interface Γ, such that its extension on Ω is bounded by $\|\varphi\|$. For simplicity, we can assume that the subdomains Ω_i are cubes with edges of length d and we have the following notations (see Figure 2):

- $\Gamma_i = \partial\Omega_i \cap \Gamma$ the local interface,
- $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a face,
- $\Xi_i = \Gamma_i \setminus \cup_j \overset{\circ}{F}_{ij}$ the wire basket composed of the edges and the vertices of the local interface.

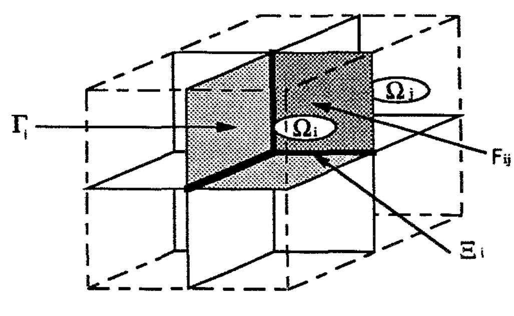


Figure 2: *Definition of the local interface.*

We construct u_{0h} in several steps:

• $\bar{\alpha}_{ij} \in \mathrm{C}^0(\mathrm{F}_{ij})$ is built as the continuous extension of α_i on F_{ij}. Indeed, α_i belongs to $\mathrm{L}^2(\Gamma_i)$ but it can only be continuous in the interior of the faces $\overset{\circ}{\mathrm{F}}_{ij}$, because we impose $\sum_i \alpha_i \equiv 1$ on Γ, thus also on Ξ_i. As weighting factors α_i, and therefore $\bar{\alpha}_{ij}$, are positive and are such that $\|\bar{\alpha}_{ij}\|_{0,\infty,\mathrm{F}_{ij}} \leq 1$. Let us also recall that in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$, one has:

$$(u,v) \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{C}^0_{\frac{1}{2}+\varepsilon}(\Gamma) \quad \Rightarrow \quad w = u \cdot v \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \text{ and } \|w\|_{\frac{1}{2},2,\Gamma} \leq \|u\|_{\frac{1}{2},2,\Gamma}\|v\|_{\epsilon,\Gamma}$$

$$\text{where} \quad \mathrm{C}^0_{\frac{1}{2}+\varepsilon}(\Gamma) = \{v \in \mathrm{C}^0(\Gamma),\ \exists c = |v|_{\epsilon,\Gamma} \ \text{ st. } \ |v(x)-v(y)| \leq c|x-y|^{\frac{1}{2}+\epsilon},\ \forall (x,y) \in \Gamma\}$$

$$\text{and } \|v\|^2_{\epsilon,\Gamma} = |v|^2_{\epsilon,\Gamma} + \|v\|^2_{0,\infty,\Gamma} \ .$$

As we define $f_{ij} = \bar{\alpha}_{ij}\mathrm{Tr}\varphi_i + \bar{\alpha}_{ji}\mathrm{Tr}\varphi_j$, we require that $\bar{\alpha}_{ij}$ belongs to $\mathrm{C}^0_{\frac{1}{2}+\varepsilon}(\Gamma)$, in order for f_{ij} to belong to the space $\mathrm{H}^{\frac{1}{2}}(\mathrm{F}_{ij})$. This is not restrictive, for in usual cases, $\bar{\alpha}_{ij}$ belongs to $\mathrm{C}^\infty(\Gamma)$.

• With f_{ij} given as above, we call f_{ijh} its discretization on F_{ij}, and we build g_{ijh}, the function that is equal to f_{ijh} in $\overset{\circ}{\mathrm{F}}_{ij}$ and vanishes on $\partial\mathrm{F}_{ij}$. As is shown in the Lemma 4.3 of Dryja [9], we can construct an extension u_{ijh} to g_{ijh} on Ω_i vanishing on $\Gamma_i \backslash \overset{\circ}{\mathrm{F}}_{ij}$ such that:

$$|u_{ijh}|_{1,2,\Omega_i} \leq C(1+\ln\frac{d}{h})\|f_{ijh}\|_{\frac{1}{2},2,\mathrm{F}_{ij}} \ . \tag{7}$$

In 3D, we take the following definition of a weighted $\mathrm{H}^{\frac{1}{2}}$-norm of v on a face F:

$$\|v\|^2_{\frac{1}{2},2,\mathrm{F}} = \int\!\!\int_{\mathrm{F}\mathrm{F}} \frac{|v(x)-v(y)|^2}{|x-y|^3}\,dx\,dy + \frac{1}{d}\int_{\mathrm{F}} |v(x)|^2\,dx \ . \tag{8}$$

On the one hand, as u_{ijh} has an homogeneous Dirichlet boundary condition on $\Gamma_i \backslash \overset{\circ}{\mathrm{F}}_{ij}$, we can use the Poincaré inequality, see Dautray and Lions [7]:

$$\|u_{ijh}\|_{1,2,\Omega_i} \leq C|u_{ijh}|_{1,2,\Omega_i} \ . \tag{9}$$

On the other hand:

$$\begin{aligned} \|f_{ijh}\|_{\frac{1}{2},2,\mathrm{F}_{ij}} &\leq (\|\bar{\alpha}_{ij}\|_{\epsilon,\mathrm{F}_{ij}}\|\varphi_i\|_{\frac{1}{2},2,\mathrm{F}_{ij}} + \|\bar{\alpha}_{ji}\|_{\epsilon,\mathrm{F}_{ij}}\|\varphi_j\|_{\frac{1}{2},2,\mathrm{F}_{ij}}) \\ &\leq C(\|\varphi_i\|_{\frac{1}{2},2,\mathrm{F}_{ij}} + \|\varphi_j\|_{\frac{1}{2},2,\mathrm{F}_{ij}}) \ . \end{aligned} \tag{10}$$

Moreover, the continuity of the trace operator on Ω_i for φ_i yields:

$$\|\varphi_i\|_{\frac{1}{2},2,\mathrm{F}_{ij}} \leq \frac{C}{d}\|\varphi_i\|_{1,2,\Omega_i} \qquad (\text{resp. for } \varphi_j \text{ on } \Omega_j) \ . \tag{11}$$

Indeed, the norm of the trace operator in $\mathrm{L}(\mathrm{H}^1, \mathrm{H}^{\frac{1}{2}})$ grows like $\frac{1}{\sqrt{d}}$ as a function of the diameter, if we use the standard non-weighted $\mathrm{H}^{\frac{1}{2}}$ norm. Therefore, it varies in $\frac{1}{d}$ for our choice of weighted norm. Such an estimate is sharp, as it can be reached by constant functions.
Then, from (7),(9),(10) and (11), one gets the expected bound for u_{ijh}

$$\|u_{ijh}\|^2_{1,2,\Omega_i} \leq \frac{C}{d}(1+\ln\frac{d}{h}) \sum_{j \in \mathcal{V}(i)} \|\varphi_j\|^2_{1,2,\Omega_j} \ . \tag{12}$$

where $\mathcal{V}(i)$ is the set of indices of the subdomains neighboring Ω_i.

• On the wire-basket Ξ, we discretize $\lambda^\varphi = \sum_i \alpha_i \mathrm{Tr}\varphi_i$ into f^c_h.
Inside each domain Ω_i, we construct the extension u^c_{ih} of f^c_h which vanishes at all the nodes of $\Omega_i \backslash \Xi_i$. For this function u^c_{ih}, which is piecewise linear on tetrahedra and vanishes at all the interior nodes, we show in Lemma 4 that:

$$\|u^c_{ih}\|_{1,2,\Omega_i} \leq C\|f^c_h\|_{0,2,\Xi_i} \ . \tag{13}$$

Then we prove the reverse inequality in Lemma 5, that is, for any edge E in Ξ_i:

$$\forall \varphi_i \in V_{ih}, \quad \|\varphi_i\|^2_{0,2,E} \leq \frac{C}{d}(1 + \ln \frac{d}{h})\|\varphi_i\|^2_{1,2,\Omega_i} \ . \tag{14}$$

As $\|\alpha_i\|_{\infty,0,\Xi_i}$ are bounded by 1, we have: $\|f_h^c\|_{0,2,\Xi_i} \leq C\sum_j \|\varphi_j\|_{0,2,\Xi_i}$,
and it easily follows from (13) and (14) that:

$$\|u^c_{ih}\|^2_{1,2,\Omega_i} \leq \left[\frac{C}{d}(1 + \ln \frac{d}{h})\right] \sum_j \|\varphi_j\|^2_{1,2,\Omega_j} \ . \tag{15}$$

• By gathering all these u^c_{ih}, we obtain u^c_h, which belongs to $V = \Pi V_{ih}$ and satisfies: $u^c_h = \lambda^\varphi$ onΓ.
• Thus, $u_{0h} = u^c_h + \sum_{j\neq i} u_{ijh}$ is piecewise linear. It is an extension of λ^φ in Ω, and it satisfies, according to (12) and (15):

$$\|u_{0h}\|^2_{1,2,\Omega} \leq \left[\frac{C}{d}(1 + \ln \frac{d}{h})\right] \|\varphi\|^2_{1,2,\Omega} + \left[\frac{C}{d}(1 + \ln \frac{d}{h})\right]^2 \|\varphi\|^2_{1,2,\Omega} \ .$$

As we wrote above, this inequality also holds for the harmonic extension z of λ^φ. If we want to highlight the behaviour of this bound when d and especially h tend to 0, we obtain for z the expected relation that proves (5):

$$\|z(\textstyle\sum_i \alpha_i \mathrm{Tr}\varphi_i, 0)\|_{1,2,\Omega} \leq \frac{C}{d}(1 + \ln \frac{d}{h})\|\varphi\|_{1,2,\Omega} \ . \qquad \blacksquare$$

Lemma 3 : *Let a be a bilinear, symmetric, elliptic form on Ω, λ be a function in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ and z be the harmonic extension of λ on Ω, i.e. the solution of:*

$$\begin{cases} a(z,v) = 0 & \text{in } \Omega, \ \forall v \in \mathrm{H}^1_0(\Omega)\ , \\ z = \lambda & \text{on } \Gamma, \quad z \in \mathrm{H}^1(\Omega)\ . \end{cases}$$

Then, we have:

$$\|z\|_{1,2,\Omega} \leq C\|u_0\|_{1,2,\Omega}, \ \forall u_0 \in \mathrm{H}^1(\Omega) \text{ with } \mathrm{Tr}_\Gamma(u_0) = \lambda \ .$$

Proof: Given u_0 in $\mathrm{H}^1(\Omega)$ with $\mathrm{Tr}_\Gamma(u_0) = \lambda$, let us introduce the following problem:

$$\begin{cases} a(w,v) = -a(u_0,v) & \text{in } \Omega, \\ \forall v \in \mathrm{H}^1_0(\Omega), & w \in \mathrm{H}^1_0(\Omega)\ . \end{cases}$$

Since a is elliptic and $v \mapsto -a(u_0,v)$ is linear, w exists and

$$J(w) = \min_{v \in \mathrm{H}^1_0(\Omega)} J(v)$$

$$\text{where } J(v) = \frac{1}{2}a(u,v) + a(u_0,v) \ .$$

Then $z = u_0 + w$ is such that $\mathrm{Tr}(z) = \lambda$ on Γ and satisfies

$$\forall v \in \mathrm{H}^1_0(\Omega), \quad a(z,v) = a(u_0,v) + a(w,v) = 0 \ .$$

Thus z is the harmonic extension of λ and

$$a(z,z) = a(u_0 + w, u_0 + w) = a(u_0,u_0) + 2J(w) \ .$$

Since the null function belongs to $\mathrm{H}^1_0(\Omega)$, and $J(w)$ is minimal, then $J(w) \leq 0$ and we have

$$a(z,z) \leq a(u_0,u_0) \ .$$

Since a is elliptic, there exist α and A such that: $\alpha\|v\|^2_{1,2,\Omega} \leq a(v,v) \leq A\|v\|^2_{1,2,\Omega}$.
This finally implies that

$$\|z\|^2_{1,2,\Omega} \leq \frac{A}{\alpha}\|u_0\|^2_{1,2,\Omega} \ .$$

Corollary of Lemma 3: *The lemma still holds at the discrete level.*

Proof: One may obviously replace in the proof the spaces $H^1(\Omega)$ by V_h, $H^{\frac{1}{2}}(\Gamma)$ by $\mathrm{Tr}_{|\Gamma}(V_h)$ and obtain the same results for the discrete functions, as needed in (6).

Lemma 4 : *Working with piecewise linear finite elements on tetrahedra, let u_h be a discretized function defined on Ξ, the wire-basket (or a part of it) of a domain Ω. We extend u_h into Ω by making it vanish at all the other nodes of Ω. Then:*

$$||u_h||_{1,2,\Omega} \leq C||u_h||_{0,2,\Xi} \ . \tag{13}$$

Proof: Let us call Ω_Ξ the set of all tetrahedra T having an edge or a vertex on Ξ. Since Ω_Ξ is the support of u_h, we have

$$||u_h||_{1,2,\Omega} = ||u_h||_{1,2,\Omega_\Xi} \ .$$

We assume that the mesh is such that no tetrahedron will touch 2 different edges E_k and E_l except possibly at the vertices of Ξ. Therefore, (see Figure 3) we can partition Ω_Ξ into $\Omega_\Xi = \bigcup_k \Omega_{E_k}$, with:

$$\begin{cases} \forall \ \text{ edge } \ E_k, \quad E_k \subset \partial\Omega_{E_k}, \\ \forall j \neq k, \quad \Omega_{E_j} \cap \Omega_{E_k} = \emptyset \ . \end{cases}$$

By dissecting each edge E of the wire basket into segments ℓ, that are edges of the finite-element mesh, we subpartition Ω_E into subregions Ω_ℓ. For a given segment ℓ, the tetrahedra belonging

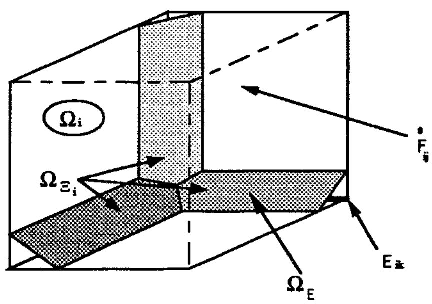


Figure 3: *Support of u_h.*

to Ω_ℓ either own ℓ among their edges (T type), or share only one vertex with ℓ (T_1 type)(see Figure 4). Then we have:

$$\begin{aligned} ||u_h||^2_{1,2,\Omega} &= ||u_h||^2_{1,2,\Omega_\Xi} = \sum_{E\in\Xi}\sum_{\ell\in E} ||u_h||^2_{1,2,\Omega_\ell} \\ &= \sum_{E\in\Xi}\sum_{\ell\in E}\left(\sum_{T\in\Omega_\ell} ||u_h||^2_{1,2,T} + \sum_{T_1\in\Omega_\ell} ||u_h||^2_{1,2,T_1}\right) \ . \end{aligned}$$

Because of the regularity of the mesh, we can associate a fixed maximum number τ of tetrahedra of the T type and τ_1 of tetrahedra of the T_1 type around the segment ℓ.

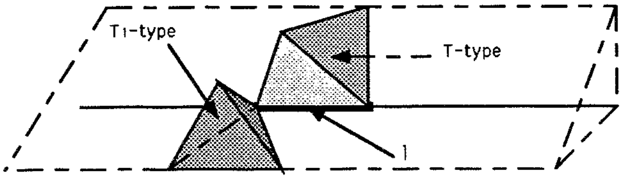


Figure 4: *Definition of 2 types of tetrahedra.*

Now, on the reference tetrahedron, all the norms $||\hat{u}_h||^2_{0,2,\hat{T}}$, $|\hat{u}_h|^2_{1,2,\hat{T}}$ and $||\hat{u}_h||^2_{0,2,\hat{\ell}}$ are equivalent for the set of functions $\hat{u}_h$ that have nonzero values on the vertices of the edge $\hat{\ell}$ and zero at all other vertices.
Indeed, one can always consider that $\hat{u}_h$ only has values on the edge $\hat{\ell} = [0, 1_x]$ and that $\hat{u}_h(0,0,0) = a$ and $\hat{u}_h(1,0,0) = b$, thus:

$$\begin{aligned} \hat{u}_h(x,y,z) &= a(1-x-y-z)+bx \\ \nabla\hat{u}_h(x,y,z) &= \begin{pmatrix} b-a \\ -a \\ -a \end{pmatrix} \\ \hat{u}_h(x,0,0) &= a(1-x)+bx \ . \end{aligned}$$

This implies that $||\hat{u}_h||^2_{0,2,\hat{T}}$, $|\hat{u}_h|^2_{1,2,\hat{T}}$ and $||\hat{u}_h||^2_{0,2,\hat{\ell}}$ correspond to formulae of the kind:

$$||\hat{u}_h||^2_{p,2,\hat{K}} = \alpha a^2 + \beta b^2 + \gamma ab \ .$$

Thus, they are all equivalent to (a^2+b^2) and there exist C such that:

$$|\hat{u}_h|^2_{1,2,\hat{T}} \leq C||u_h||^2_{0,2,\ell} \quad \text{and} \quad ||\hat{u}_h||^2_{0,2,\hat{T}} \leq C||u_h||^2_{0,2,\ell} \ .$$

Now, let us call $\hat{v}_h$ the function that has the value a at O and null at all other nodes. All its norms will only depend on a, thus we also have:

$$|\hat{v}_h|^2_{1,2,\hat{T}} \leq C||\hat{u}_h||_{0,2,\ell} \quad \text{and} \quad ||\hat{v}_h||^2_{0,2,\hat{T}} \leq C||\hat{u}_h||_{0,2,\ell} \ . \tag{16}$$

On the other hand, general results of approximations given in Raviart-Thomas [16] p.101 hold and show that, if $x = B\hat{x} + b$ is a linear transformation from the reference element $\hat{K}$ to the current element K, then we have

$$\begin{aligned} |\hat{v}|_{l,2,\hat{K}} &\leq \gamma ||B||^l |\det(B)|^{-\frac{1}{2}} |v|_{l,2,K} \ , \\ |v|_{l,2,K} &\leq \gamma\prime ||B^{-1}||^l |\det(B)|^{\frac{1}{2}} |\hat{v}|_{l,2,\hat{K}} \ . \end{aligned}$$

This yields in our case:

$$\begin{aligned} |u_h|_{1,2,T} &\leq \gamma\prime h^{\frac{1}{2}} |\hat{u}_h|_{1,2,\hat{T}} \ , \\ ||u_h||_{0,2,T} &\leq \gamma\prime h^{\frac{3}{2}} ||\hat{u}_h||_{0,2,\hat{T}} \ , \\ ||\hat{u}_h||_{0,2,\hat{\ell}} &\leq \gamma h^{-\frac{1}{2}} ||u_h||_{0,2,\ell} \ . \end{aligned}$$

As we can assume that $h < 1$ (it tends to 0) on each tetrahedron sharing an edge with E, we finally have

$$\begin{aligned} ||u_h||_{1,2,T} &\leq \gamma\prime h^{\frac{1}{2}} (|\hat{u}_h|_{1,2,\hat{T}} + C|\hat{u}_h|_{0,2,\hat{T}}) \\ &\leq Ch^{\frac{1}{2}} ||\hat{u}_h||_{0,2,\hat{\ell}} \\ &\leq C||u_h||_{0,2,\ell} \ . \end{aligned}$$

Another property of regular conforming finite elements forces the internal radius and the internal angles to be bounded in each tetrahedron. This implies that for each T_1 type, an integral along one of its edges or along ℓ have the same order. So, as a result of (16), we can also bound:

$$||u_h||_{1,2,T_1} \leq C||u_h||_{0,2,\ell} \ .$$

Then we have, by summation of the squares:

$$\begin{aligned}\|u_h\|^2_{1,2,\Omega} &= \sum_{E\in\Xi}\sum_{\ell\in E}\sum_{T\in\Omega_\ell}\|u_h\|^2_{1,2,T}\\ &\le \sum_{E\in\Xi}\sum_{\ell\in E}C(\tau+\tau_1)\|u_h\|^2_{0,2,\ell}\\ &\le C(\tau+\tau_1)\|u_h\|^2_{0,2,\Xi}\ . \qquad \blacksquare\end{aligned}$$

Lemma 5 *: Let E be an edge of a cube Ω. Then for tetrahedric piecewise linear finite elements, one has:*

$$\forall v\in V_h(\Omega),\qquad \|v\|^2_{0,2,E}\le \frac{C}{d}(1+\ln\frac{d}{h})\|v\|^2_{1,2,\Omega}\ .$$

Proof: We are interested in an edge E belonging to the interface between subdomains. Thus, it is not restrictive to consider E at the border of a face F belonging to the interface. In turn, let E be the Ox_1 axis of coordinates ($x_1\in[0,d]$), and let F be defined by:

$$(x_1,x_2), 0\le x_i\le d, \forall i\in[1,2]\ .$$

Let v be a piecewise linear finite-element function, defined in Ω, on F and on E. Then:

$$\begin{aligned}\|v\|^2_{0,2,E} &= \int_E |v(t)|^2dt\\ &\le \max_{0\le x_2\le d}\|v(.,x_2)\|^2_{0,2,E}\\ &\le \left[\max_{x_2}\|v(.,x_2)\|_{0,2,E}\right]^2\ .\end{aligned}$$

Let $u(x_2)=\|v(.,x_2)\|_{0,2,E}$, then the inequality above can be written:

$$\|v\|^2_{0,2,E}\le \|u\|^2_{0,\infty,Ox_2}\ .$$

From Lemma 4.1 of Dryja [9], we have the following relation in one dimension:

$$\|u\|_{0,\infty,Ox_2}\le C(1+\ln\frac{d}{h})^{\frac12}\|u\|_{\frac12,2,Ox_2}\ .$$

Let us estimate this last norm. The definition of $\|u\|_{\frac12,2,Ox_2}$ in one dimension, is given by:

$$\|u\|^2_{\frac12,2,Ox_2}=\int_0^d\int_0^d\frac{|u(x_2)-u(y_2)|^2}{|x_2-y_2|^2}\,dx_2\,dy_2+\frac1d\int_0^d|u(x_2)|^2\,dx_2\ .$$

The following triangular inequality: $|\ \|a\|-\|b\|\ |^2\ \le\ \|a-b\|^2$, applied in the Hilbert space $L^2(E)$, can then be written:

$$|\ \|v(.,x_2)\|_{0,2,E}-\|v(.,y_2)\|_{0,2,E}\ |^2\le \|v(.,x_2)-v(.,y_2)\|^2_{0,2,E}\ ,$$

and it implies:

$$\int_0^d\int_0^d\frac{|u(x_2)-u(y_2)|^2}{|x_2-y_2|^2}\,dx_2\,dy_2\le\int_0^d\int_0^d\frac{\|v(.,x_2)-v(.,y_2)\|^2_{0,2,E}}{|x_2-y_2|^2}\,dx_2\,dy_2\ .$$

The weighted $H^{\frac12}$-norm for 3D that we have chosen in (8), is equivalent on a square to the following norm, see Nečas [15]:

$$\begin{aligned}\|v\|^2_{\frac12,2,F} &= \int_0^d\int_0^d\frac{\|v(x_1,.)-v(y_1,.)\|^2_{0,2,Ox_2}}{|x_1-y_1|^2}\,dx_1\,dy_1\\ &+ \int_0^d\int_0^d\frac{\|v(.,x_2)-v(.,y_2)\|^2_{0,2,E}}{|x_2-y_2|^2}\,dx_2\,dy_2\quad+\frac1d\|v\|^2_{0,2,F}\ .\end{aligned}$$

Thus, a rough estimate gives finally:

$$||u||^2_{\frac{1}{2},2,O_{x_2}} \leq ||v||^2_{\frac{1}{2},2,F} \ .$$

On the other hand, there exists an extension theorem between $H^{\frac{1}{2}}(F)$ and $H^1(\Omega)$:

$$||v||_{\frac{1}{2},2,F} \leq \frac{C}{d}||v||_{1,2,\Omega} \ .$$

The whole set of inequalities justified above leads to:

$$\begin{aligned} ||v||^2_{0,2,E} &\leq ||u||^2_{0,\infty,O_{x_2}} \\ &\leq C(1+\ln\frac{d}{h})||u||^2_{\frac{1}{2},2,O_{x_2}} \\ &\leq \frac{C}{d}(1+\ln\frac{d}{h})||v||^2_{1,2,\Omega} \ . \end{aligned}$$

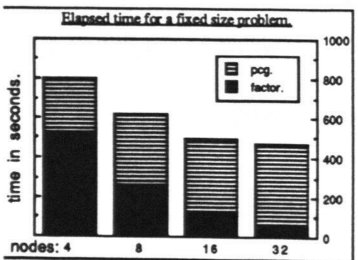
4 Numerical results:

Our tests were performed on a stiff structural analysis problem, namely a beam consisting of composite parallel pencils. Each of them is composed of a central carbon fibre wrapped in a very soft elastomere (see Figure 6). The ratio between the elasticity moduli of the two materials is very high, and locally the carbon has high orthotropic coefficients in the main direction of the beam. Furthermore, in order to simulate the incompressibility of the elastomere, we use a very high Poisson coefficient (ν close to 0.5). The effect of these parameters on the ill-conditioning of the problem, and on the relative efficiency of the preconditioner have been shown in [8] and [12]. In that last report, the implementation on a distributed memory machine is described in detail, along with considerations about the data structure.

In these reports, we have also shown the impressive effect of a full reorthogonalisation during the PCG, see also Roux [17]. This operation is still cheap because only the interface vectors are stored.

4.1 Test over d:

The beam, consisting here of one pencil, is sliced along its leading dimension, see Figure 5. Computing on a hypercube, the Intel iPSC/2 32 SX, we show results on a fixed-size problem. Namely, we have kept constant the total number of degrees of freedom, and using 2 processors per subdomain (one for the Dirichlet subproblem, one for the Neumann subproblem), we show the effect of lowering d (increasing the number of subdomains).



dom.	nodes	dof in Ω	dof on Γ	iter.
2	4	9675	75	**9**
4	8	9675	225	**25**
8	16	9675	525	**49**
16	32	9675	1125	**98**

Figure 5: *Test over d.*

As expected, the convergence of the conjugate gradient becomes worse linearly with n, the number of subdomains. However, as the size of the local problems decreases (both for the factorization and for the iterations), an actual speed-up is obtained by using more nodes.
Let us recall that in this practical case, where the domain is heterogeneous and anisotropic, we cannot use any coarser grid for coarse grid smoothing in the preconditioner, which would have freed the algorithm from its dependency on d.

4.2 Test over h:

1D

step	dof in Ω	dof on Γ	iter1	iter2
h	975	225	**24**	**15**
$\frac{h}{2}$	6375	765	**32**	**18**
$\frac{h}{4}$	46011	2817	**36**	**20**

1D

step	dof in Ω	dof on Γ	iter1	iter2
h	3315	507	**26**	**16**
$\frac{h}{2}$	23475	1875	**26**	**18**
$\frac{h}{4}$	176547	7203	**n.a.**	**n.a.**

2D

step	dof in Ω	dof on Γ	iter1	iter2
h	3315	507	**16**	**30**
$\frac{h}{2}$	23475	1875	**20**	**32**
$\frac{h}{4}$	176547	7203	**24**	**34**

3D

step	dof in Ω	dof on Γ	iter1	iter2
h	3315	507	**34**	**52**
$\frac{h}{2}$	23475	1875	**58**	**58**
$\frac{h}{4}$	176547	7203	**69**	**n.a.**

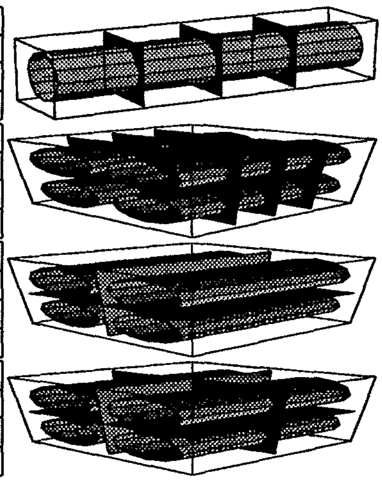


Figure 6: *Test over h.*

We have tested the dependency over h with 3 types of interface:
-One-dimensional slicing, thus no cross-points.
-Two-dimensional slicing, with edges as cross-points.
-Three-dimensional slicing, with vertices and edges as cross-points.
Tests were performed on a CRAY-2 because up to 180 Megawords of memory were needed. The aspect ratio of the finite elements is kept identical throughout these tests, i.e. for the different slicings and when h decreases. The iteration counts **iter1** and **iter2** correpond respectively to heterogeneous and homogeneous materials.
Memory space problems occur for the very large test cases, because our local direct solvers are very sensitive to the band-width of the structure of the matrices. The dependency over h, which is in $(1+\ln \frac{d}{h})$, can be detected: it yields to an affine increase of the number of iterations versus the exponential refinement of h. However, it works better for homogeneous material, because for an heterogeneous material, a finer discretization contributes to a better conditioning of the problem.

4.3 Same problem, same cube, different splittings:

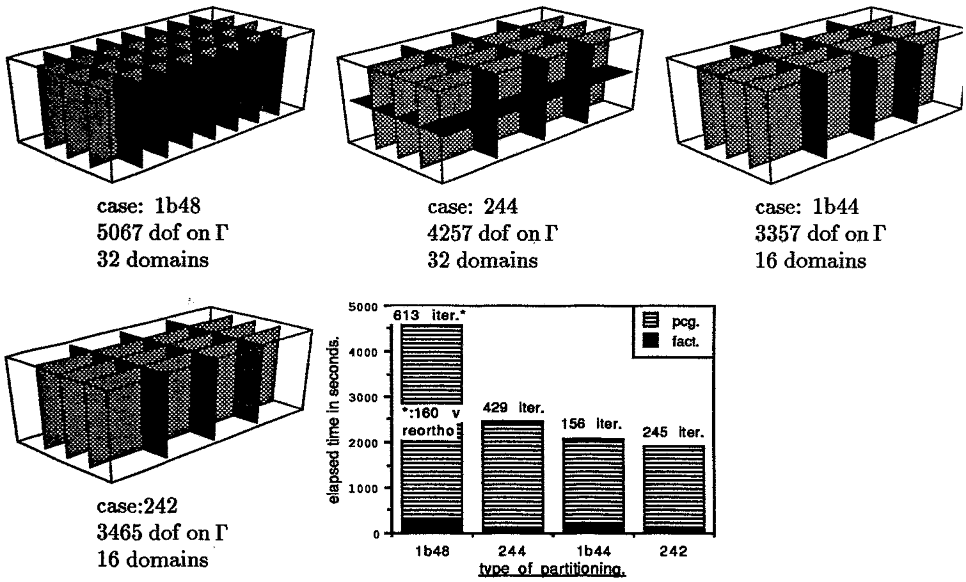


Figure 7: *Different splitting strategies.*

In our implementation on the hypercube, a domain can be allocated to two computing nodes. Each node computes and stores the factorization, and then performs the forward and backward substitutions for the Dirichlet or the Neumann subproblem respectively. During the initialisation process, the potential for parallelism is thus doubled. During the PCG iterations, the Dirichlet and the Neumann solvers cannot run simultaneously, but we still benefit from the data for one domain being divided between two nodes. This is, however, a fruitful approach because it overcomes the limits due to the small size of the local memories (4 Megabytes).
The global dot-products and the reorthogonalisation procedure are performed on all nodes in parallel, with some redundancy in order to minimize the number of communications.
The test is performed with 16137 degrees of freedom, on the 32 nodes of the iPSC/2. Thus, we split the domain into 32 subdomains with a two-dimensional slicing in the case **1b48**, and a three-dimensional slicing in the case **244**. Then, using 2 nodes per subdomain, we partition into 16 subdomains, with a two-dimensional slicing in the case **1b44**, and a three-dimensional slicing in the case **242**.
In terms of iteration count, we can again remark that: the simpler the interface, the faster the PCG convergence. Indeed, a two-dimensional splitting must be preferred to a three-dimensional one, and having half the number of subdomains lead to twice as fast convergence. However, one must pay attention to the local solvers. For instance, in case **1b48**, the storage requirement becomes so large that only 160 interface vectors can be saved for reorthogonalisation. As we have shown in [8], a complete reorthogonalisation is needed, therefore, the 613 iterations could decrease to a lower number if there were enough space for more interface vectors.

The influence of the local solvers is also highlighted by the best performance obtained in case **242**. Namely, although the number of iterations is 1.5 times that of case 1b44, the local geometry allows a smaller fill-in for the solver, leading to faster factorization and quicker subsequent solutions. Notice again that the best performance is obtained by an implementation that divides the data over 32 nodes, but has a parallelism of only 16 for the time-consuming parts of the computations (forward and backward substitutions with the Dirichlet and the Neumann solvers).

Conclusion:
We want to point out again that this preconditioner is based upon the idea of the locality of the data. For this reason, it leads to a simple data structure, even with three-dimensional unstructured finite-elements. For instance, the initialisation procedure does not exceed the time for the iterations of the PCG. It is also well suited to computers with a distributed memory architecture, except if there are very many computing nodes. And finally, even in the case of a very stiff problem, the singular term in $(1 + \ln \frac{d}{h})$ in the condition number has no practical effect on the convergence behavior when the mesh is refined.

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