CHAPTER 22

Application of Domain Decompostion to Elliptic Problems With Discontinuous Coefficients

S. V. Nepomnyaschikh*

Abstract. This paper suggests a technique for constructing the preconditioner for iterative solution of systems of grid equations approximating elliptic boundary value problems in domains with piecewise-smooth boundaries. The tecnique suggested is based on decomposition of the original domain into subdomains in which the coefficients of equations inconsiderably vary and on construction of preconditioners for these subdomains. In addition, the paper determines the preconditioner which corresponds to grid functions defined at the subdomain boundaries. The resultant preconditioner is obtained by summing up the preconditioners constructed. The convergence rate of the iterative process which uses the preconditioner suggested is independent of both the grid step size and the equation coefficients. The number of arithmetic operations required for multiplication of the preconditioner by the vector is proportional to the number of nodes of the grid domain.

1. Introduction. This paper suggests a technique for construction of the preconditioner for solving systems of grid equations approximating boundary value problems for second-order elliptic equations with discontinuous coefficients. The technique suggested is based on decomposition of the original energy space into a vector sum of subspaces and on determination of preconditioners in these subspaces. The splitting into subspaces involves partitioning of the original domain into non-overlapping subdomains inside which

* Computational Center, Siberian Branch, USSR Academy of Sciences,Novosibirsk, USSR.

the equation coefficients inconsiderably vary. The subspace corresponding to values of grid functions at the subdomain boundaries is constructed in a special way. In this case, we mainly follow the publications [1-3]. To construct preconditioners corresponding to subspaces of grid functions which are nonzero only inside subdomains, these subdomains are agian partitioned into non-overlapping subdomains and their number can depend on the grid step size. A similar technique for overlapping domains was suggested in [4].

2. Problem formulation. Let Ω be a bounded domain on the plane with the piecewise-smooth boundary Γ of class C^2, which satisfies the Lipschitz condition. In the domain Ω let us consider the boundary value problem

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x) u = f(x), \quad x \in \Omega$$

$$u(x) = 0, \quad x \in \Gamma_0 \tag{2.1}$$

$$\frac{\partial u}{\partial N} + \sigma(x) u = 0, \quad x \in \Gamma_1$$

where $\partial u / \partial N$ is the derivative in the conormal. Let the domain Ω be a union of n non-overlapping subdomains $\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$. Assume that subdomains Ω_i also have piecewise-smooth boundaries.

Denote by $S = \bigcup_{i=1}^{n} \partial\Omega_i$ a union of boundaries of the subdomains Ω_i. Let $W_2^1(\Omega)$ be a Sobolev space with the norm $\|u\|_{W_2^1(\Omega)}$ and $W_2^1(\Omega,\Gamma_0)$ be its subspace

$$W_2^1(\Omega,\Gamma_0) = \{v \in W_2^1(\Omega) \mid v(x) = 0, x \in \Gamma_0\}.$$

Let $a(u,v)$ be a bilinear form corresponding to problem (2.1) Assume that $a(u,v)$ is a symmetric form and the following inequalities are valid:

$$\alpha_0 a(v,v) \leqslant \int_{\Omega} p(x)(|\nabla v|^2 + v^2) dx \leqslant \alpha_1 a(v,v) \quad \forall v \in W_2^1(\Omega,\Gamma_0).$$

Here, $p(x) \equiv p_i = \text{const} > 0$, $x \in \Omega_i$, α_0, α_1 are positive constants.

Let $\Omega^h = \bigcup_{i=1}^{n} \Omega_i^h$ be a triangulation of Ω [5], and also $S^h = \bigcup_{i=1}^{n} \partial\Omega_i^h$. Assume that the side lengths of triangles of the triangulation are order h and the areas of triangles are of order h^2. For simplicity, assume that $\Omega_i^h = \Omega_i$. At the points where the type of boundary condition changes there are nodes of the triangulation, Γ_0^h is the part of $\partial\Omega^h$ which approximates Γ_0, and also $\Gamma^h = \partial\Omega^h = \Gamma_1^h \cup \Gamma_0^h$.

Denote by $W_{2,h}^1(\Omega^h)$ a space of real-valued continuous functions linear on triangles of the triangulation Ω^h and by W its subspace:

$$W = \{v^h \in W^1_{2,h}(\Omega^h) \mid v^h(x) = 0, x \in \Gamma_0^h\}.$$

Using the finite element method pass on from problem (2.1) to the linear alebraic system

$$Au = f.$$

This paper is aimed at the construction of the preconditioner B such that the following inequalities should be valid:

$$\check{c}(Bu,u) \leqslant (Au,u) \leqslant \hat{c}(Bu,u) \quad \forall u \in R^N$$

where the positive constants $\check{c}$ and $\hat{c}$ are independent of h and p, and the multiplication of B^{-1} by the vector can be cost-effectively realized.

3. Construction of preconditioner. The construction of the preconditioner will be realized on the basis of the space splitting method [2,3,6]. To this end, construct the decomposition $W = W_0 + W_1$. Set

$$\begin{aligned} W_0 &= \{u^h \in W \mid u^h(x) = 0,\ x \in S^h\} \\ W_{0,i} &= \{u^h \in W_0 \mid u^h(x) = 0,\ x \notin \Omega_i^h\}, \quad i = 1,2,\dots,n. \end{aligned} \tag{3.1}$$

Assume that we can determine symmetric operators $B_{0,i}$ such that $\mathrm{Im} B_{0,i} = W_{0,i}$,

$$\check{c}_2(B_{0,i}v,v) \leqslant \|v^h\|^2_{W_2^1\Omega_i^h} \leqslant \hat{c}_2(B_{0,i}v,v) \qquad \forall v \in W_{0,i} \tag{3.2}$$

where the positive constants $\check{c}_2$ and $\hat{c}_2$ are independent of h (here and henceforth, the subspaces of vectors from R^N are identified with subspaces of their piecewise-linear prolongations). Set $B_0 = p_1 B_{0,1} + \dots + p_n B_{0,n}$. It is obvious that $B_0^+ = (p_1 B_{0,1})^+ + \dots + (p_n B_{0,n})^+$. The operator B_0 defines norms in the subspace W_0.

Now define subspace W_1 and the operator B_1. The dimension of this subspace is equal to the number of nodes of the triangulation Ω^h which lie on $S^h \backslash \Gamma_0^h$, and the subspace $W_{1/2}$ itself can be defined in the following way. First, define $W_{1/2}$ which is a space of traces of functions from W on S^h:

$$W_{1/2} = \{\varphi^h \mid \varphi^h = u^h|_{S^h}, u^h \in W\}.$$

Let T be the operator of continuation of grid functions from the set S^h onto Ω^h whose structure was defined in [7,8] and therefore we omit it in this paper. Note that the operator T performs norm-preserving continuation, and the multiplication of T and T^* by the vector is realized in $\mathcal{O}(h^{-2})$ arithmetic operations; here T^* is an operator adjoint to T for the Euclidean scalar product. Set $W_1 = TW_{1/2}$. It is obvious that $W = W_0 + W_1$. To define the operator B_1, define the norms in the space $W_{1/2}$. Associate each function $\varphi \in W_{1/2}$ with functions φ_i, $i = 1,2,...,n$, whose values are equal to the values of φ at the nodes lying on $\partial\Omega_i^h$. Assume that on $\partial\Omega_i^h$ there are m_i nodes. Define the following matrices. Let $\tilde{\Sigma}_i$ be a symmetric tri-diagonal matrix with the periodic conditions [1]

$$\tilde{\Sigma}_i = \text{Tridiag}\{-1;2+1/m_i^2;-1\}, \quad \Sigma_i = (\tilde{\Sigma}_i)^{1/2}$$

$$(\Sigma\varphi,\psi) = p_1(\Sigma_1\varphi_1,\psi_1) + ... + p_n(\Sigma_n\varphi_n,\psi_n)$$

$$(D\varphi,\psi) = p_1(\varphi_1,\psi_1) + ... + p_n(\varphi_n,\psi_n) \quad \forall\varphi,\psi \in W_{1/2}.$$

It is obvious that

$$\check{c}_3(D\varphi,\varphi) \leqslant (\Sigma\varphi,\varphi) \leqslant \hat{c}_3(D\varphi,\varphi) \quad \forall\varphi \in W_{1/2}$$

$$\check{c}_3 = \frac{1}{m_{\max}}, \quad \hat{c}_3 = (4 + 1/m_{\min}^2)^{1/2}$$

where $m_{\min}$ and $m_{\max}$ are correspondingly minimal and maximal numbers from m_i, $i = 1,2,...,n$.

The operator Σ generates norms in the space of traces $W_{1/2}$. However, the explicit inversion of the operator Σ on the vector, i.e. the solution of the system of equations

$$\Sigma\varphi = \psi \tag{3.3}$$

is a complicated problem. Instead of solving system (3.3) exactly consider for its solution, for example, with accuracy $\varepsilon = 0.5$ the iterative process with the Tchebyshev set of iteration parameters τ_k [9]:

$$\begin{aligned} &\varphi^0 = 0 \\ &\varphi^{k+1} - \varphi^k = -\tau_k D^{-1}(\Sigma\varphi^k - \psi). \end{aligned} \tag{3.4}$$

Set

$$B_{1/2}^{-1} = (I - \prod_{j=0}^{n(\varepsilon)} (I - \tau_j D^{-1}\Sigma))\Sigma^{-1}$$

$$n(\varepsilon) \geqslant \frac{\ln(2/\varepsilon)}{\ln(1/q)}, \quad q = \frac{\hat{c}_3^{1/2} - \check{c}_3^{1/2}}{\hat{c}_3^{1/2} + \check{c}_3^{1/2}}.$$

Then, $\varphi^{n(\varepsilon)} = B_{1/2}^{-1}\psi$

$$\tfrac{1}{2}(B_{1/2}\varphi,\varphi) \leqslant (\Sigma\varphi,\varphi) \leqslant \tfrac{3}{2}(B_{1/2}\varphi,\varphi) \quad \forall\varphi \in W_{1/2}.$$

Set $B_1^+ = TB_{1/2}^{-1}T^*$ and $B^{-1} = B_0^+ + B_1^+$. The following theorem is valid.

Theorem 3.1. There exist positive constants $\check{c}_4$ and $\hat{c}_4$, independent of h and p such that

$$\check{c}_4(Bv,v) \leqslant (Av,v) \leqslant \hat{c}_4(Bv,v) \quad \forall v \in W.$$

Remark 3.1. If the discrete fast Fourier transform algorithm is used for multiplication by the matrix Σ in (3.4), the multiplication by $B_{1/2}^{-1}$ can be performed in $\mathcal{O}(h^{-3/2}\ln h^{-1})$ arithmetic operations and the multiplication by B_1^{-1} can be performed in $\mathcal{O}(h^{-2})$ arithmetic operations.

4. Construction of operator $B_{0,i}$. For operators $B_{0,i}$ satisfying condition (3.2) we could choose, for example, the grid counterpart of the operator $-\Delta$, where Δ is the Laplace operator in the subdomain Ω_i, and continue this operator with zero outside Ω_i^h. The inversion of the grid counterpart of the Laplace operator for complex subdomains Ω_i^h is however a labour-consuming problem. It is thus necessary to construct an easily invertible operator which could be spectrally equivalent [9] to the grid counterpart of the Laplace operator in Ω_i^h. In this section we suggest the construction of such operator by decomposing the subdomain Ω_i into non-overlapping subdomains. The particular feature is the fact that the number of subdomains into which Ω_i is partitioned can depend on the grid step size. Henceforth, the subscript i will be omitted.

Let Ω be a union of m non-overlapping subdomains $\bar{\Omega} = \bigcup_{i=1}^m \bar{D}_i$, $D_i \cap D_j = \varnothing$, $i \neq j$. We make the following assumption on the subdomains D_i. Let ε be a positive parameter. Then the subdomains with the change of variables $x_1 = \varepsilon s_1$ and $x_2 = \varepsilon s_2$ are transformed into subdomains $\tilde{D}_i$ with piecewise-smooth boundaries of class C^2 satisfying the Lipschitz condition, and the characteristics of $\tilde{D}_i$ are independent of ε. This assumption in particular implies that $\operatorname{diam} D_i = \mathcal{O}(\varepsilon)$, $\operatorname{meas}(\partial D_i) = \mathcal{O}(\varepsilon)$. Denote by $S_\varepsilon = \bigcup_{i=1}^m \partial D_i$ a union of boundaries D_i. Let $\Omega^h = \bigcup_{i=1}^m D_i^h$ be a triangulation of Ω, and also $S_e^h = \bigcup_{i=1}^m \partial D_i^h$. Assume that S_ε^h approximates S_ε with the second order in h.

Denote by W a space of real-valued continuous functions linear on triangles of the triangulation and vanishing on $\partial\Omega^h$. As in the previous section, split the space W into a vector sum of subspaces $W = W_0^\varepsilon + W_1^\varepsilon$. Set

$$W_0^\varepsilon = \{u^h \in W \mid u^h(x) = 0,\ x \in S_\varepsilon^h\}$$

$$W_{0,i}^\varepsilon = \{u^h \in W_0^\varepsilon \mid u^h(x) = 0,\ x \notin D_i^h\}, \qquad i = 1,2,\ldots,m.$$

Denote by $W_{1/2}^\varepsilon$ a space of traces of the functions from W on S_ε^h:

$$W_{1/2}^\varepsilon = \{\varphi^h \mid \varphi^h = u^h|_{S_\varepsilon^h},\ u^h \in W\}.$$

Now define the operator T_ε which is the operator of continuation of grid functions from the set S_ε^h onto Ω^h. Let a function φ^h be given on S_ε^h. Set

$$\varphi_{i,0}^h(x) \equiv \frac{1}{\text{meas}\,(\partial D_i^h)} \int_{\partial D_i^h} \varphi^h(x)\,\mathrm{d}x$$

$$\varphi_{i,1}^h(x) = \varphi^h(x) = \varphi_{i,0}^h(x), \qquad x \in \partial D_i^h .$$

The continuation of the function $\varphi_{i,0}^h$ inside of D_i^h is constructed in the trivial way $u_{i,0}^h(x) \equiv \varphi_{i,0}^h$, $x \in D_i^h$, and the construction of $u_{i,1}^h$ which is the continuation of $\varphi_{i,1}^h$ is carried out as in the previous section following [7,8]. Define $T_\varepsilon^h \varphi^h = u_{i,0}^h(x) + u_{i,1}^h(x)$, $x \in D_i^h$. It is obvious that $T_\varepsilon \varphi^h |_{S^h} = \varphi^h$. Set $W_1^\varepsilon = T_\varepsilon W_{1/2}^\varepsilon$.

To construct norms in the subspaces $W_{0,i}^\varepsilon$, we make use of the technique contained in [8]. As a result, define the symmetric operators $B_{0,i}^\varepsilon$ such that $\operatorname{Im} B_{0,i}^\varepsilon = W_{0,i}^\varepsilon$,

$$\check{c}_5(B_{0,i}^\varepsilon u,u) \leqslant \|u^h\|^2_{W_2^1(D_i^h)} \leqslant \hat{c}_5(B_{0,i}^\varepsilon u,u) \qquad \forall u \in W_{0,i}^\varepsilon$$

where the positive constants $\check{c}_5$ and $\hat{c}_5$ are independent of h, and the main operation in multiplying $(B_{0,i}^\varepsilon)^+$ by the vector is an inversion of two Laplace operators (for simply connected subdomains D_i^h) in the rectangular domain on the uniform grid or an inversion of any operators spectrally equivalent to them.

Define $B_0^\varepsilon = B_{0,1}^\varepsilon + \dots + B_{0,m}^\varepsilon$. Then $(B_0^\varepsilon)^+ = (B_{0,1}^\varepsilon)^+ + \dots + (B_{0,m}^\varepsilon)^+$. To construct norms in the space W_1^ε, consider $W_{1/2}^\varepsilon$. According to [1-3] partition the set $\Gamma_\varepsilon^h = S_\varepsilon^h \backslash \partial\Omega^h$ into parts in the following way. Let p_i, $i = 1,2,\dots,m_1$, be branching points of Γ_ε^h (the branching points are points belonging simultaneously to boundaries of three subdomains D_i^h, $i = 1,2,\dots,m$). Set $K_i = \Gamma_\varepsilon^h \cap B(p_i,r_i)$, $i = 1,2,\dots,m_1$, where $B(p_i,r_i)$ is a ball of radius r_i with the centre at p_i. In this case, choose radii r_i of order ε in such a way that the spacing between K_i and K_j, $i \neq j$, should equal at least $\mathcal{O}(\varepsilon)$. Between each two branching points put curvilinear segments L_i, $i = 1,2,\dots,m_2$, of length of order ε and such that the spacing between L_i and K_j equals at least $\mathcal{O}(\varepsilon)$. And finally, let K_j, $j = m_1 + 1,\dots,m_3$, be curvilinear segments such that any point of the set Γ_ε^h together with its neighbourhood of order ε belongs to a set K_i, $i = 1,2,\dots,m_3$. Split the space $W_{1/2}^\varepsilon$ into a vector sum of subspaces

$$W_{1/2}^\varepsilon = V_0^\varepsilon + V_1^\varepsilon + \dots + V_{m_3}^\varepsilon$$

$$V_i^\varepsilon = \{\varphi^h \in W_{1/2}^\varepsilon \,|\, \varphi^h(x) = 0, x \notin K_i\}, \qquad i = 1,2,\dots,m_3$$

and the subspace V_0 consists of functions which take constant values on K_i, $i = 1,2,\dots,m_1$, and L_j, $j = 1,2,\dots,m_2$, and are linearly continued onto the remaining part of Γ_ε^h.

Associate each function $\varphi^h \in V_0^\varepsilon$ with the vector $\varphi \in R^{m_4}$, $m_4 = m_1 + m_2$, whose components are equal to the values of the function φ^h on K_i and L_j, and vice versa, associate each vector $\varphi \in R^{m_4}$ with $\varphi^h \in V_0^\varepsilon$ which takes given values on K_i and L_j. The last correspondence will be denoted by T_0^ε.

Following [1-3] in subspaces V_i^ε, $i = 1,2,\dots,m_3$, define the symmetric operators Σ_i^ε such that $\operatorname{Im}\Sigma_i^\varepsilon = V_i^\varepsilon$,

$$\check{c}_6(\Sigma_i^\varepsilon\varphi,\varphi) \leqslant \sum_{j=1}^{m}\left[\varepsilon\int_{\partial D_j^h}(\varphi^h(x))^2\,dx + \int_{\partial D_j^h}\int_{\partial D_j^h}\frac{(\varphi^h(x)-\varphi^h(y))^2}{|x-y|^2}\,dx\,dy\right] \leqslant \hat{c}_6(\Sigma_i^\varepsilon\varphi,\varphi)$$

$$\forall\varphi \in V_i^\varepsilon, \quad i = 1,2,\dots,m_3.$$

To multiply $(\Sigma_i^\varepsilon)^+$ by the vector using the discrete fast Fourier transform algorithm, it is sufficient to perform $\mathcal{O}(\frac{\varepsilon}{h}\ln\frac{\varepsilon}{h})$ arithmetic operations [if $\varepsilon = \mathcal{O}(h)$, it is sufficient to perform $\mathcal{O}(1)$ operations]. Note also that $m_3 = \mathcal{O}(1/\varepsilon^2)$. Now define the norms in the space V_0^ε. To this end, associate each vector $\varphi \in R^{m_4}$ with vectors $\varphi_i = (c_i^{(1)},\dots,c_i^{(k_i)})^{\mathrm{T}}$, $i = 1,2,\dots,m$, whose components correspond to the values of φ on ∂D_i^h. If $\partial\Omega^h \cap \partial D_i^h = \varnothing$, i.e. D_i^h is an internal subdomain, define the symmetric matrix $S_0^{(i)}$ in the following way:

$$(S_0^{(i)}\varphi_i,\varphi_i) = \sum_{j=1}^{k_i}(\varepsilon^2(c_i^{(j)})^2 + (c_i^{(j)} - c_i^{(j-1)})^2) \tag{4.1}$$

$$c_i^{(0)} = c_i^{(k_i)}.$$

If $\partial\Omega^h \cap \partial D_i^h \neq \varnothing$, it is necessary to add summands with $c_i^{(0)} = c_i^{(k_i+1)} = 0$ to (4.1). Set

$$(S_0^{(\varepsilon)}\varphi,\psi) = (S_0^{(1)}\varphi_1,\psi_1) + \dots + (S_0^{(m)}\varphi_m,\psi_m) \quad \forall\varphi,\psi \in R^{m_4}$$

$$(\Sigma_0^\varepsilon)^+ = T_0^\varepsilon(S_0^\varepsilon)^{-1}(T_0^\varepsilon)^*$$

$$\Sigma^{-1} = (\Sigma_0^\varepsilon)^+ + (\Sigma_1^\varepsilon)^+ + \dots + (\Sigma_{m_3}^\varepsilon)^+$$

$$(B_1^\varepsilon)^+ = T_\varepsilon\Sigma^{-1}T_\varepsilon^*$$

and finally,

$$B_0^{-1} = (B_0^\varepsilon)^+ + (B_1^\varepsilon)^+.$$

The following theorem is valid.

Theorem 4.1. There exist positive constants $\check{c}_7$ and $\hat{c}_7$ independent of h and ε such that we have

$$\check{c}_7\|u^h\|^2_{W_2^1(\Omega^h)} \leqslant (B_0u,u) \leqslant \hat{c}_7\|u^h\|^2_{W_2^1(\Omega^h)} \quad \forall u \in W.$$

Proof. The proof is based on the theory of the space splitting method [1-3] and makes use of the following lemma and its corollary.

Lemma 4.1. There exist positive constants $\check{c}_8$ and $\hat{c}_8$ independent of ε such that we have

$$\check{c}_8 \left[\varepsilon \int_{\Gamma_i} \varphi^2(x)\,\mathrm{d}x + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right] \leqslant \|u\|^2_{W_2^1(D_i)}$$

$$\hat{c}_8 \left[\left[\int_{\Gamma_i} \varphi(x)\,\mathrm{d}x \right]^2 + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right] \leqslant \|u\|^2_{W_2^1(D_i)}$$

for any function $u \in W_2^1(D_i)$, where φ is the trace of u on ∂D_i. And vice versa, for any φ there exists $u \in W_2^1(D_i)$:

$$u(x) = \varphi(x), \qquad x \in \partial D_i$$

$$\|u\|^2_{W_2^1(D_i)} \leqslant \check{c}_8 \left[\varepsilon \int_{\Gamma_i} \varphi^2(x)\,\mathrm{d}x + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right]$$

$$\|u\|^2_{W_2^1(D_i)} \leqslant \hat{c}_8 \left[\left[\int_{\Gamma_i} \varphi(x)\,\mathrm{d}x \right]^2 + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right]$$

$$i = 1,2,\dots,m.$$

Here, $\Gamma_1 \subset \partial D_i$, $\operatorname{meas}\Gamma_i = \mathscr{O}(\varepsilon)$.

Corollary 4.1. Let $W_0^{1/2}(\partial D_i)$ be a subspace of traces of functions from $W_2^1(D_i)$ on ∂D_i such that we have

$$\int_{\Gamma_i} \varphi(x)\,\mathrm{d}x = 0 \quad \forall \varphi \in W_0^{1/2}(\partial D_i).$$

Then there exist positive constants $\check{c}_9$ and $\hat{c}_9$ independent of ε such that we have

$$\check{c}_9 \left[\frac{1}{\varepsilon} \int_{\Gamma_i} \varphi^2(x)\,\mathrm{d}x + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right] \leqslant \|u\|^2_{W_2^1(D_i)}$$

where $\varphi \in W_0^{1/2}(\partial D_i)$ is the trace of $u \in W_2^1(D_i)$ on Γ_i'. And vice versa, for any $\varphi \in W_0^{1/2}(\partial D_i)$ there exists $u \in W_2^1(D_i)$:

$$u(x) = \varphi(x), \qquad x \in \partial D_i$$

$$\|u\|^2_{W_2^1(D_i)} \leqslant \hat{c}_9 \left[\frac{1}{\varepsilon} \int_{\Gamma_i'} \varphi^2(x)\,\mathrm{d}x + \int_{\partial D_i}\int_{\partial D_i} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \right]$$

$$i = 1,2,\dots,m.$$

Here, $\Gamma_1' \subset \partial D_i$, $\operatorname{meas}\Gamma_i = \mathscr{O}(\varepsilon)$.

Remark 4.1. Under the assumptions made on the triangulation D_i^h there is a finite element counterpart of this lemma and its corollary.

An important fact in analysing the splitting of $W_{1/2}$ into a vector sum of subspaces is the use of the Steklov-averaging operation for projection from a fine grid to a coarser one (in this case for determining values on K_i and L_j). Such construction as applied to domain decomposition methods seems to have been first used in [10] for the case of the space $W_2^1(\Omega)$. In this paper, it is used and analysed in the space of traces.

Note also that the sets K_i in defining Σ_0 have been used only for the sake of simplicity. It is sufficient that at the boundary of each subdomain D_i^h only one set L_i lie and a constructive technique be indicated for continuation of constants with L_i onto the entire norm-preserving set S_ε^h.

Let us now briefly consider how to realize the multiplication of B_0^{-1} by the vector. Let $\varepsilon = h^\alpha$, $0 \leqslant \alpha \leqslant 1$. The multiplication of $(\Sigma_i^\varepsilon)^+$, $i = 1,\dots,m_3$, by the vector has already been discussed. The structure and properties of the matrix Σ_0 are the same as those of the original system of grid equations. Therefore, for Σ_0 as well as for $B_{0,1},\dots,B_{0,m}$ we can use the operators suggested in [8] replacing the grid counterpart of the Laplace operator in the rectangle with the rectangular uniform grid with the operator suggested in [11]. If $\alpha = 1/2$, we obtain a series of identical problems of the same dimension (in order). Then to multiply B_0^{-1} by the vector, it is sufficient to perform $\mathcal{O}(h^{-2})$ arithmetic operations. We can suggest also a somewhat different approach, i.e. the use of the technique of [11] for constructing operators spectrally equivalent to the operators $B_{0,1},\dots,B_{0,m}$, Σ_0 on the basis of splitting of appropriate grid spaces as it has been done for Ω^h and use inner Tchebysev iterative processes. In this case, unlike [11] the total cost of multiplication by B_0^{-1} is $\mathcal{O}(h^{-2})$ arithmetic operations for any finite number of inner iterations. The last fact is important for solving problems of the elasticity theory [12].

This technique is also applicable in the three-dimensional case.

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