Domain Decomposition Method: Some Results of Theory and Applications

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Abstract. The paper suggests the application of special basis functions to approximating an abstract variational problem. To solve the problem, domain decomposition algorithms are formulated. It is proved that with special basis functions used, the convergence rate of these algorithms does not depend on the grid step size and a number of other parameters of the problem. The application of the algorithms to specific problems of mathematical physics is considered.

1. Basic assumptions and problem formulation. Let D be a bounded domain in R^n with the boundary ∂D. Denote by $\mathbb{W}_2^1(D) \equiv (W_2^1(D))^N$ a Hilbert space of vector functions $u = (u_1,...,u_N)$ with components $u_i(x) \in W_2^1(D)$, $x = (x_1,...,x_n) \in D$ (below we consider only real functions, vectors and numbers). The scalar product and norm in $\mathbb{W}_2^1(D)$ are of the form

$$(u,v)_{\mathbb{W}_2^1(D)} = \sum_{i=1}^{N} (u_i,v_i)_{W_2^1(D)}, \qquad \|u\|_{\mathbb{W}_2^1(D)} = (u,u)^{1/2}_{\mathbb{W}_2^1(D)}.$$

Let $\tilde{\mathbb{W}}_2^1(D)$ be either $\mathring{\mathbb{W}}_2^1(D)_1 \equiv (\mathring{W}_2^1(D))^N$ or $\tilde{\mathbb{W}}_2^1(D) \equiv \mathbb{W}_2^1(D)$. Assume that on $\mathbb{W}_2^1(D) \times \mathbb{W}_2^1(D)$ we have defined the bilinear form $a(u,v)$ satisfying the relations

$$c_1\|u\|^2_{\mathbb{W}_2^1(D)} \leq a(u,u) \quad \forall u \in \tilde{\mathbb{W}}_2^1(D)$$

$$|a(u,v)| \leq c_2\|u\|_{\mathbb{W}_2^1(D)} \cdot \|v\|_{\mathbb{W}_2^1(D)} \quad \forall u,v \in \tilde{\mathbb{W}}_2^1(D) \tag{1.1}$$

$$c_1,c_2 = \text{const} > 0.$$

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Formulate the following problem: find a function $u \in \tilde{\mathbb{W}}_2^1(D)$ satisfying the equality

$$a(u,v) = f(v) \quad \forall v \in \tilde{\mathbb{W}}_2^1(D) \tag{1.2}$$

where $f(v)$ is the linear functional over $\tilde{\mathbb{W}}_2^1(D)$, and also $\|f\|_{\tilde{\mathbb{W}}_2^{-1}(D)} < \infty$. It is not difficult to show that under the constraints made this problem has the unique solution $u \in \tilde{\mathbb{W}}_2^1(D)$. We will formulate domain decomposition algorithms as applied to problem (1.2) (or to its particular cases).

Let us introduce partitioning of D into subdomains and formulate additional constraints on the form $a(u,v)$.

Let D be partitioned by a set $\gamma \subset R^{n-1}$ into subdomains D_k, $k = 1,\dots,K < \infty$. Assume that the boundary ∂D_k of the subdomain D_k, $k = 1,\dots,K$, is a Lipschitz one and $\bar{D} = D \cup \partial D = (\cup_{k=1}^{K} \bar{D}_k) \cup \gamma$. Assume that $a(u,v)$ satisfies the following relations:

(1) for $\forall u \in \mathring{\mathbb{W}}_2^1(D_k) \equiv (\mathring{W}_2^1(D_k))^N$, $\forall v \in \mathring{\mathbb{W}}_2^1(D_{k'})$, $k \neq k'$

$$a(u,v) = 0 \tag{1.3}$$

(here and henceforth, we assume the functions from $\mathring{W}_2^1(D_k)$, $k = 1,\dots,K$, to be continued with zeroes on $\bar{D}\backslash D_k$);

(2) for $\forall u,v \in \tilde{\mathbb{W}}_2^1(D_k)$

$$\begin{gathered} p_0 \sum_{k=1}^{K} [u]_k^2 \leq a(u,u), \qquad p_0 = \text{const} > 0 \\ |a(u,v)| \leq p_1 \left[\sum_{k=1}^{K} [u]_k^2\right]^{1/2} \left[\sum_{k=1}^{K} [v]_k^2\right]^{1/2}, \qquad p_1 = \text{const} < \infty \end{gathered} \tag{1.4}$$

where

$$\mathbb{L}_2(D_k) \equiv (L_2(D_k))^N, \qquad \mathbb{L}_2(\partial D_k) \equiv (L_2(\partial D_k))^N$$

$$[w]_k^2 = \varepsilon q_k \|\nabla w\|^2_{\mathbb{L}_2(D_k)} + r_k \|w\|^2_{\mathbb{L}_2(D_k)} + m_k \|w\|^2_{\mathbb{L}_2(\partial D_k)}$$

$$\varepsilon, q_k, r_k, m_k = \text{const}, \qquad 0 < \varepsilon \leq 1, \qquad q_k > 0 \tag{1.5}$$

$$m_k, r_k \geq 0, \qquad m_k + r_k > 0, \qquad k = 1,\dots,K$$

$$\|\nabla w\|^2_{\mathbb{L}_2(D_k)} = \sum_{i=1}^{N} \|\nabla w_i\|^2_{L_2(D_k)} = \sum_{i=1}^{N} \sum_{j=1}^{n} \left\| \frac{\partial w_i}{\partial x_j} \right\|^2_{L_2(D_k)}.$$

In what follows we assume constraints (1.3) and (1.4) to be satisfied.

2. Approximation of the problem. Cases of basis functions. For each subdomain D_k $(k = 1,\dots,K)$ let us introduce a system of linearly independent functions $\{w_i^{(k)}\}_{i=1}^{N_k}$, $w_i^{(k)} \in \overset{\circ}{W}{}_2^1(D_k)$, $i = 1,\dots,N_k$ (each function $w_i^{(k)}$ is assumed to be continued with zeroes on $\bar{D}\backslash D_k$). This system is assumed to be dense in $\overset{\circ}{W}{}_2^1(D_k)$:

$$\inf_{w_N} \|w - w_N\|_{W_2^1(D_k)} \equiv \varepsilon_k(N_k, w) \to 0, \qquad N_k \to \infty$$

where

$$w \in \overset{\circ}{W}{}_2^1(D_k), \qquad w_N = \sum_{i=1}^{N_k} b_i w_i^{(k)}, \qquad b_i = \text{const}.$$

Introduce also a system of linearly independent functions $\{W_i^{(\gamma)}\}_{i=1}^{N_\gamma}$, $W_i^{(\gamma)} \in \tilde{W}_2^1(D)$, $i = 1,\dots,N_\gamma$, which we will call the system corresponding to $\gamma \cup \partial D$ if $\tilde{W}_2^1(D) = W_2^1(D)$ and to γ if $\tilde{W}_2^1(D) = \overset{\circ}{W}{}_2^1(D)$. Assume that the system

$$\{w_i^{(1)}\}_{i=1}^{N_1}, \dots, \{w_i^{(K)}\}_{i=1}^{N_K}, \{W_i^{(\gamma)}\}_{i=1}^{N_\gamma} \tag{2.1}$$

also consists of linearly independent functions and the following density condition is satisfied in $\tilde{\mathbb{W}}_2^1(D)$:

$$\inf_{v_N} \|v - v_N\|_{\mathbb{W}_2^1(D)} \equiv \varepsilon(\tilde{N}, v) \to 0, \qquad \tilde{N} \to \infty \tag{2.2}$$

where

$$\begin{gathered} v_n = \sum_{k=1}^{K} \sum_{i=1}^{N_k} b_i^{(k)} w_i^{(k)} + \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)} \\ b_i^{(k)} = (b_{i,1}^{(k)},\dots,b_{i,N}^{(k)}), \qquad b_i = (b_{i,1},\dots,b_{i,N}) \\ \tilde{N} = \min(N_1,\dots,N_K,N_\gamma). \end{gathered} \tag{2.3}$$

An approximate solution to problem (1.2) will be sought in the form

$$\begin{gathered} u_n = \sum_{k=1}^{K} \sum_{i=1}^{N_k} a_i^{(k)} w_i^{(k)} + \sum_{i=1}^{N_\gamma} a_i W_i^{(\gamma)} \\ a_i^{(k)} = (a_{i,1}^{(k)},\dots,a_{i,N}^{(k)}), \qquad a_i = (a_{i,1},\dots,a_{i,N}). \end{gathered} \tag{2.4}$$

The unknowns $\{a_i^{(k)}\}$ and $\{a_i\}$ will be found from the equation

$$a(u_N, v_N) = f(v_N) \tag{2.5}$$

where v_N is an arbitrary function of form (2.3). This equations is equivalents to the system

$$\begin{bmatrix} A_1 & & \mathbf{0} & & U_1 \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ \mathbf{0} & & & A_K & U_K \\ L_1 & \cdot & \cdot & L_K & A_\gamma \end{bmatrix} \begin{bmatrix} a^{(1)} \\ \cdot \\ \cdot \\ a^{(k)} \\ a \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \cdot \\ \cdot \\ f^{(k)} \\ f^{(\gamma)} \end{bmatrix} . \tag{2.6}$$

By virtue of constraints imposed on $a(u,v)$ and functions (2.1), system (2.6) has a unique solution which uniquely defines the approximate solution u_N, and additionally

$$\|u - u_N\|_{W_2^1(D)} \leq c\varepsilon(\tilde{N},u) \to 0, \quad \tilde{N} \to \infty$$

where $c = \text{const} > 0$.

Let us outline some cases of choosing functions (2.1) which satisfy the conditions of linear independence and density introduced above. (The arguments concerning the verification of these conditions in the cases to be considered below are simple and therefore they are not given).

Case 1. Prescribe on $\gamma \cup \partial D$ a system of linearly independent functions $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ such that $w_i^{(\gamma)} \in W_2^{1/2}(\partial D_k)$, $k = 1,\dots,K$, and also $w_i^{(\gamma)}|_{\partial D} = 0$ for $\tilde{W}_2^1(D) = \mathring{W}_2^1(D)$, $i = 1,\dots,N_\gamma$. It is assumed that for all $w \in \tilde{W}_2^1(D)$ we have

$$\inf_{w_N} \|w - w_N\|_{W_2^{1/2}(\partial D_k)} \to 0, \quad N_\gamma \to \infty$$

where

$$w_N = \sum_{i=1}^{N_\gamma} b_i w_i^{(\gamma)} .$$

Take for functions $\{W_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ the generalized solutions to the problems

$$-\varepsilon q_k \Delta W_i^{(\gamma)} + r_k W_i^{(\gamma)} = 0 \text{ in } D_k$$

$$W_i^{(\gamma)} = w_i^{(\gamma)} \text{ on } \partial D_k \cap \operatorname{Supp} w_i^{(\gamma)} \tag{2.7}$$

$$W_i^{(\gamma)} = 0 \text{ on } \partial D_k \backslash (\partial D_k \cap \operatorname{Supp} w_i^{(\gamma)})$$

$$k = 1,\dots,K.$$

(The ways of constructive derivation of $W_i^{(\gamma)}$ for some subdomains $\{D_k\}$ will be considered below).

Case 2. Let the form $a(u,v)$ satisfy (1.3) and (1.4) for $\varepsilon = 1$ and $r_k = 0$, $k = 1,\ldots,K$. Assume that for $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ we choose the same functions as in Case 1. Then for $W_i^{(\gamma)}$ we can take the solutions to problems of the form

$$\Delta W_i^{(\gamma)} = 0 \text{ in } D_k$$

$$W_i^{(\gamma)} = w_i^{(\gamma)} \text{ on } \partial D_k \cap \operatorname{Supp} w_i^{(\gamma)} \tag{2.8}$$

$$W_i^{(\gamma)} = 0 \text{ on } \partial D_k \backslash (\partial D_k \cap \operatorname{Supp} w_i^{(\gamma)})$$

$$k = 1,\ldots,K.$$

Case 3. For simplicity, consider this case and the others for $R^n = R^2$.

Let on $\bar{D}_k$, $k = 1,\ldots,K$, a certain system of nodal points $x_i^{(k)} = (x_{1,i}^{(k)}, x_{2,i}^{(k)})$ be introduced and the partitioning of $\bar{D}_k$ into finite elements $\{D_{k,j}\}$ be carried out, and also $\max_{k,j} \operatorname{diam}(D_{k,j}) \leqslant h = \text{const}$ (h is the parameter of the introduced grid). Assume that if D_k and $D_{k'}$ are tangible along a certain line, then the grid nodes on this line are common for $\bar{D}_k$ and $\bar{D}_{k'}$ (Fig. 1).

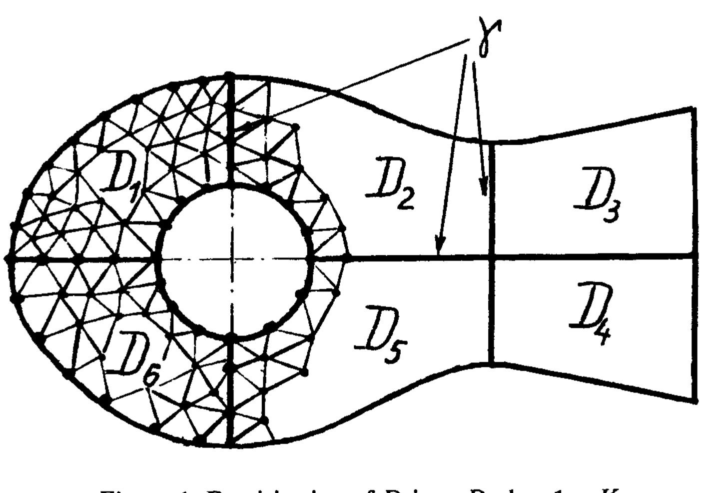

Figure 1. Partitioning of D into D_k, $k = 1,\ldots,K$.

Associate the nodes $\{x_i^{(k)}\}$ with one of the possible bases made up of finite piecewise-polynomial functions $\{w_i^{(k)}\}_{i=1}^{N_k}$, $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$, $k = 1,\ldots,K$, where $\{w_i^{(k)}\}_{i=1}^{N_k}$ correspond to nodes from D_k, and the functions $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ correspond to all nodes on $\partial D \cup \gamma$ if $\tilde{W}_2^1(D) = W_2^1(D)$ and to the nodes on γ if $\tilde{W}_2^1(D) = \mathring{W}_2^1(D)$. For example, if $\{D_{k,j}\}$ are triangles, these functions are piecewise-linear and if $\{D_{k,j}\}$ are rectangles, these functions are bilinear. Assume that $\forall w_i^{(k)} \in \mathring{W}_2^1(D_k)$ and also $\forall w_i^{(\gamma)} \in \tilde{W}_2^1(D)$. It is assumed that the functions $\{w_i^{(k)}\}_{i=1}^{N_k}$, $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$,

$k = 1,\dots,K$, are linearly independent and obey estimates of the form

$$\inf_{\{c_i^{(k)}\},\{c_i\}} \left\| u - \sum_{k=1}^{K} \sum_{i=1}^{N_k} c_i^{(k)} w_i^{(k)} - \sum_{i=1}^{N_\gamma} c_i w_i^{(\gamma)} \right\|_{W_2^1(D)} \leqslant ch \|u\|_{W_2^2(D)}$$

$$\forall u \in (W_2^2(D) \cap \tilde{W}_2^1(D)) .$$

If we set

$$W_i^{(\gamma)} \equiv w_i^{(\gamma)} , \qquad i = 1,\dots,N_\gamma \tag{2.9}$$

we obtain system (2.1) which in this case coincides with the 'ordinary' system of basis functions.

Case 4. Let $\tilde{W}_2^1(D) = \overset{\circ}{W}{}_2^1(D)$ and $D \subset R^2$ be composed of rectangles (Fig. 2). Take for $\{w_i^{(k)}\}_{i=1}^{N_k}$, $k = 1,\dots,K$, $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ the bilinear (or piecewise-linear) functions mentioned in Case 3. We choose functions $\{W_i^{(\gamma)}\}$ as solutions to problems (2.7). To construct them, we can make use of fundamental functions of Poincaré-Steklov operators of problems of the form

$$\begin{gathered} -\varepsilon q \Delta \Phi + r\Phi = 0 \ \text{ in } D_0 \\ \Phi\big|_\Gamma = 0, \qquad \lambda\varepsilon q \frac{\partial \Phi}{\partial n}\Big|_\gamma = \Phi\big|_\gamma \end{gathered} \tag{2.10}$$

where $\varepsilon, q, r = \text{const} > 0$, n is the unit vector of the external normal, and D_0, Γ and γ are shown in Fig. 3.

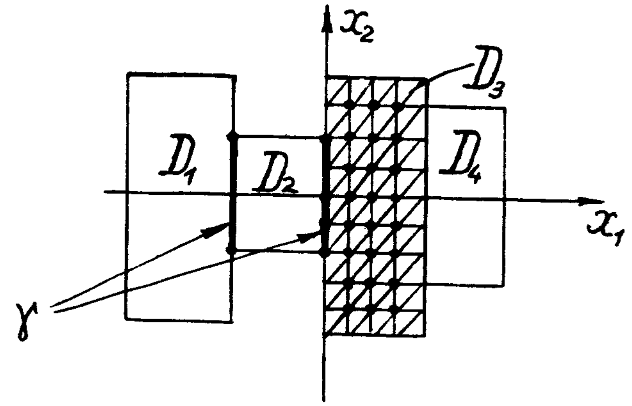

Figure 2. Domain composed of rectangles.

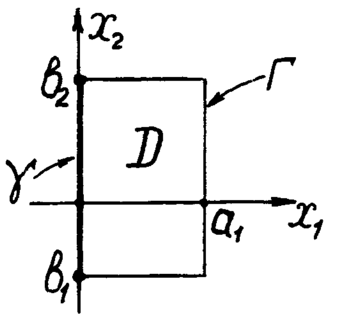

Figure 3. Domain D_0 in which fundamental functions are computed.

These fundamental functions $\{\Phi_m\}$ and the eigenvalues $\{\lambda_m\}$ corresponding to them are of the form

$$\Phi_m(x_1,x_2) = \mathrm{sh}(h_m(a_1 - x_1))\sin\frac{m\pi(x_2 - b_2)}{b_1 - b_2}$$

$$h_m = \left[\frac{m^2\pi^2}{(b_1 - b_2)^2} + \frac{r}{\varepsilon q}\right]^{-1/2} \tag{2.11}$$

$$\lambda_m = \frac{\mathrm{th}(h_m a_1)}{\varepsilon q h_m}, \qquad m = 1,2,\ldots.$$

If on γ we now prescribe a function $w_i^{(\gamma)}$ with the support on γ, the solution to

$$-\varepsilon q \Delta W_i^{(\gamma)} + rW_i^{(\gamma)} = 0 \text{ in } D_0$$

$$W_i^{(\gamma)} = w_i^{(\gamma)} \text{ on } \partial D_0 \cap \mathrm{Supp}\, w_i^{(\gamma)} \tag{2.12}$$

$$W_i^{(\gamma)} = 0 \text{ on } \partial D_0 \backslash (\partial D_0 \cap \mathrm{Supp}\, w_i^{(\gamma)})$$

can be presented in the form of a series

$$W_i^{(\gamma)} = \sum_{m=1}^{\infty} \alpha_m^{(i)} \Phi_m(x_1,x_2) \tag{2.13}$$

where

$$\alpha_m^{(i)} = \int_{\partial D_0 \cap \mathrm{Supp} w_i^{(\gamma)}} w_i^{(\gamma)} \Phi_m \, d\Gamma / \|\Phi_m\|^2_{L_2(\partial D_0)}.$$

The functions $\{W_i^{(\gamma)}\}$ are constructed in a similar way in the case where $\{w_i^{(\gamma)}\}$ are given on $\gamma = \{(x_1,x_2): x_1 = a_1, b_1 < x_2 < b_2\}$ [here, we have to make use of the functions $\psi_m(x_1,x_2) = \mathrm{sh}(h_m x_1)\sin(m\pi(x_2 - b_2)/(b_1 - b_2))$]. The above-outlined arguments imply the process of construction of $\{W_i^{(\gamma)}\}$ for the domain D shown in Fig. 2. These functions in each subdomain D_k can be presented, for example, in the form of a series in appropriate fundamental functions (of types $\{\Phi_m\}$ and $\{\Psi_m\}$) and they satisfy (2.7).

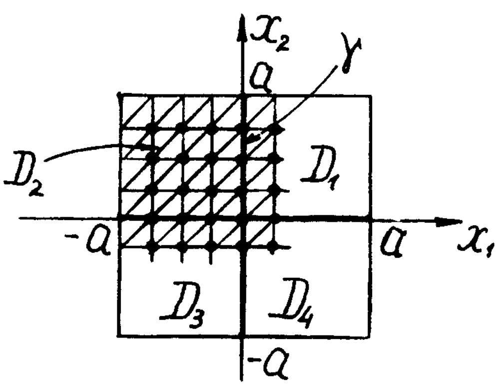

Figure 4. A case of a grid in D.

Case 5. Let $\tilde{W}_2^1(D) = \mathring{W}_2^1(D)$ and the domain $D = (\cup_{k=1}^4 D_k) \cup \gamma$ be shown in Fig. 4. Assume that in (1.3) and (1.4) we have $\varepsilon = 1$, $m_k = r_k = 0$. Introduce on D a grid and associate its internal nodes with the piecewise-linear functions $\{w_i^{(k)}\}_{i=1}^{N_k}$, $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$, $k = 1,2,3,4$. Consider the subdomain D_1 and the following problem of finding fundamental functions of Poincaré-Steklov operators:

$$\Delta\Phi = 0 \quad \text{in } D_1$$

$$\Phi = 0 \quad \text{on } \partial D_1 \backslash \gamma \tag{2.14}$$

$$\lambda\frac{\partial\Phi}{\partial n} \quad \text{on } \partial D_1 \cap \gamma.$$

These functions and the eigenvalues λ in this problem are of the form

$$\Phi_0(x_1,x_2) = (a - x_1)(a - x_2)$$

$$\Phi_m^{(1)}(x_1,x_1) = \mathrm{sh}(\mu_m(a - x_1))\sin(\mu_m(a - x_2))$$

$$\Phi_m^{(2)}(x_1,x_2) = \mathrm{sh}(\mu_m(a - x_2))\sin(\mu_m(a - x_1)) \tag{2.15}$$

$$\lambda_0 = a, \qquad \lambda_m = (\mathrm{tg}(\mu_m a))/\mu_m, \qquad m = 1,2,\ldots$$

and μ_m are solutions to the equation

$$\operatorname{tg}(\mu_m a) = \operatorname{th}(\mu_m a).$$

Using these functions we can construct $W_i^{(\gamma)}$ in D_1:

$$W_i^{(\gamma)} = \alpha_0 \Phi_0 + \sum_{m=1}^{\infty} (\alpha_{m,i}^{(1)} \Phi_m^{(1)} + \alpha_{m,i}^{(2)} \Phi_m^{(2)})$$

$$\alpha_{m,i}^{(i)} = \int_{\partial D_1 \cap \operatorname{Supp} w_i^{(\gamma)}} w_i^{(\gamma)} \Phi_m^{(j)} \, d\Gamma \,/\, \| \Phi_m^{(j)} \|^2_{L_2(\partial D_1)} . \tag{2.16}$$

By using the symmetric reflection with respect to the coordiante axes we can easily find the form of fundamental functions in other subdomains and exploiting them we can find the form $\{W_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ in D_k, $k = 2,3,4$.

Note that the algorithm outlined here for constructing $\{W_i^{(\gamma)}\}$ can be extended to the case where each of the subdomains $\{D_k\}$ is a rectangle (as the fundamental functions given here are also known).

Case 6. Let $\varepsilon = q_k = m_k = 1$, $r_k = 0$, and the domain $D \subset R^2$ be composed of rectangles (Fig. 5). Assume that $\{w_i^{(k)}\}_{i=1}^{N_k}$, $k = 1,\dots,K$, $\{w_i^{(\gamma)}\}_{i=1}^{N_\gamma}$, are piecewise-linear functions, and for $\{W_i^{(\gamma)}\}$ we take the solutions to model problems (2.8). It can be easily seen that the construction of $\{W_i^{(\gamma)}\}$ in this case can be reduced to the algorithm of construction of $\{W_i^{(\gamma)}\}$ which is outlined in Case 5.

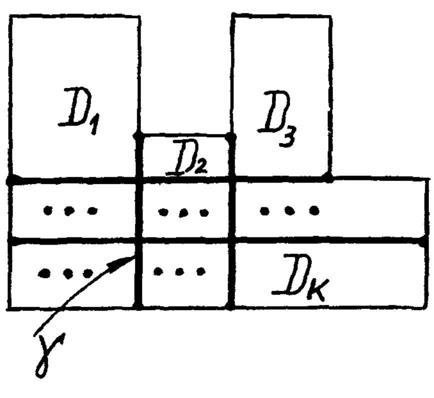

Figure 5. Domain with internal points of intersection γ.

3. Equation on γ. Properties of matrices.

3.1. Removing in (2.6) the vectors

$$a^{(k)} = -A_k^{-1}U_k a + A_k^{-1}f^{(k)}, \quad k = 1,\dots,K \tag{3.1}$$

we obtain the 'equation on γ':

$$Aa = F \tag{3.2}$$

where

$$A = A_\gamma - \sum_{k=1}^{K} L_k A_k^{-1} U_k, \qquad F = f^{(\gamma)} - \sum_{k=1}^{K} L_k A_k^{-1} f^{(k)}.$$

Let us formulate propositions on the properties of the matrices A, A_γ and A_k.

Proposition 3.1. Under an arbitrary choice of functions $\{w_i^{(k)}\}_{i=1}^{N_k}$, $k = 1,\dots,K$, $\{W_i^{(\gamma)}\}_{i=1}^{N_\gamma}$ satisfying the conditions of linear independence and density

(1) the matrices A_γ, A_k, $k = 1,\dots,K$ and A are positive definite;

(2) the following estimates are valid

$$c_1\beta_1 \sum_{k=1}^{K} \Big\| \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)} \Big\|^2_{\mathring{W}_2^{1/2}(\partial D_k)} \leqslant (Ab,b)_2$$

$$c_1\beta_1 \sum_{k=1}^{K} \Big\| \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)} \Big\|^2_{\mathring{W}_2^{1/2}(\partial D_k)} \leqslant (A_\gamma b,b)_2 \tag{3.3}$$

$$\forall b_i = (b_{i,1},\dots,b_{i,N}), \quad i = 1,\dots,N_\gamma$$

where $\beta_1 = \min\beta_{1,k}$ and $\beta_{1,k}$ is a constant in the inequality $\|u\|^2_{W_2^1(D_k)} \geqslant \beta_{1,k}\|u\|^2_{W_2^{1/2}(\partial D_k)}$ $\forall u \in W_2^1(D_k)$;

(3) if the form $a(u,v)$ is symmetric on $\tilde{\mathbb{W}}_2^1(D) \times \tilde{\mathbb{W}}_2^1(D)$, the matrices A_γ, A_k and A are symmetric, and also $(L_k)^{\mathrm{T}} = U_k$. The sufficient condition of simmetricity of the matrices A_k is the requirement of simmetricity of $a(u,v)$ on $\mathring{\mathbb{W}}_2^1(D_k) \times \mathring{\mathbb{W}}_2^1(D_k)$, $k = 1,\dots,K$.

Let $b = (b_1,\dots,b_{N_\gamma})$ [for $b_i = (b_{i,1},\dots,b_{i,N})$] be an arbitrary vector. By using this vector construct vectors of the form

$$b^{(k)} = -A_k^{-1}U_k b, \quad k = 1,\dots,K \tag{3.4}$$

and then using b, $b^{(k)}$, $k = 1,\dots,K$, define a function of the form

$$v_N = \sum_{k=1}^{K} v_N^{(k)} + v^{(\gamma)} \tag{3.5}$$

where

$$v_N^{(k)} = \sum_{i=1}^{N_k} b_i^{(k)} w_i^{(k)}, \qquad v^{(\gamma)} = \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)}. \tag{3.6}$$

In what follows, if not specified otherwise, by v_N, $v_N^{(k)}$ and $v^{(\gamma)}$ we mean the functions defined by expressions (3.4)-(3.6).

Proposition 3.2. The equalities $a(v_N,w_k) = 0$, $k = 1,...,K$, are valid for arbitrary linear combinations $w_k = \sum_{i=1}^{N_k} c_i^{(k)} w_i^{(k)}$.

Proposition 3.3. For an arbitrary vector b the following equality is valid:

$$(Ab,b)_2 = a(v_N,v_N). \tag{3.7}$$

If additionally the form $a(u,v)$ is symmetric, we have

$$\begin{aligned}(Ab,b)_2 &= (A_\gamma b,b)_2 - \sum_{k=1}^{K} (U_k b, A_k^{-1} U_k b)_2 \\ &= a(v^{(\gamma)},v^{(\gamma)}) - \sum_{k=1}^{K} a(v_N^{(k)},v_N^{(k)}).\end{aligned} \tag{3.8}$$

Corollary. If the form $a(u,v)$ is symmetric, then

$$c_1\beta_1 \sum_{k=1}^{K} \left\| \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)} \right\|^2_{\mathbb{W}_2^{1/2}(\partial D_k)} \leqslant (Ab,b)_2 \leqslant (A_\gamma b,b)_2 = a(v^{(\gamma)},v^{(\gamma)}) \quad \forall b. \tag{3.9}$$

3.2. In this subsection we assume that the form $a(u,v)$ is symmetric and no additional constraints (i.e. in addition to those introduced at the beginning of this paper) are imposed on the partitioning of D into $\{D_k\}$ if not specified otherwise.

Proposition 3.4. Let system (2.1) correspond to Case 1. Then,

$$p_0 \leqslant \frac{(Ab,b)_2}{(A_\gamma^{(1)}b,b)_2} \leqslant p_1 \qquad \forall b \neq 0 \tag{3.10}$$

where the matrix $A_\gamma^{(1)}$ is defined by the relation

$$\begin{aligned}(A_\gamma^{(1)}b,c)_2 &= \sum_{k=1}^{K} [\varphi,\psi]_k \quad \forall \varphi = \sum_{i=1}^{N_\gamma} b_i W_i^{(\gamma)},\ \forall \psi = \sum_{i=1}^{N_\gamma} c_i W_i^{(\gamma)} \\ [\varphi,\psi]_k &= \varepsilon q_k(\nabla\varphi,\nabla\psi)_{\mathbb{L}_2(D_k)} + r_k(u,v)_{\mathbb{L}_2(D_k)} + m_k(\varphi,\psi)_{\mathbb{L}_2(\partial D_k)}.\end{aligned} \tag{3.11}$$

Proposition 3.5. Assume that in (1.3) and (1.4) we have $\varepsilon = 1$, $r_k = 0$, $k = 1,\dots,K$. If system (2.1) correponds to Case 2, then

$$p_0 \leqslant \frac{(Ab,b)_2}{(A_\gamma^{(2)}b,b)_2} \leqslant p_1 \qquad \forall b \neq 0 \tag{3.12}$$

where the matrix $A_\gamma^{(2)}$ is defined by the relation

$$(A_\gamma^{(2)}b,c)_2 = \sum_{k=1}^{K} (q_k(\nabla\varphi,\nabla\psi)_{\mathbb{L}_2(D_k)} + m_k(\varphi,\psi)_{\mathbb{L}_2(\partial D_k)}) \tag{3.13}$$

for all φ and ψ from (3.11).

Proposition 3.6. Let (1) $\widetilde{\mathbb{W}}_2^1(D) = \mathring{\mathbb{W}}_2^1(D)$; (2) $\mathrm{mes}(\partial D \cap \partial D_k) \neq 0$, $k = 1,\dots,K$; (3) $\varepsilon = 1$, $r_k = m_k = 0$, $k = 1,\dots,K$. If system (2.1) corresponds to Case 2, then estimates (3.12) are valid.

Let $\beta_{2,k}$ be a constant in the inequality

$$\int_{D_k} |\nabla u|^2 \mathrm{d}x + \int_{\partial D_k} |u|^2 \mathrm{d}\Gamma \geqslant \beta_{2,k} \|u\|^2_{\mathbb{W}_2^{1/2}(\partial D_k)} \quad \forall u \in W_2^1(D_k) \tag{3.14}$$

and the constant d is determined as follows:

$$d = \min_k \min_\varphi (\|\varphi\|^2_{W_2^{1/2}(\partial D_k)} / (\|\nabla\varphi\|^2_{L_2(\delta D_k)} + \|\varphi\|^2_{L_2(\partial D_k)})) \tag{3.15}$$

where $\varphi = \sum_{i=1}^{N_\gamma} b_i w_i^{(\gamma)}$, $\delta D_k = D_k \cap (\bigcup_{i=1}^{N_\gamma} \mathrm{Supp}\, w_i^{(\gamma)})$.

Proposition 3.7. Let $\varepsilon = q_k = m_k = 1$, $r_k = 0$, $k = 1,\dots,K$, and an 'ordinary' system of functions (Case 3) be chosen for system (2.1). Then,

$$p_0 d \min_k \beta_{2,k} \leqslant \frac{(Ab,b)_2}{(A_\gamma^{(3)}b,b)_2} \leqslant p_1 \qquad \forall b \neq 0 \tag{3.16}$$

where the matrix $A_\gamma^{(3)}$ is defined by the relation

$$\begin{gathered}(A_\gamma^{(3)}b,c)_2 = \sum_{k=1}^{K} ((\nabla\varphi,\nabla\psi)_{L_2(\delta D_k)} + (\varphi,\psi)_{L_2(\partial D_k)}) \\ \forall\varphi = \sum_{i=1}^{N_\gamma} b_i w_i^{(\gamma)}, \qquad \forall\psi = \sum_{i=1}^{N_\gamma} c_i w_i^{(\gamma)}.\end{gathered} \tag{3.17}$$

Remark 3.1. Specifying the form of subdomains and assuming the grids introduced to be quasi-uniform we can estimate from below the quantity d by a constant (which may prove independent of grid step sizes). A number of estimates for d was also obtained in [6].

3.3. Assume now that the form $a(u,v)$ is not symmetric. Introduce the matrix $A_\gamma^{(4)}$ by the following relation:

$$(A_\gamma^{(4)}b,c)_2 = \sum_{k=1}^{K} \left[\sum_{i=1}^{N_\gamma} b_i w_i^{(\gamma)}, \sum_{i=1}^{N_\gamma} c_i w_i^{(\gamma)} \right]_{W_2^{1/2}(\partial D_k)} . \tag{3.18}$$

Proposition 3.8. If system (2.1) corresponds to Case 1, it is possible to indicate positive constants $æ_1$ and $æ_2$ independent of the grid parameters, which are such that

$$\begin{gathered} p_0 æ_1 (A_\gamma^{(4)}b,b)_2 \leq p_0 (A_\gamma^{(1)}b,b)_2 \leq (Ab,b)_2 \\ (Ab, A_\gamma^{(1)^{-1}}Ab)_2 \leq \frac{p_1^2}{p_0}(Ab,b)_2 \\ (Ab, A_\gamma^{(4)^{-1}}Ab)_2 \leq æ_2 \frac{p_1^2}{p_0}(Ab,b)_2 . \end{gathered} \tag{3.19}$$

Remark 3.2. The statement similar to that formulated in Proposition 3.8 is also valid for the functions $\{W_i^{(\gamma)}\}$ defined by (2.8) in the case where, for example, $r_k = 0$, $m_k > 0$, $k = 1,...,K$.

4. Iterative algorithms. Estimates of convergence rate. Let us formulate some iterative algorithms for solving equation (3.2).

Method of steepest descent:

$$\begin{gathered} w^m = B^{-1}\xi^m , \quad p^m = Aw^m \\ a^{m+1} = a^m - \tau_{m+1} w^m \\ \xi^{m+1} = \xi^m - \tau_{m+1} A \xi^m \\ \tau_{m+1} = (\xi^m, w^m)_2/(p^m, w^m)_2 , \quad m = 0,1,... \end{gathered} \tag{4.1}$$

where $\xi^m = Aa^m - F$.

Theorem 4.1. Let (1) the form $a(u,v)$ be symmetric; (2) system (2.1) correspond to Case 1; (3) $B = A_\gamma^{(1)}$. Then the following estimate is valid:

$$\|a - a^m\|_A \leq \left[\frac{p_1 - p_0}{p_1 + p_0} \right]^m \|a - a^0\|_A . \tag{4.2}$$

If we write down the stages of realization of process (4.1), we obtain the generally accepted form of the domain decomposition algorithm.

Conjugate correction method:

$$Ba^{m+1} = \alpha_{m+1}(B - \tau_{m+1}A)a^m + (1-\alpha_{m+1})Ba^{m+1} + \alpha_{m+1}\tau_{m+1}F, \qquad m = 1,2,...$$

$$\begin{aligned} Ba^1 &= (B - \tau_1 A)a^0 + \tau_1 F \\ \tau_{m+1} &= (Aw^m,w^m)_2/(B^{-1}Aw^m,Aw^m)_2 \\ \alpha_{m+1} &= \left[1 - \frac{\tau_{m+1}}{\tau_m}\frac{(Aw^m,w^m)_2}{(Aw^{m-1},w^{m-1})_2}\frac{1}{\alpha_m}\right]^{-1} \\ w^m &= B^{-1}\xi^m, \qquad \xi^m = Aa^m - F, \qquad \alpha_1 = 1. \end{aligned} \tag{4.3}$$

Using the theory of iterative processes and the estimates from Section 3 we conclude that the domain decomposition algorithm based on the conjugate correction method obeys

Theorem 4.2. Let the hypotheses of Theorem 4.1 be satisfied. Then,

$$\|a - a^m\|_A \leqslant 2\left[\frac{p_1^{1/2} - p_0^{1/2}}{p_1^{1/2} + p_0^{1/2}}\right]^m \|a - a^0\|_A. \tag{4.4}$$

Minimal correction method:

$$\begin{aligned} w^m &= B^{-1}\xi^m \\ a^{m+1} &= a^m - \tau_{m+1}w^m \\ \tau_{m+1} &= (Aw^m,w^m)_2/(B^{-1}Aw^m,Aw^m), \qquad m = 1,2,... \end{aligned} \tag{4.5}$$

where $\xi^m = Aa^m - F$, $B = B^{\mathrm{T}}$ is a positive definite matrix.

Theorem 4.3. Let conditions (1.3) and (1.4) be satisfied for the form $a(u,v)$ (which can be nonsymmetric) and system (2.1) correspond to Case 1. Then for $B = A_\gamma^{(4)}$ the following estimate is valid:

$$\|a - a^m\|_B \leqslant \left[\frac{p_1^2 æ_2 - p_0^2 æ_1}{p_1^2 æ_2 + p_0^2 æ_1}\right]^m \|a - a^0\|_B \tag{4.6}$$

where $\|a\|_B = (Ba,a)_2^{1/2}$, $æ_1$ and $æ_2$ are constants from (3.19).

It is obvious that it is not difficult to formulate a number of other iterative algorithms and estimate their convergence rate using the estimates obtained before.

5. Application for certain problems. For the purpose of illustrating the above-outlined algorithms based on using special basis functions $\{W_i^{(\gamma)}\}$ consider their application to certain specific problems.

5.1. Assume that in the domain given in Fig. 4 we solve the problem of finding $u \in \overset{\circ}{W}{}_2^1(D)$ which satisfies the equality

$$a(u,v) = \int_D p(x) \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \mathrm{d}x = (f,v)_{L_2(D)} \quad \forall v \in \overset{\circ}{W}{}_2^1(D) \tag{5.1}$$

where $p(x) = p_k = \text{const} > 0$ in D_k, $k = 1,2,3,4$.

To solve this problem, let us make use of the system corresponding to Case 5. It can be easily seen that in this case we have $U_k = 0$, $L_k = 0$ and the solution of system (2.6) reduces to the solution of systems $A_k a^{(k)} = f^{(k)}$ and the system $A_\gamma a = f^{(\gamma)}$ with the 'one-dimensional' matrix A_γ, which can be easily carried out by appropriate methods.

5.2. In the domain D shown in Fig. 2 let us consider the problem

$$\frac{\partial u}{\partial t} - \mu \left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right] + \sum_{i=1}^{2} v_i \frac{\partial u}{\partial x_i} + \lambda u = f \tag{5.2}$$

$$u\Big|_{\partial D} = 0, \qquad u\Big|_{t=0} = u_{(0)}$$

where $\mu = \text{const} > 0$; $\{v_i\}$ are bounded functions, and additionally $v_i\big|_{\partial D} = 0$, $\partial v_1/\partial x_1 + \partial v_2/\partial x_2 = 0$; $\lambda = \text{const} \geqslant 0$. Using the approximation $(\partial u/\partial t)(x,t_j) \cong (u(x,t_j) - u(x,t_{j-1}))/\tau$, $t_j = j\tau$ we obtain a sequence of problems (in the generalized formulation)

$$a(u^j,w) = f_j(w) \quad \forall w \in \overset{\circ}{W}{}_2^1(D), \qquad j = 1,2,\ldots \tag{5.3}$$

where

$$a(u,v) = \int_D \left[\mu \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \sum_{i=1}^{2} v_i \frac{\partial u}{\partial x_i} v + (\lambda + 1/\tau) uv \right] \mathrm{d}x .$$

For the form $a(u,v)$ we can take

$$\varepsilon = \mu , \qquad q_k = 1 , \qquad r_k = \lambda + 1/\tau$$

$$p_0 = 1 , \qquad p_1 = \max \left[2, 1 + \frac{\tau |v|^2}{\mu(1 + \lambda\tau)} \right]$$

where $|v|^2 = \sup\operatorname{vrai} (v_1^2 + v_2^2)$.

Assume that for solving problems (5.3) we use system (2.1) corresponding to Case 4 and the domain decomposition algorithm based on the minimal correction method for $B = A_{\gamma}^{(1)}$. Then the algorithm convergence rate is characterized with the estimate

$$\|a_j - a_j^m\|_B \leqslant \left[\frac{p_1^2 - p_0^2}{p_1^2 + p_0^2}\right]^m \|a_j - a_j^0\|_B$$

(where the coefficients a_j determine the behaviour of the solution to the problem on γ). Note that if, for example, $\tau < \mu/|\nu|^2$, the convergence rate is independent of τ, μ and $|\nu|$. But if in (5.2) we have $\nu_i \equiv 0$, the algorithm convergence rate without any constraints on τ is independent of this parameter.

Remark. The paper [7] considered also algorithms based on the use of special basis functions as applied to a number of other problems of mathematical physics.

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