## CHAPTER 8

# Domain Decomposition Method in Partial Symmetric Eigenvalue Problem

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**Abstract.** Domain decomposition method algorithms are described for solving the partial generalized symmetric eigenvalue problem with $2\times2$ block operators. A practical approach is suggested to computing initial guess and the results of numerical experiments are given for model problems.

**1. Introduction.** Modified iterative methods in subspace were presented in [1] for solving ordinary spectral problems which in a specific situation can be regarded as domain decomposition methods. Thus, for problems with the grid Laplace operator in domains composed of rectangles and parallelepipeds the iterations in these methods are performed at common boundaries of subdomains, and their convergence rate does not decrease with the grid becoming finer.

This paper describes a very simple iterative method in subspace for the generalized eigenvalue problem, suggests a practical approach to choosing initial guess and gives the results of numerical experiments for model problems.

Let us formulate the problem in the abstract form.

Let $H$ be an Euclidean space (i.e. real, finite-dimensional) with the scalar product $(\cdot,\times)$ and the norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. Consider in $H$ the eigenvalue problem

$$Mu = \lambda Lu, \quad M = M^*, \quad L = L^* > 0, \quad u \in H \tag{1.1}$$

and number the eigenvalues of problem (1.1) in the non-increasing order: $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{\min}$. The Rayleigh quotient for problem (1.1) will be denoted by $\lambda(\cdot) \equiv (M\cdot,\cdot)/(L\cdot,\cdot)$.

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Present $H$ in the form of an orthogonal sum of subspaces $H_1$ and $H_2$: $H = H_1 \oplus H_2$. Such partition of the space $H$ is associated with unique representations of the vectors $u \in H$: $u \equiv (u_1,u_2)^T = u_1 + u_2$, $u_i \equiv H_i$ and the operators $M$ and $L$

$$M = \begin{bmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_{12} \\ L_{21} & L_2 \end{bmatrix}, \quad \begin{matrix} M_i, L_i : H_i \to H_i \\ M_{ij}, L_{ij} : H_j \to H_i \,. \end{matrix}$$

The partition of the operators $M$ and $L$ is assumed to be such that the estimate from above $\bar{\lambda}$, $\bar{\lambda} > 0$, is known for the maximal eigenvalue of the problem

$$M_1 u_1 = \lambda L_1 u_1, \quad u_1 \in H_1. \tag{1.2}$$

It is also assumed that the strict inequality $\lambda_1 > \bar{\lambda}$ is valid.

**2. Computation of maximal eigenvalue.** Let $\lambda$ be a parameter, $\lambda > \bar{\lambda}$. Then the condition $\lambda > \bar{\lambda}$ is assumed to be always satisfied and that is why it is not mentioned. Then we have $M_1 - \lambda L_1 < 0$.

Denote by $S_\lambda$ the Schur supplement to the block $\lambda L_1 - M_1 > 0$ of the operator $\lambda L - M$:

$$S_\lambda = \lambda L_2 - M_2 - (\lambda L_{21} - M_{21})(\lambda L_1 - M_1)^{-1}(\lambda L_{12} - M_{12}).$$

Let us introduce into consideration an auxiliary self-adjoint operator $S_B = S_B^* > 0$ acting from $H_2$ into $H_2$ and satisfying the following two conditions.

(a) The operator $S_B$ is spectrally equivalent to the operator $S_\infty = L_2 - L_{21}L_1^{-1}L_{12}$ which is the Schur supplement to the block $L_1$ of the operator $L$, i.e. there exist positive constants $\beta_0$ and $\beta_1$ such that

$$0 < \beta_0 S_B \leqslant S_\infty \leqslant \beta_1 S_B.$$

(b) The system of equations with the operator $S_B$ can be 'cost-effectively' solved.

For some classes of grid problems such operators are known (see, for example, [2,3]). Define the functional $\hat{\lambda}(\cdot)$ of $u_2 \in H_2 \backslash 0$:

$$\hat{\lambda}(u_2) = \lambda - \frac{(S_\lambda u_2, u_2)}{(L_\lambda u_2, u_2)} \tag{2.1}$$

where

$$L_\lambda = L_2 - 2L_{21}C_{12} + C_{12}^* L_1 C_{12}$$

$$C_{12} = (\lambda L_1 - M_1)^{-1}(\lambda L_{12} - M_{12})$$

which has the sense of the Rayleigh quotient $\lambda(\cdot)$ on the vector

$$u_\lambda = (-(\lambda L_1 - M_1)^{-1}(\lambda L_{12} - M_{12})u_2, u_2)^{\mathrm{T}}.$$

Let $\delta_1(\lambda)$ be a quantity from the inequality $\|u_\lambda\|_L^2 \leqslant \delta_1(\lambda)\|u_2\|_{S_B}^2$. As shown in [1], for the grid problems for $M = I$ the quantity $\delta_1(\lambda)$ is bounded from above by a constant independent of the grid step size.

Similarly to Method 2 from Section 4 in [1], to compute $\lambda_1$, we can suggest

*Method 2.1.* (Modified one-step method in subspace).
(1) Choose $u_2^0 \neq 0$ and the parameter $\lambda^0 > \bar{\lambda}$.
(2) For $k = 0,1,\ldots$:
(a) compute the vector

$$u_2^{k+1} = (-S_B^{-1}S_{\lambda^k} + \gamma_k)u_2^k$$

where $\gamma_k = \delta_1(\lambda^k)(\lambda^k - \lambda_{\min})$;
(b) compute $\hat{\lambda}(u_2^{k+1})$ by formula (2.1) with $\lambda = \lambda^k$ and set $\lambda^{k+1} = \hat{\lambda}(u_2^{k+1})$.

The convergence of Method 2.1. for $M = I$, $\lambda_{\min} = 0$, is established by Theorem 4.3 in [1].

**3. Numerical experiments.** This section contains the results of computation of the maximal eigenvalue $\lambda_1$ of problem (1.1) for some model problems. This section therefore can be regarded as a supplement to Section 5 in [1].

Let us describe a practical approach to computing the initial guess $u^0$ in the methods from Section 4 in [1]. It is based on the following idea. Let the vector $\bar{u}_1 \in H_1$ be the eigenvector of problem (1.2) corresponding to $\bar{\lambda}$, and also $u^1 = (\bar{u}_1 0)^{\mathrm{T}}$. Choose $m-1$ vectors $u^i = (u_1^i, u_2^i)$, $i = 2,\ldots,m$, and form a subspace $W = \mathrm{span}\{u^1,\ldots,u^m\}$. Assume that $\dim W = m$ and using the Rayleigh-Ritz procedure find the maximum of $\lambda(\cdot)$ on $W$, the maximal Ritz number $\lambda(W)$, and the corresponding Ritz vector $w = (w_1, w_2)^{\mathrm{T}}: \lambda(w) = \lambda(W)$. Since $\bar{u}_1 \in W$, then $\lambda(w) \geqslant \bar{\lambda}$ and (if vectors $u^i$ are chosen well) $\lambda(w) > \bar{\lambda}$. In this case, we can set $u^0 = w$ ($u_2^0 = \omega_2^0$) and $\lambda^0 = \lambda(w)$.

Consider the problem (in the generalized sense)

$$\begin{aligned} -\lambda \Delta u &= u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3.1}$$

where the model domains $\Omega \subset \mathbb{R}^2$, $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ are given in Fig. 1, and $\Delta$ is the Laplace operator. Cover the domains $\Omega$ with an $h$-step grid uniform in both variables. To problem (3.1) put into correspondence problem (1.1), where $L$ is an ordinary five-point approximation of the operator $-\Delta$ and $M = I$ and single out the nodes lying on $\Gamma$. In this case, the matrix $L$ assumes the $2\times 2$ block structure, where $L_1$ is the $2\times 2$ block-diagonal matrix, and also

$L_2 = h^{-2}\text{tridiag}\{-1,4,-1\}$. A transition from problem (3.1) to problem (1.1) is described in more detail in Section 5 in [1]. Note only that for $S_B$ in all experiments we chose the matrix [1,3]

$$S_B = (\Lambda + \Lambda^2/4)^{1/2}, \qquad \Lambda = \text{tridiag}\{-1,2,-1\}.$$

Describe now the way how the initial guess was found in the numerical experiments. The quantity $\bar{\lambda}$ is the largest one out of the two maximal eigenvalues of subblocks of the $2\times 2$ block-diagonal matrix $L_1$. Denote by $\bar{\bar{\lambda}}$ the least of these eigenvalues. Let $\bar{u}_1$, $\|\bar{u}_1\| = 1$ and $\bar{\bar{u}}_1$, $\|\bar{\bar{u}}_1\| = 1$ be eigenvalues known in the explicit form which correspond to $\bar{\lambda}$ and $\bar{\bar{\lambda}}$. Choose $u^1 = (\bar{u}_1,0)^{\mathrm{T}} \in H_1$ and $u^2 = (\bar{\bar{u}}_1,0)^{\mathrm{T}} \in H_1$. Then, $(u^1,u^2) = (Lu^1,u^2) = 0$. For $u^3$ we choose $u^3 = (-L_1^{-1}L_{12}v_2,v_2)^{\mathrm{T}}$, where $(0,v_2)^{\mathrm{T}} \in H_2$, $v_2^j = \sqrt{2h}\sin \pi jh$, $j = 1,\dots,h^{-1}-1$, and form a subspace $W = \text{span}\{u^1,u^2,u^3\}$. The projection of problem (1.1) onto $W$ is of the form

$$\hat{M}a = \lambda \hat{L}a, \qquad a = (a_1,a_2,a_3)^{\mathrm{T}} \tag{3.2}$$

where

$$\hat{M} = \begin{bmatrix} 1 & 0 & m_{13} \\ 0 & 1 & m_{23} \\ m_{13} & m_{23} & m_3 \end{bmatrix}, \qquad \hat{L} = \begin{bmatrix} \bar{\lambda}^{-1} & 0 & 0 \\ 0 & \bar{\bar{\lambda}}^{-1} & 0 \\ 0 & 0 & (S_\infty v_2,v_2) \end{bmatrix}$$

$$m_{13} = -\bar{\lambda}(L_{21}\bar{u}_1,v_2), \qquad m_{23} = -\bar{\bar{\lambda}}(L_{21}\bar{\bar{u}}_1,v_2), \qquad m_3 = ((I_2 + L_{21}L_1^{-1}L_{12})v_2,v_2).$$

The relations given above imply that the entries of the matrices $\hat{M}$ and $\hat{L}$ are computed by using only vectors from the subspace $H_2$. The maximal eigenvalue of problem (3.2) equal to $\lambda(w)$ can be found by using a subroutine, and also $w_2 = u_2^0 = v_2$.

The numerical experiments were carried out for domains (see Fig. 1) composed of unit squares. Then, $n = h^{-1} - 1$ is the number of grid nodes on $\Gamma$.

Table 1 describes the qualitative feature of the computation of $\lambda^0 = \lambda(w)$ by the scheme proposed above. To make comparisons, we give also the value of $\lambda_1^{-1}$ for $n = 8191$.

Table 2 shows the results of computation of $\lambda_1^{-1}$ for a certain $n$. Method 5 from Section 4 in [1] was realized with $l_k = 2$ for all $k$ (the Lanczos method with one internal iteration). In all cases, to attain the accuracy indicated in Table 2, it required 6-7 iterations by this method. Note that in these model cases the cost of one iteration is $\mathcal{O}(n \ln n)$ arithmetic operations which is considerably less than the estimate in [1] suitable for more complicated cases (for example, for 'red-black' partitioning). The experiments were carried out on HP-3000 computer with double precision. For the case of Fig. 1a it took 140 sec. of computer time to perform one iteration for $n = 8191$.

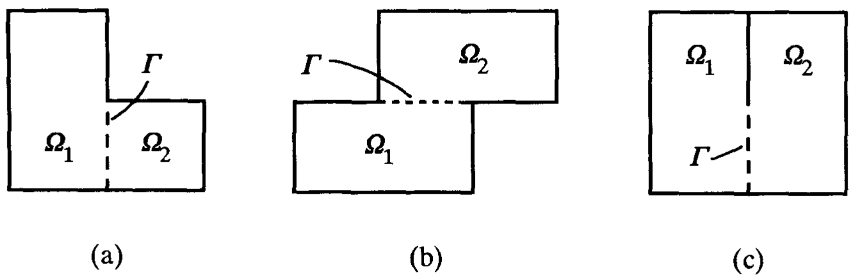


*Figure 1.* Model domains composed of unit squares.

| | $\bar{\lambda}^{-1}$ | $\lambda^{-1}(w)$ | $\lambda_1^{-1}$ |
|---|---|---|---|
| Fig. 1a | 12.3 | 9.92 | 9.64 |
| Fig. 1b | 12.3 | 9.03 | 8.67 |
| Fig. 1c | 12.3 | 9.03 | 8.37 |

*Table 1.* Quality of computation of initial guess.

| $n$ | Fig. 1a | Fig. 1b | Fig. 1c |
|---|---|---|---|
| 511 | 9.64024031896 | 8.66848350440 | 8.376559847 |
| 1023 | 9.63993175185 | 8.66805911949 | 8.373947877 |
| 2047 | 9.63980708831 | 8.66788899168 | 8.372639568 |
| 4095 | 9.63975706352 | 8.66782104893 | 8.371984833 |
| 8191 | 9.63973707318 | 8.66779397895 | 8.371657320 |

*Table 2.* Values of $\lambda_1^{-1}$ for certain $n$ for model cases.

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## REFERENCES

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