CHAPTER 5

Domain Decomposition in Boundary Element Methods*

George C. Hsiao†
Wolfgang L. Wendland‡

Abstract. An arbitrary substructuring technique is presented for solving elliptic boundary value problems via a symmetric boundary element Galerkin formulation. The Steklov–Poincaré operator is expressed explicitly by boundary integral equations and can be approximated by boundary element methods. Asymptotic stability and convergence results are given for simple model problems. The method is suited for parallel processing, since the corresponding boundary integral equations for the subdomains can be solved in parallel.

1. **Introduction.** The basic idea of domain decomposition methods or substructuring techniques for elliptic boundary value problems consists of reducing the solution of the boundary value problem on a domain to the solution of problems of same type on the subdomains. This idea is not new (see e.g. [18],[19],[20]), but gets considerable attention in recent years because of modern development of parallel computers; it is attractive for parallel processing, since the problem can be decoupled in independent subproblems and the communication needed will be only for the interface values [2], [4],[16]. The computation of these interface values is of central importance to the method.

The approach proposed in this paper uses boundary integral representations for solutions on the subdomains and reduces subproblems to boundary integral equations over the boundaries of subdomains. This allows one to define the Steklov–Poincaré operator [1],[3] explicitly in terms of boundary integral operators on the boundaries of subdomains (see §2). The Steklov–Poincaré operator represents a Dirichlet–Neumann map [23] (the capacity operator in the case of the Laplacian

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† Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA

†† Mathematisches Institut A, Universität Stuttgart, D–7000 Stuttgart 80, Germany

[7]) and is represented after discretization by the so-called Schur complement matrix (sometimes called the capacitance matrix). In forming this matrix, as usual, a boundary element Galerkin formulation of the problem is employed. Because of the integral representations of the solutions, the bilinear forms over the subdomains can be replaced by the boundary integral forms over the boundaries of the subdomains (see §2). Here we adopt a symmetric weak formulation which allows us to treat a larger class of elliptic partial differential equations including the Lamé system in elasticity. This formulation corresponds to the Hellinger–Reissner principle [5],[8],[9],[17] and involves a special coupling of boundary integral operators (see §2 and §3).

We present asymptotic stability and convergence results for the simple model problem in §4. It is worth mentioning that for the present substructuring technique, no assumption (or restriction) is made on the size of the subdomains and that the formulation contains only boundary elements. In a forthcoming paper [13], we shall discuss substructuring techniques concerning domain decomposition into a family of macro–elements whose size tends to zero; in addition, each macro–element can be modelled either with finite or with boundary elements, and the coupling requires only a few parameters on the common macro–element boundaries.

2. **Model problem.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. We begin with the Dirichlet problem for the Laplacian,

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u|_\Gamma = \varphi \quad \text{on} \quad \Gamma, \tag{2.1}$$

and consider its weak formulation: *Given* $\varphi \in H^{1/2}(\Gamma)$, *find* $u \in H^1(\Omega)$ *such that* $u|_\Gamma = \varphi$ *and*

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx = 0 \quad \forall\, v \in \overset{\circ}{H}{}^1(\Omega). \tag{2.2}$$

Here $\overset{\circ}{H}{}^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ is a subspace of the Sobolev space $H^1(\Omega)$ and $H^{1/2}(\Gamma)$ stands for the usual trace space.

To describe the substructuring technique, we partition Ω into N subdomains $\Omega_j, j = 1, \ldots, N$ as illustrated in Figure 1. Now let u_j be the restriction of u to Ω_j and denote by

$$\lambda_j := Du_j|_{\Gamma_j}$$

the exterior normal derivative of u_j on the boundary Γ_j of Ω_j. Then we may rewrite $a(u,v)$ in the form:

$$a(u,v) = \sum_{j=1}^N \int_{\Omega_j} \nabla u_j \cdot \nabla v dx = \sum_{j=1}^N \int_{\Gamma_j} \lambda_j v ds,$$

provided $\Delta u_j = 0$ in Ω_j. We note that in this way the bilinear form $a(\cdot,\cdot)$ can be replaced by the sum of the *boundary integral forms* (see e.g. [12])

$$< \lambda_j, v >_{\Gamma_j} := \int_{\Gamma_j} \lambda_j v ds. \tag{2.3}$$

For this purpose, we now represent u_j by the integral representation

$$u_j(x) = \int_{\Gamma_j} E(x,y)\lambda_j(y)ds_y - \int_{\Gamma_j} D_y E(x,y)\mu_j(y)ds_y, \; x \in \Omega_j \tag{2.4}$$

with $\mu_j := u_j|_{\Gamma_j}$ and

$$E(x,y) = \frac{-1}{2\pi}\log|x-y|,$$

the fundamental solution of the Laplacian. We remark that in the representation formula (2.4), both λ_j and μ_j are generally unknown (except that $\mu_j = \varphi$ on $\Gamma_j \cap \Gamma$). These are the Cauchy data of the solution of $\Delta u_j = 0$ in Ω_j, and are related by

$$\begin{pmatrix}\mu_j\\ \lambda_j\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\mathbf{I} - \mathbf{K}_j & \mathbf{V}_j \\ \mathbf{D}_j & \frac{1}{2}\mathbf{I} + \mathbf{K}_j'\end{pmatrix}\begin{pmatrix}\mu_j\\ \lambda_j\end{pmatrix} =: \mathbf{C}_j\begin{pmatrix}\mu_j\\ \lambda_j\end{pmatrix}. \tag{2.5}$$

Here $\mathbf{C}_j$ is the Calderón projector (with respect to Ω_j) in terms of the boundary integral operators:

$$\begin{aligned} \mathbf{V}_j\lambda(x) &:= \int_{\Gamma_j} E(x,y)\lambda(y)ds_y; & \mathbf{K}_j\mu(x) &:= \int_{\Gamma_j} D_y E(x,y)\mu(y)ds_y, \\ \mathbf{K}_j'\lambda(x) &:= \int_{\Gamma_j} D_x E(x,y)\lambda(y)ds_y; & \mathbf{D}_j\mu(x) &:= -D_x\int_{\Gamma_j} D_y E(x,y)\mu(y)ds_y, \end{aligned} \tag{2.6}$$

whose mapping properties on Sobolev spaces are now well known (see e.g. [14] and [6]). In particular, from the first equation of (2.5) we see that for diam $\Omega_j < 1$,

$$\mathbf{V}_j^{-1}\left(\frac{1}{2}\mathbf{I} + \mathbf{K}_j\right) : H^{\frac{1}{2}}(\Gamma_j) \ni \mu_j \to \lambda_j \in H^{-\frac{1}{2}}(\Gamma_j)$$

represents a *Dirichlet–Neumann map* [23] and is an *isomorphism* from the quotient space $H^{\frac{1}{2}}(\Gamma_j)/\mathbb{R}$ onto the subspace of $H^{-\frac{1}{2}}(\Gamma_j)$ defined by

$$\overset{\circ}{H}{}^{-\frac{1}{2}}(\Gamma_j) := \{\lambda_j \in H^{-\frac{1}{2}}(\Gamma_j) : <\lambda_j, 1>_{\Gamma_j} = 0\}.$$

Consequently, one may substitute λ_j in the boundary integral form (2.3) by

$$\lambda_j = \mathbf{V}_j^{-1}\left(\frac{1}{2}\mathbf{I} + \mathbf{K}_j\right)\mu_j \tag{2.7}$$

and define the Steklov–Poincaré operator on the traces of the solution u along all the interfaces of subdomains.

Alternatively, one may also use the second equation of (2.5), namely,

$$\lambda_j = \mathbf{D}_j\mu_j + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}_j'\right)\lambda_j$$

and rewrite (2.3) in the form

$$(2.8)\qquad \begin{aligned} <\lambda_j, v>_{\Gamma_j} := \int_{\Gamma_j} \lambda_j v ds &= \int_{\Gamma_j} \left(\mathbf{D}_j\mu_j + \frac{1}{2}\lambda_j + \mathbf{K}'_j\lambda_j\right) vds \\ &= <\mathbf{D}_j\mu_j, v>_{\Gamma_j} + \frac{1}{2} <\lambda_j, v>_{\Gamma_j} + <\lambda_j, \mathbf{K}_j v>_{\Gamma_j} . \end{aligned}$$

In fact, (2.7) and (2.8) taken togather is the basis for our symmetric boundary element Galerkin formulation in the next two sections. It is worth pointing out that this approach does not depend on the compactness properties of the boundary integral operators $\mathbf{K}_j$ and $\mathbf{K}'_j$. This gives us the flexibility for partitioning the domain and allows us to handle a larger class of elliptic boundary value problems including the ones for the Lamé system in linear elasticity.

3. **Symmetric formulation.** In the following let us introduce the function spaces:

$$\begin{aligned} &U = \{\mu = (\mu_1, \ldots, \mu_N) : \mu_j \in H^{\frac{1}{2}}(\Gamma_j);\ \mu_j|_{\Gamma_j \cap \Gamma} = \varphi|_{\Gamma_j\cap\Gamma}\} \\ &\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_N) : \lambda_j \in H^{-\frac{1}{2}}(\Gamma_j), \quad \textit{the dual of} \quad H^{\frac{1}{2}}(\Gamma_j)\} \\ &\overset{\circ}{U} = \{\mu = (\mu_1, \ldots, \mu_n) : \mu \in U \text{ with } \varphi = 0\}. \end{aligned}$$

The spaces U(*also* $\overset{\circ}{U}$) and Λ are equipped with the norms

$$\|\mu\|_U := \left\{\sum_{j=1}^N \|\mu_j\|^2_{H^{\frac{1}{2}}(\Gamma_j)}\right\}^{\frac{1}{2}} \text{ and } \|\lambda\|_\Lambda := \left\{\sum_{j=1}^N \|\lambda_j\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}\right\}^{\frac{1}{2}} .$$

Then our symmetric substructuring technique for the model problem (2.1) can be formulated as: *Find* $\mu \in U$ *and* $\lambda \in \Lambda$ *such that*

$$(3.1)\qquad \sum_{j=1}^N \left\{\frac{1}{2} <\lambda_j, v_j>_{\Gamma_j} + <\lambda_j, \mathbf{K}_j v_j>_{\Gamma_j} + <\mathbf{D}_j\mu_j, v_j>_{\Gamma_j}\right\} = 0 \quad \forall v \in \overset{\circ}{U},$$

and for $j = 1, \ldots, N,$

$$(3.2)\qquad <\eta_j, \mathbf{V}_j\lambda_j>_{\Gamma_j} - \frac{1}{2} <\eta_j, \mu_j>_{\Gamma_j} - <\eta_j, \mathbf{K}_j\mu_j>_{\Gamma_j} = 0 \quad \forall \eta \in \Lambda .$$

In this formulation, the bracket $<\cdot,\cdot>_{\Gamma_j}$ denotes the duality pairing on $H^{-\frac{1}{2}}(\Gamma_j)\times H^{\frac{1}{2}}(\Gamma_j)$ and is an extension of the $L^2(\Gamma_j)$–inner product defined by (2.3) for smooth functions.

Now if we introduce the boundary bilinear form,

$$(3.3)\qquad \begin{aligned} \tilde{a}(\mu, \lambda; v, \eta) := \sum_{j=1}^N \Big\{ &<\eta_j, \mathbf{V}_j\lambda_j>_{\Gamma_j} + <\mathbf{D}_j\mu_j, v_j>_{\Gamma_j} \\ &+\frac{1}{2} <\lambda_j, v_j>_{\Gamma_j} - \frac{1}{2} <\eta_j, \mu_j>_{\Gamma_j} + <\lambda_j, \mathbf{K}_j v_j>_{\Gamma_j} - <\eta_j, \mathbf{K}_j\mu_j>_{\Gamma_j} \Big\}, \end{aligned}$$

then we see that

$$\tilde{a}(v,\eta;v,\eta) = \sum_{j=1}^{N}\Big\{ < \eta_j, \mathbf{V}_j\eta_j >_{\Gamma_j} + < \mathbf{D}_j v_j, v_j >_{\Gamma_j} \Big\}.$$

Hence by the coerciveness properties of $\mathbf{V}_j$ and $\mathbf{D}_j$ on Lipschitz domains [6], and by mapping each of the subdomains to a master element with the help of scaling, one can establish that $\tilde{a}$ is $\overset{\circ}{U} \times \Lambda$*–elliptic*. More precisely, suppose that the configuration of the subdomain Ω_j satisfies the assumption: *For each* Ω_j *there exist two positive constants* r_j *and* R_j *such that*

$$\text{(H)} \qquad 2r_j \le (\operatorname{diam}\Omega_j) \le 2R_j \quad \text{and} \quad 0 < c_1 \le R_j/r_j \le c_2,$$

where c_1 *and* c_2 *are fixed constants for all* $j = 1, \ldots, N$ (see Figure 2). Then we have the following result.

Lemma 1. *Under the assumption* (H), *there exists a constant* $\gamma_0 > 0$ *such that*

$$\begin{aligned}\tilde{a}(v,\eta;v,\eta) &\ge \gamma_0\{||v||_U^2 + ||\eta||_\Lambda^2\} \\ &= \gamma_0 \left\{ \sum_{j=1}^{N} \left(||v_j||^2_{H^{\frac{1}{2}}(\Gamma_j)} + ||\eta_j||^2_{H^{-\frac{1}{2}}(\Gamma_j)} \right) \right\}\end{aligned}$$

for all $v \in \overset{\circ}{U}$ *and* $\eta \in \Lambda$.

As an easy consequence of the Lax–Milgram Theorem, we then have the existence and uniqueness results for the solution of the system (3.1) and (3.2).

We remark that the present symmetric formulation for the model problem (2.1) can be easily generalized to a larger class of elliptic boundary value problems. In the case of linear elasticity, we see that the bilinear form $a(\cdot,\cdot)$ assumes the form:

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})dx\,,$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress tensors defined by

$$\boldsymbol{\epsilon}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$$

and

$$\boldsymbol{\sigma}(\mathbf{u}) := \frac{2\nu G}{1-2\nu}(\operatorname{div}\mathbf{u})\mathbf{I} + 2G\boldsymbol{\epsilon}(\mathbf{u})\,,$$

respectively; here the material constants ν and G are referred to as the Poisson's ratio and shear modulus in elasticity. Hence, if $\mathbf{u}_j = \mathbf{u}|_{\Gamma_j}$ satisfies the Lamé system

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}_j) = \mathbf{0} \quad \text{in} \quad \Omega_j,$$

we may again replace the bilinear form $a(\mathbf{u}, \mathbf{v})$ by the sum of the boundary integral forms

$$< \boldsymbol{\lambda}_j, \mathbf{v} >_{\Gamma_j} := \int_{\Gamma_j} \boldsymbol{\lambda}_j \cdot \mathbf{v} ds,$$

where $\boldsymbol{\lambda}_j := D\mathbf{u}_j|_{\Gamma_j}$ is now the traction on the boundary Γ_j,

$$D\mathbf{u}_j|_{\Gamma_j} = \boldsymbol{\sigma}(\mathbf{u}_j) \cdot \mathbf{n}|_{\Gamma_j} ,$$

where $\mathbf{n}$ denotes the exterior normal to Γ_j. Then one can proceed in the same manner as for the model problem (2.1). The corresponding boundary integral operators are defined explicitly in [11],[15],[22] and details will be available in the forthcoming paper [13].

4. **Boundary element Galerkin method**. We now consider the discretization of (3.1) and (3.2). To this end, let us introduce the finite–dimensional subspaces

$$\begin{aligned} U_h &:= \{\mu_h = (\mu_{1h}, \ldots, \mu_{Nh}) : \mu_{jh}|_{\Gamma_j \setminus \Gamma} \in S_h^d(\Gamma_j)\} , \\ \overset{\circ}{U}_h &:= \left\{\mu_h = (\mu_{ih}, \ldots, \mu_{Nh}) : \mu_h \in U_h \cap \overset{\circ}{U}\right\} , \\ \Lambda_h &:= \{\lambda_h = (\lambda_{ih}, \ldots, \lambda_{Nh}) : \lambda_{jh} \in S_h^d(\Gamma_j)\} , \end{aligned}$$

where $S_h^d(\Gamma_j)$ denotes the family of $(d-1)$–times continuously differentiable splines of polynomial degree d associated with the mesh width h_j on Γ_j.

The BEM–Galerkin method for the equations (3.1) and (3.2) can be simply formulated: *Find* $(\mu_h, \lambda_h) \in U_h \times \Lambda_h$ *such that*

$$\tilde{a}(\mu_h, \lambda_h; v_h, \eta_h) = 0 \quad \forall \ (v_h, \eta_h) \in \overset{\circ}{U}_h \times \Lambda_h . \tag{4.1}$$

Here the boundary bilinear form $\tilde{a}$ is defined by (3.3). In componentwise the functions u_{jh} and λ_{jh}, $j = 1, \ldots, N$ are solutions of equations (3.1) and (3.2) in $U_h \times \Lambda_h$ with test functions $(v_h, \eta_h) \in \overset{\circ}{U}_h \times \Lambda_h$. In what follows we shall refer to these equations as $(4.1)_1$ and $(4.2)_2$ of the Galerkin equations (4.1). We now summarize our results concerning the solutions μ_{jh} and λ_{jh} in the following theorem.

Theorem 1. *Let* $h_j \le c_0 H$, $j = 1, \ldots, N$ *for* $0 < H < 1$, *where* c_0 *is a constant independent of* H. *Then the Galerkin equations* (4.1) (*or* $(4.1)_1$, *and* $(4.1)_2$) *are always uniquely solvable. Moreover, the Galerkin solutions* (μ_{jh}, λ_{jh}) *for the domain decomposition satisfy the asymptotic estimate*

$$\left\{ \sum_{j=1}^N \left(\|u|_{\Gamma_j} - \mu_{jh}\|^2_{H^{\frac12}(\Gamma_j)} + \|Du|_{\Gamma_j} - \lambda_{jh}\|^2_{H^{-\frac12}(\Gamma_j)} \right) \right\}^{1/2} \le c(c_0 H)^{\ell - 1} \|u\|_{H^\ell(\Omega)}$$

for $1 < \ell \le d + 3/2$, *where* u *is the exact solution of* (2.2).

The significance of the condition of $h_j \le c_0 H$ in the theorem needs some explanation. Consider first a family of decompositions with $H \to 0$ for fixed constant c_0 independent of H. This is the case indicated in Figure 1(a). The number of decompositions increases while the numbers of individual elements on all subboundaries Γ_j can always be chosen to be the same. In this situation, for congruent Ω_j one needs only to invert the BEM equations $(4.1)_2$ for λ_{jh} once for all on a master domain, and for arbitrary Ω_j, these equations can be inverted *in parallel.*

The unknowns λ_{jh} can be eliminated and the system is condensated to the first equation $(4.1)_1$ of the variational equations for the μ_{jh}'s. This leads to the usual *capacitance system* for the interface unknowns which can be solved by preconditioned conjugate gradient methods (see e.g. [2],[16],[24]).

On the other hand, as indicated in Figure 1(b), for a fixed decomposition as in boundary element substructering, H is fixed. One may then require $c_0 \to 0$ to allow $h_j \to 0$ for the mesh refinement(see e.g. [10]).

To conclude the paper, we now comment on the proof of the asymptotic estimate in Theorem 1. The essence of the proof is to show that the Galerkin projection $\mathbf{G}_h$ defined by

$$\mathbf{G}_h : (\mu, \lambda) \to (\mu_h, \lambda_h)$$

is uniformly bounded on $U \times \Lambda$. This follows from the standard arguments now in boundary element methods (see e.g. [14],[21],[22]). From the boundedness of the Galerkin projection, we then obtain an inequality of Ceá's type from which the estimate follows easily from the convergence property of the splines.

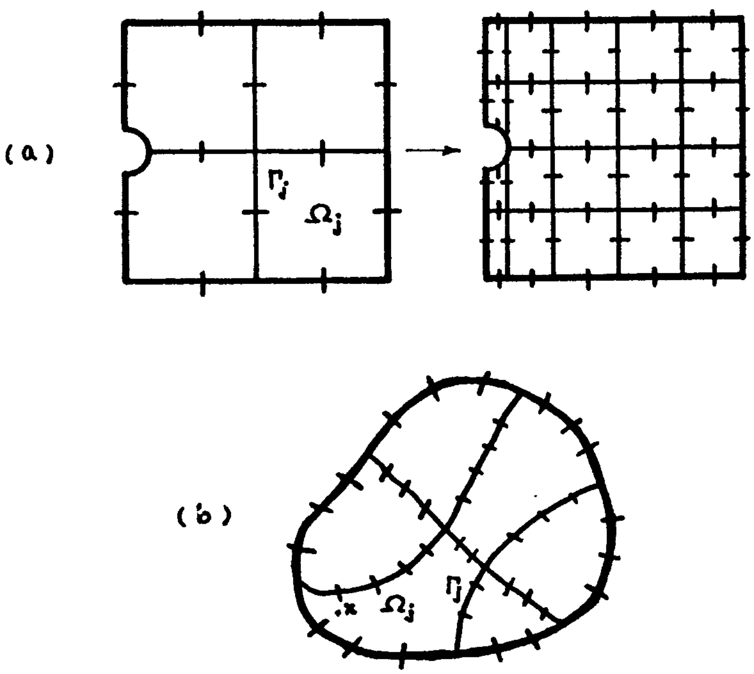

Figure 1: The domain and its partition: (a) c_0 fixed and $H \to 0$; (b) H fixed and $c_0 \to 0$.

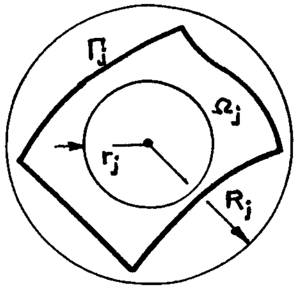

Figure 2: Assumptions on the subdomains Ω_j.

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