Iterative Methods for Solving Equations with Highly Varying Coefficients

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Abstract. Using as an example the Dirichlet problem for the diffusion equation with piecewise-constant coefficients a relation is established between iterative algorithms of two main versions of the decomposition method with iterations of gradients and fluxes typical of the fictitious domain method.

1. Introduction. The problem of solution of elliptic equations and systems with highly varying coefficients is typical of the mathematical modelling of behaviour of structures composed of materials with constant properties, for example, in mechanics and electrical physics.

One of the approaches to solving such problems is based on decomposition of the original structure into its constituent substructures with homogeneous properties. It is particularly efficient in the case where the number of substructures is not large and all of them are of the simple form (the latter permits application of fast algorithms to solving auxilary subproblems arising in decomposition methods). In addition, in a number of cases the computational characteristics of some decomposition methods do not become worse with the coefficient discontinuity infinitely increasing.

On the other hand, in numerical modelling it is sometimes reasonable to combine some substructures of complex form into one simple-form structure adding probably 'fictitious' parts as in the fictitious domain method thereby again leading to a great difference between coefficients. Similar equations also arise in

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the theory of averaging compositional materials. To solve problems of such kind, it is natural to replace the decomposition method with methods involved in solving the auxiliary boundary value problem in the composition domain as it has a simple form (in the fictitious domain method by construction).

The duality of these two approaches is reflected in a definite relation of the transition operators of the iterative processes corresponding to them. This paper is devoted to finding this relation.

In Section 2 according to [1] we give two main realizations of the decomposition method on the surface of coefficient discontinuity by using Poincaré-Steklov operators for a very simple case of the diffusion equation with piecewise-constant coefficients. In the first realization there arises an equation for the trace of the solution on this surface, in the second one there arises an equation for the normal component of the flux.

In Section 3 for the problems from Section 1 we present a known iterative method whose grid version is regarded as the method with spectrally equivalent operators. It is shown that under a special choice of the initial guess this method, first, uniformly converges in all possible values of coefficients of the equation solved and, second, permits realization on the functions defined only at the coefficient discontinuity boundary. Moreover, its representation in terms of the Poincaré-Steklov operators coincides with one of the iterative algorithms in the first-type decomposition method considered in Section 1. A particular case of infinitely large coefficient discontinuity was studied earlier in [3].

In Section 4 we have obtained similar results for the flux iteration method with the only difference that its representation in terms of the Poincaré-Steklov operators coincides with one of the iterative algorithms in the decomposition method of the second type but not of the first one.

2. Formulation of domain decomposition main equations. Let us prescribe non-overlapping simply-connected bounded Lipschitz domains Ω_0 and Ω_1 such that

$$\operatorname{mes}\Gamma > 0, \qquad \Gamma \equiv \partial\Omega_0 \cap \partial\Omega_1$$

and let the domain Ω defined by the equality $\bar{\Omega} = \bar{\Omega}_0 \cup \bar{\Omega}_1$ be also a Lipschitz one. Consider the equation

$$\operatorname{div}(k\nabla u - f) = 0, \qquad u \in \overset{\circ}{W}{}_2^1(\Omega) \tag{2.1}$$

with the piecewise-constant coefficient

$$k = k_i \text{ in } \Omega_i, \qquad k_i \equiv \text{const} > 0, \qquad i = 0,1.$$

As known, the generalized solution to problem (2.1) exists and is unique for any right-hand side $f \in \mathbb{L}_2(\Omega)$. The following continuity conditions are valid at the boundary Γ:

$$[u] = 0, \qquad [(k\nabla u - f)_\perp] = 0. \tag{2.2}$$

These conditions imply equations of the two main realizations of the domain decomposition method at the 'cutting line' (common boundary of subdomains) Γ [1].

Introduce restriction spaces $H_i \equiv \overset{\circ}{W}{}_2^1(\Omega)|_{\Omega_i}$, $i = 0,1$, and operators γ_i of taking the trace on Γ of functions from H_i. For solenoidal vector functions (with zero divergence) in the domain Ω_i define the operator ν_i of taking the normal component on Γ. Then conditions (2.2) can be written in the form

$$\begin{aligned} \gamma_0 u_0 &= \gamma_1 u_1 \\ \nu_0(k_0 \nabla u_0 - f_0) + \nu_1(k_1 \nabla u_1 - f_1) &= 0 \end{aligned} \tag{2.3}$$

here and henceforth, the subscripts of functions mean that the restirictions of the functions are taken on Ω_i, $i = 0,1$.

Define Poincaré-Steklov operators S_i by the relations [1,5]

$$S_i \psi = \varphi \Leftrightarrow \psi \equiv \nu_i(\nabla v), \quad \varphi \equiv \gamma_i v, \quad \Delta v = 0 \text{ in } \Omega_i, \quad v \equiv H_i .$$

Note that here we have $\nu_i(\nabla v) = \partial v/\partial n$ on Γ.

Let us derive the equation of the first version of the decomposition method. To make use of the Poincaré-Steklov operators, it is necessary to pass from u to the piecewise-harmonic function. To this end, define the function v by its restrictions v_i which are the solutions to the problems

$$\operatorname{div}(k_i \nabla v_i - f_i) = 0 \text{ in } \Omega_i, \quad v_i \in \overset{\circ}{W}{}_2^1(\Omega_i). \tag{2.4}$$

Then the difference $w \equiv u - v$ is a piecewise-harmonic function and from conditions (2.3) we derive the conditions for the difference

$$\begin{aligned} \gamma_0(w_0) &= \gamma_1(w_1) \\ k_0 \nu_0(\nabla w_0) + k_1 \nu_1(\nabla w_1) &= \nu_0(k_0 \nabla v_0 + f_0) + \nu_1(k_1 \nabla v_1 + f_1) \end{aligned} \tag{2.5}$$

as $\gamma_0(v_0) = \gamma_1(v_1) = 0$ by the definition of v. Set

$$\varphi \equiv \gamma_0(w_0) = \gamma_1(w_1)$$

and write down the left-hand side of the second equality in (2.5) by using the Poincaré-Steklov operators

$$(k_0 S_0^{-1} + k_1 S_1^{-1})\varphi = \ldots . \tag{2.6}$$

This condition is considered in the decomposition method as equation in φ. To solve it, we can use various iterative methods, for example, the stationary Richardson method

$$A \frac{\varphi^{n+1} - \varphi^n}{\tau} + (k_0 S_0^{-1} + k_1 S_1^{-1})\varphi^n - \ldots = 0 . \tag{2.7}$$

Method (2.7) with an appropriate $\tau > 0$ converges in $W_2^{1/2}(\Gamma)$, at least, at the rate of geometric progression for any initial guess $\varphi^0 \in \mathring{W}_2^{1/2}(\Gamma)$ if an auxiliary operator A is properly chosen. Moreover, the common ratio is independent of k_0 and k_1. The typical choice of A is $A = S_0^{-1}$ or $A = S_1^{-1}$ [1,5]. The choice of A in (2.7) is considered in more detail in Section 3.

In the second version it is convenient to define the function v in a somewhat different way:

$$\begin{gathered} \operatorname{div}(k_i \nabla v_i - f_i) = 0 \ \text{ in } \ \Omega_i, \qquad v_i \in H_i \\ \nu_i (k_i \nabla v_i - f_i) = 0 . \end{gathered} \tag{2.8}$$

The function v may have a discontinuity on Γ. From (2.3) and the definition of v we have relations for the difference $w \equiv u - v$:

$$\begin{gathered} \gamma_0(w_0) - \gamma_1(w_1) = \gamma_0(v_0) - \gamma_1(v_1) \\ k_0 \nu_0(\nabla w_0) + k_1 \nu_1(\nabla w_1) = 0 . \end{gathered} \tag{2.9}$$

Set

$$\psi \equiv k_0 \nu_0(\nabla w_0) = - k_1 \nu_1(\nabla w_1)$$

and write down the first equality in (2.9) by using the Poincaré-Steklov operators.

$$(k_0^{-1} S_0 + k_1^{-1} S_1)\psi = \gamma_0(v_0) - \gamma_1(v_1). \tag{2.10}$$

In the decomposition method this condition is treated as an equation in ψ. To solve this equation, we can also use the stationary Richardson method

$$B \frac{\psi^{n+1} - \psi^n}{\tau} + (k_0^{-1} S_0 + k_1^{-1} S_1)\psi^n - \gamma_0(v_0) + \gamma_1(v_1) = 0. \tag{2.11}$$

Method (2.11) with an appropriate $\tau > 0$ converges in $W_2^{-1/2}(\Gamma)$, at least, at the rate of geometric progression for any initial guess $\psi^0 \in W_2^{-1/2}(\Gamma)$ if an auxiliary operator B is properly chosen. Similarly to method (2.7) the common ratio is independent of k_0 and k_1. If the appropriate operator A in (2.7) is known, we can choose $B = A^{-1}$, specifically, $B = S_0$ or $B = S_1$. This problem is considered in detail in Section 4.

3. Iterations of gradients and their realization at the 'cutting line'. To solve problem (2.1), let us consider the iterative process

$$\Delta \frac{u^{n+1} - u^n}{\tau} + \operatorname{div}(k \nabla u^n - f) = 0, \qquad u^n \in \mathring{W}_2^1(\Omega) \tag{3.1}$$

with the initial guess u^0 which is the solution to the problem

$$\operatorname{div}(\nabla u^0 - g) = 0 \text{ in } \Omega$$

$$g \equiv \begin{cases} f/k_0 \text{ in } \Omega_0 \\ f/k_1 \text{ in } \Omega_1. \end{cases} \tag{3.2}$$

Method (3.1) is equivalent to the iterations of gradients:

$$\frac{\nabla u^{n+1} - \nabla u^n}{\tau} + P(k\nabla u^n - f) = 0, \quad u^n \in \overset{\circ}{W}{}_2^1(\Omega) \tag{3.3}$$

where $P \equiv \nabla\Delta^{-1}\operatorname{div}$ and $\Delta^{-1}: W_2^{-1}(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega)$.

It is known that for an appropriate $\tau > 0$ (for example, for any positive $\tau < 2/\max k_i$) iterative method (3.1) converges in $\overset{\circ}{W}{}_2^1(\Omega)$ with an arbitrary initial guess $u^0 \in \overset{\circ}{W}{}_2^1(\Omega)$, at least, at the rate of geometric progression. However, the common ratio is dependent on k_0 and k_1, in particular, for $k_0/k_1 \to 0$ the convergence dramatically decreases.

Let us show that with the special initial guess u^0 from (3.2) the common ratio estimating the convergence rate of process (3.1) is independent of k_0 and k_1. Introduce the error function $\varepsilon^n \equiv u^n - u$. We can directly verify the fact that the function ε^0 is piecewise-harmonic, i.e. $\Delta\varepsilon_i^n = 0$ in Ω_i, $i = 0,1$, hence, by induction all ε^n are piecewise-harmonic as (3.1) implies

$$\Delta\frac{\varepsilon^{n+1} - \varepsilon^n}{\tau} + \operatorname{div} k\nabla\varepsilon^n = 0, \quad \varepsilon^n \in \overset{\circ}{W}{}_2^1(\Omega). \tag{3.4}$$

The convergence rate of iterations (3.4) is determined by the relation $\varkappa \equiv \underline{\varkappa}/\overline{\varkappa}$ of the constants from the inequalities

$$\begin{gathered} 0 < \underline{\varkappa} \leqslant (k\nabla\varepsilon,\nabla\varepsilon)/(\nabla\varepsilon,\nabla\varepsilon) \leqslant \overline{\varkappa} \\ \varepsilon \in \overset{\circ}{W}{}_2^1(\Omega), \quad \Delta\varepsilon_i = 0 \text{ in } \Omega_i. \end{gathered} \tag{3.5}$$

Here and henceforth, $(\cdot,\cdot)$ is an ordinary scalar product of vector functions in $\mathbb{L}_2(\Omega)$. The condition of piecewise harmonicity in (3.5) plays a dicisive part as it is this condition which makes it possible to obtain the relation $\varkappa$ independent of k_0 and k_1. We have

$$\begin{aligned} (k\nabla\varepsilon,\nabla\varepsilon) &= k_0(\nabla\varepsilon_0,\nabla\varepsilon_0)_0 + k_1(\nabla\varepsilon_1,\nabla\varepsilon_1)_1 \\ &= k_0\langle \nu_0(\nabla\varepsilon_0),\gamma_0\varepsilon_0\rangle + k_1\langle \nu_1(\nabla\varepsilon_1),\gamma_1\varepsilon_1\rangle \\ &= \langle \{k_0S_0^{-1} + k_1S_1^{-1}\}\gamma\varepsilon,\gamma\varepsilon\rangle \end{aligned} \tag{3.6}$$

where $\gamma\varepsilon \equiv \gamma_0\varepsilon = \gamma_1\varepsilon$ as $\varepsilon \in \overset{\circ}{W}{}_2^1(\Omega)$. Here and henceforth, $\langle \cdot,\cdot \rangle$ is an ordinary scalar product in $L_2(\Gamma)$ and also the duality relation for the pair of spaces $W_2^{-1/2}(\Gamma) \times \overset{\circ}{W}{}_2^{1/2}(\Gamma)$. Likewise, for the denominator of the fraction in (3.5) we have the representation

$$(\nabla\varepsilon, \nabla\varepsilon) = <\{k_0 S_0^{-1} + k_1 S_1^{-1}\}\gamma\varepsilon, \gamma\varepsilon> .$$

Thus, (3.5) is equivalent to

$$0 < \underline{\varkappa} \leqslant \frac{<\{k_0 S_0^{-1} + k_1 S_1^{-1}\}\varphi,\varphi>}{<\{S_0^{-1} + S_1^{-1}\}\varphi,\varphi>} \leqslant \bar{\varkappa}, \quad \varphi \in \overset{\circ}{W}{}_2^{1/2}(\Gamma). \tag{3.7}$$

Inequalities (3.7) are in turn implied by the important property of Poincaré-Steklov operators [5] that the functional $<S_i^{-1}\varphi,\varphi>^{1/2}$, $i = 0,1$, is the norm on $\overset{\circ}{W}{}_2^{1/2}(\Gamma)$, which is equivalent to the ordinary norm $\|\varphi\|_{W_2^{1/2}(\Gamma)}$. Indeed, we have in the numerator

$$<\{k_0 S_0^{-1} + k_1 S_1^{-1}\}\varphi,\varphi> \asymp \max\{k_i\} \|\varphi\|^2_{W_2^{1/2}(\Gamma)}$$

and in the denominator

$$<\{S_0^{-1} + S_1^{-1}\}\varphi,\varphi> \asymp \|\varphi\|^2_{W_2^{1/2}(\Gamma)}$$

hence, both estimates $\underline{\varkappa}$ and $\bar{\varkappa} \asymp \max\{k_i\}$, and their relation $\varkappa$ is independent of k_0 and k_1.

Returning to method (3.1) we can state that the error in (3.4) with a definite choice of τ decreases in $\overset{\circ}{W}{}_2^1(\Omega)$, at least, at the rate of geometric progression with the common ratio independent of k_i. For example, for $\tau = 1/\max\{k_i\}$ the following estimate is valid:

$$\|\varepsilon^n\|_{W_2^1(\Omega)} \leqslant (1 - \varkappa)^n \|\varepsilon^0\|_{W_2^1(\Omega)} . \tag{3.8}$$

Of interest is a comparison of the behaviour of errors for decomposition method (2.7) and method (3.1). Similarly to transformations (3.6) we obtain a process equivalent to (3.4) for traces of the errors $\gamma\varepsilon^n$:

$$\{S_0^{-1} + S_1^{-1}\} \frac{\gamma\varepsilon^{n+1} - \gamma\varepsilon^n}{\tau} + \{k_0 S_0^{-1} + k_1 S_1^{-1}\}\gamma\varepsilon^n = 0. \tag{3.9}$$

On the other hand, from (2.7) we directly derive the formula

$$A \frac{\gamma\varepsilon^{n+1} - \gamma\varepsilon^n}{\tau} + \{k_0 S_0^{-1} + k_1 S_1^{-1}\}\gamma\varepsilon^n = 0 \tag{3.10}$$

in which we denote $\gamma\varepsilon^n \equiv \varphi^n - \gamma u$. Iterative method (3.1) thus leads to formula (3.10) for traces of the errors $\gamma\varepsilon^n$, which is a particular case of similar formula (3.10) with $A = S_0^{-1} + S_1^{-1}$ for decomposition method (2.7). The particular case of $k_0 = 0$ and $k_1 = 1$ in (3.9) and $A = S_1^{-1}$, $k_0 = k_1 = 1$, in (3.10) was analysed in [3].

Similarly to the above-given arguments concerning the convergence estimate of method (3.9) we can establish the fact that method (3.10) with an appropriate τ converges, at least, at the rate of geometric progression with the common ratio independent of k_i if the linear operator A: $\overset{\circ}{W}{}_2^{1/2}(\Gamma) \to W_2^{-1/2}(\Gamma)$ is symmetric in $L_2(\Gamma)$ and the functional $<A\varphi,\varphi>^{1/2}$ defines the equivalent norm on $\overset{\circ}{W}{}_2^{1/2}(\Gamma)$ [5].

Methods of type (3.1) were analysed in the paper by N. S. Bakhvalov and A. V. Knyazev which is called 'Methods of cost-effective computation of averaged characteristics of composites of a periodic structure composed of essentially heterogeneous materials' and published in 'Computational Processes and Systems', No.8, Nauka, Moscow, 1990.

4. Iterations of fluxes and their realization at the 'cutting line'. In [4] we suggested an approach to solving the problem of type (2.1), which is based on iterations of fluxes. Such iterative methods by their efficiency are approximately equivalent to methods of type (3.1), only the requirements for memory are stronger as in iterations of gradients (3.3). In particular, they converge, at least, at the rate of geometric progression with the common ratio independent of k_i. Let us consider one of the simplest methods following [6] and the paper by the same authors mentioned at the end of Section 3. Transform problem (2.1) to the equivalent form

$$\begin{gathered} \operatorname{div}(\sigma - f) = 0, \quad k^{-1}\sigma = \nabla u \\ u \in \overset{\circ}{W}{}_2^1(\Gamma), \quad \sigma \in \mathbb{L}_2(\Omega). \end{gathered} \tag{4.1}$$

As in (3.3), introduce the operator

$$P \equiv \nabla\Delta^{-1}\operatorname{div}, \quad \text{where } \Delta^{-1}: W_2^{-1}(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega).$$

It is known that P is the orthoprojector in $\mathbb{L}_2(\Omega)$ onto the subspace $\mathbb{P} \equiv \nabla\overset{\circ}{W}{}_2^1(\Omega) \subset \mathbb{L}_2(\Omega)$, and the operator $P^{\perp} \equiv I - P$ is the orthoprojector in $\mathbb{L}_2(\Omega)$ onto the subspace $\mathbb{P}^{\perp}$ of functions with zero divergence. Write down (4.1) using these operators:

$$P(\sigma - f) = 0, \quad P^{\perp}k^{-1}\sigma = 0. \tag{4.2}$$

Note the possibility of geometric interpretation of equations (4.2):

$$\sigma \in (Pf + \mathbb{P}^{\perp}) \cap \ker(P^{\perp}k^{-1}).$$

To solve system (4.2), make use of the method of iterations of fluxes σ^n:

$$\frac{\sigma^{n+1} - \sigma^n}{\tau} + P^{\perp}k^{-1}\sigma^n = 0, \quad \sigma^0 = Pf. \tag{4.3}$$

For errors $\varepsilon^n \equiv \sigma^n - \sigma$ the following recurrent formula is also valid:

$$\frac{\varepsilon^{n+1}-\varepsilon^{n}}{\tau}+P^{\perp}k^{-1}\varepsilon^{n}=0. \tag{4.4}$$

The initial error $\varepsilon^0 = Pf - \sigma \in \mathbb{P}^{\perp}$, i.e. $\operatorname{div}\varepsilon^0 = 0$ in Ω and in each subdomain Ω_i is a gradient of a function $v_i \in H_i \equiv \overset{\circ}{W}{}_2^1(\Omega)|_{\Omega_i}$, and this can be verified in a straightforward way. Then by virtue of (4.4) the same properties belong to all errors ε^n. Define the subspace $\mathbb{G} \subset \mathbb{L}_2(\Omega)$ by the relations

$$\varepsilon \in \mathbb{G} \Leftrightarrow \operatorname{div}\varepsilon = 0 \text{ in } \Omega, \quad \varepsilon_i = \nabla v_i \text{ in } \Omega_i,\ v_i \in H_i$$

which contain all errors $\varepsilon^n \in \mathbb{G}$. The rate of convergence of iterations (4.4) in $\mathbb{L}_2(\Omega)$ is determined by the relation $\text{æ} \equiv \underline{\text{æ}}/\bar{\text{æ}}$ of the constants in the inequalities

$$0 < \underline{\text{æ}} \leqslant (k^{-1}\varepsilon,\varepsilon)/(\varepsilon,\varepsilon) \leqslant \bar{\text{æ}}, \quad \varepsilon \in \mathbb{G}. \tag{4.5}$$

From the definition of the subspace $\mathbb{G}$ and the Poincaré-Steklov operators we derive

$$\begin{aligned}(k^{-1}\varepsilon,\varepsilon) &= k_0^{-1}(\nabla v_0,\nabla v_0)_0 + k_1^{-1}(\nabla v_1,\nabla v_1)_1 \\ &= k_0^{-1}< v_0\varepsilon_0,\gamma_0 v_0> + k_1^{-1}< v_1\varepsilon_1,\gamma_1 v_1> \\ &= <\psi,\{k_0^{-1}S_0 + k_1^{-1}S_1\}\psi>\end{aligned} \tag{4.6}$$

where $\psi \equiv v_0\varepsilon_0 = -v_1\varepsilon_1$. Making use of the similar representation for the denominator we arrive at the inequalitites equivalent to (4.5):

$$0 < \underline{\text{æ}} \leqslant \frac{<\psi,\{k_0^{-1}S_0 + k_1^{-1}S_1\}\psi>}{<\psi,\{S_0+S_1\}\psi>} \leqslant \bar{\text{æ}}, \quad \varphi \in W_2^{-1/2}(\Gamma). \tag{4.7}$$

As known [5], the functional $<\psi,S_i\psi>^{1/2}$, $i=0,1$, is the norm on $W_2^{-1/2}(\Gamma)$ which is equivalent to the ordinary norm $\|\psi\|_{W_2^{-1/2}(\Gamma)}$. Therefore, in (4.6) we have in the numerator

$$<\psi,\{k_0^{-1}S_0 + k_1^{-1}S_1\}\psi> \asymp \max\{k_i^{-1}\}\|\psi\|^2_{W_2^{1/2}(\Gamma)}$$

and in the denominator

$$<\psi,\{S_0+S_1\}\psi> \asymp \|\psi\|^2_{W_2^{1/2}(\Gamma)}$$

hence, both estimates $\underline{\text{æ}}$ and $\bar{\text{æ}} \asymp \max\{k_i^{-1}\}$, and their relation æ is independent of k_i.

It is not accidental that the same symbol æ is used for the relations of constants in (3.5) and in (4.5) as æ in (3.5) and æ in (4.5) can be chosen identical. It can be easily established taking into account the equality of extreme values of the functionals

$$<S_0^{-1}\varphi,\varphi>/<S_1^{-1}\varphi,\varphi>, \quad <\psi,S_1\psi>/<\psi,S_0\psi>.$$

Thus, the errors in (4.4) for an appropriate τ decrease in $\mathbb{L}_2(\Omega)$, at least, at the rate of geometric progression with the common ratio independent of k_i. In particular, the following estimate is valid for $\tau = \min\{k_i\}$:

$$\|\varepsilon^n\|_{\mathbb{L}_2(\Omega)} \leqslant (1 - æ)^n \|\varepsilon^0\|_{\mathbb{L}_2(\Omega)} . \tag{4.8}$$

Let us compare the behaviour of errors in decomposition method (2.11) with that in method (4.8). Similarly to (4.6) we obtain a process equivalent to (4.4) for the normal component $\nu\varepsilon^n \equiv \nu_0\varepsilon_0^n = -\nu_1\varepsilon_1^n$ of the errors ε^n on Γ:

$$\{S_0 + S_1\}\frac{\nu\varepsilon^{n+1} - \nu\varepsilon^n}{\tau} + \{k_0^{-1}S_0 + k_1^{-1}S_1\}\nu\varepsilon^n = 0 . \tag{4.9}$$

On the other hand, from (2.11) we derive

$$B\frac{\nu\varepsilon^{n+1} - \nu\varepsilon^n}{\tau} + \{k_0^{-1}S_0 + k_1^{-1}S_1\}\nu\varepsilon^n = 0 \tag{4.10}$$

where we have changed the notation $\nu\varepsilon^n \equiv \psi^n - \psi$.

We arrive at the conclusion that formula (4.9) can be treated as a particular case of formula (4.10) with $B = S_0 + S_1$.

The choice of the operator B suitable for iterations (2.11) in the sense indicated at the end of Section 2 is the choice B: $W_2^{-1/2}(\Gamma) \to \mathring{W}_2^{1/2}(\Gamma)$ symmetric in $L_2(\Gamma)$ under which the functional $<\psi,B\psi>^{1/2}$ defines the equivalent norms on $W_2^{-1/2}(\Gamma)$.

The results of the paper can be extended to the case where Ω_0 and Ω_1 are not simply connected and even not necessarily connected under the condition that Γ remains a Lipschitz boundary.

The results can also be extended to the case of other boundary conditions on $\partial\Omega$ and to the case of elliptic systems, equations of the elasticity theory and the Stokes problem.

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