CHAPTER 19

Mixed Finite Element Solutions of Second Order Elliptic Problems on Grids with Regular Local Refinement*

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Abstract. A discretization of second-order elliptic problems using rectangular Raviart-Thomas mixed finite elements on grids with local refinement is presented. Two-grid BEPS type preconditioners for the resulting linear system of equations are constructed and studied. The iterations are performed in a subspace after elimination of the velocity unknowns. Numerical experiments that demonstrate the fast convergence of the conjugate gradient method with the constructed preconditioner are presented.

1. Introduction

In many applications, it is desirable to have high accuracy for both the primary unknown and some function of its gradient. Solving for both quantities simultaneously often gives higher-order approximations to variables like fluid velocities, fluxes, or stresses while directly coupling important associated physical properties. Mixed finite element techniques have been applied for this purpose to elliptic or parabolic problems alone [4] or in systems, coupled with other equations [1]. Various theoretical results have been obtained with regard to accuracy [4] and efficient implementation [2,3,8] of these methods.

Often the applications are of sufficiently large scale that local grid refinement techniques are required to resolve important local phenomena. Efficient solution

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techniques for solving the algebraic equations resulting from the mixed methods on locally-refined grids are required. A domain decomposition type of approach is presented to construct efficient algorithms of this type.

A mixed finite element discretization of second-order elliptic problems on locally-refined rectangular grids using Raviart-Thomas elements of arbitrary degree has been proposed in [8]. The case of repeatedly refined meshes has been studied in [11]. This technique uses "slave" nodes in a manner that ensures the necessary continuity of the velocity, satisfies the Babŭska-Brezzi condition, and thus yields a unique solution and an error bound.

In this paper, we consider the linear algebraic system arising from this discretization and construct a BEPS type preconditioner for the reduced Schur composite-grid matrix of the problem for the pressure unknowns. Here, we investigate the BEPS-preconditioner using the algebraic approach from [5]. The main difficulty with the mixed finite element discretization is to show that the explicitly reduced system for the pressure unknowns (in this case we use the lumped mass approximation) can be assembled element by element, so that the corresponding element stiffness matrices are semi-definite and their null-spaces contain just the constant vectors. This fact was verified for the Raviart-Thomas elements of arbitrary order $r \geq 0$ in [4]. This enables us to prove that the Shur complement of the reduced matrix, when eliminating the nodes that were added in the local refinement, is spectrally equivalent to the reduced matrix for the pressure unknowns on the global coarse grid. Then the algebraic theory from [5] applies and shows that the BEPS preconditioner is spectrally equivalent to the composite-grid reduced matrix for the pressure unknowns. We consider this to be the main result of the present paper; to our knowledge, this seems to be a new result.

2. Preliminaries and Mixed Finite Element Discretization

Let $W = L^2(\Omega)$ and

$$\mathbf{V} = H(div;\Omega) = \left\{\mathbf{v} = (v_1, v_2), v_1, v_2 \in L^2(\Omega), div\mathbf{v} \in L^2(\Omega)\right\},$$

where $\Omega = [0,1]^2$. For any $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $w \in W$ we define the following bilinear forms

$$a(\mathbf{u},\mathbf{v}) = (\alpha^{-1}u_1, v_1) + (\alpha^{-1}u_2, v_2), \quad b(\mathbf{u}, w) = (div\ \mathbf{u}, w)\ ,$$

where $\alpha(x,y) \geq \alpha_0 \geq 0$ is a given bounded function in Ω and $(\cdot,\cdot)$ stands for the standard inner product in $L^2(\Omega)$.

We consider the following variational saddle-point problem: find $\mathbf{u} \in \mathbf{V}$ and $p \in W$ such that

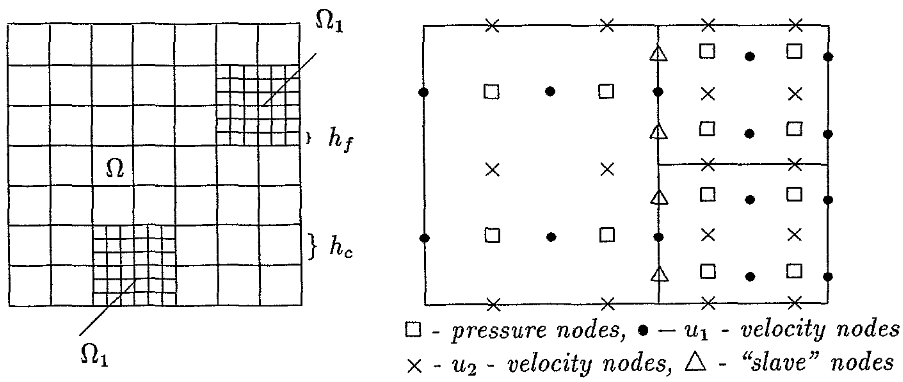
$$a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) + b(\mathbf{u},w) = (f,w) \tag{2.1}$$

for all $\mathbf{v} \in \mathbf{V}$ and $w \in W$, where $f \in L^2(\Omega)$ is a given function.

The problem (2.1) corresponds to the weak formulation of the homogeneous Dirichlet boundary value problem for the equation $div\,(\alpha\ grad\ p) = -f$ split-up as a system $-\alpha\ grad\ p = \mathbf{u}$, $div\ \mathbf{u} = f$. In the following, we call p the pressure and $\mathbf{u}$ the velocity vector. In a similar way, one can consider a nonhomogeneous Dirichlet boundary condition or problem with a mixed boundary condition, when on a part of the boundary, a Neumann boundary condition is specified.

In [8], a finite element approximation to (2.1) on rectangular grids with local refinement was described and studied. The goal of this paper is to investigate the convergence properties of the PCG method for this discretization with the BEPS preconditioner. We first briefly describe the finite element approximation.

Let $\tilde{T}_h$ be the initial coarse partition of Ω into square finite elements of size h_c, and let Ω_1, a subdomain of Ω, be a union of a certain number of coarse finite elements. We partition the elements in Ω_1, introducing a finer mesh as shown in Figure 1. We suppose that the refinement is uniform with a fine-grid step size h_f and is consistent, i.e. any two adjacent elements in Ω_1 have the same partition along their common side. We call this partition of Ω into coarse and fine elements a composite partition (or composite grid) and denote it by T_h.



(a) Grid with regular local refinement *(b) Interface finite elements*

Figure 1. Finite element partitioning of Ω

We denote by $\tilde{W}_h^r$ and $\tilde{\mathbf{V}}_h^r$ the Raviart-Thomas finite element spaces associated with the coarse partition $\tilde{T}_h$ (see, e.g. [8]). Now we construct the composite-grid finite element spaces W_h^r and $\mathbf{V}_h^r$. Since the elements in W_h^r do not require any continuity across the finite element boundaries, an obvious definition of W_h^r is

$$W_h^r = \{w(x,y) : w(x,y)|_e \in Q(r,r), \text{ for any } e \in T_h\} .$$

For the construction of $\mathbf{V}_h^r$, we use the simple idea of enriching the coarse-grid space $\tilde{\mathbf{V}}_h^r$ with some functions which are nonzero only in Ω_1. In the standard FEM, this construction has been described in a compact form by Dryja and Widlund in [10]. For the mixed finite element method, the space $\mathbf{V}_h^r$ has been described in detail in [8], (see also [11]). This construction can be summarized as follows. The space $\mathring{\mathbf{V}}_h^r(\Omega_1)$ consists of vector-functions $\mathbf{v}$ which: (i) vanish outside of Ω_1; (ii) have zero normal component at the boundary of Ω_1; and (iii) have restrictions on Ω_1 that are elements of the Raviart-Thomas space on the fine-grid partition T_h in Ω_1. Then,

$$\mathbf{V}_h^r = \tilde{\mathbf{V}}_h^r + \mathring{\mathbf{V}}_h^r(\Omega_1) .$$

Nodes that will give a nodal basis of $\mathbf{V}_h^r$ at the interface elements with "slave" nodes are shown for $v = 1$ in Figure 1b. In all other elements, the Gauss and Lobatto points are chosen for nodal points (see [8] for a detailed description).

We denote by $a_h(\cdot,\cdot)$ an approximation to $a(\cdot,\cdot)$ using quadrature formulas for evaluating the integrals so that in each finite element we have the same nodes as the nodal basis in $\mathbf{V}_h^r$. For example, in the case $r = 1$, for $(\alpha^{-1}v_1, u_1)$ these are the points marked by "•" and for $(\alpha^{-1}v_2, v_2)$ the points marked by "×" in Figure 1b. These quadratures produce the so-called "lumped mass" approximation of the form $a(\cdot,\cdot)$. Then the finite element approximation to (2.1) is: find $\mathbf{u}_h \in \mathbf{V}_h^r$ and $p_h \in W_h^r$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) + b(\mathbf{u}_h, w) = (f, w), \quad \text{for all } \mathbf{v} \in \mathbf{V}_h^r \text{ and } w \in W_h^r . \tag{2.2}$$

The stability and uniqueness of the discrete solution and the error bounds of this mixed finite element discretization were studied in [8].

If we introduce vector notations U_1, U_2, P for the unknown values of u_{1h}, u_{2h}, p_h, correspondingly, at the nodes and the following matrix notations for the bilinear forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{i=1}^{2} V_i^T M_i U_i, \quad -b(\mathbf{v}_h, p_h) = \sum_{i=1}^{2} V_i^T N_i P, \quad -b(\mathbf{u}_h, w) = \sum_{i=1}^{2} W^T N_i^T U_i ,$$

then the matrix form of (2.2) will be

$$\begin{bmatrix} M_1 & 0 & N_1 \\ 0 & M_2 & N_2 \\ N_1^T & N_2^T & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -F \end{bmatrix} . \tag{2.3}$$

The matrices M_i are block-diagonal with blocks which are diagonal matrices with an exception along the interface between the coarse and fine-grid regions. Then the velocity unknowns in (2.3) can be eliminated explicitly, and we obtain the following system for the pressure unknowns P.

$$AP = (N_1^T M_1^{-1} N_1 + N_2^T M_2^{-1} N_2)P = F . \tag{2.4}$$

The structure of the reduced Schur matrix A will be studied in the next section.

3. BEPS-Preconditioner for the Composite-Grid System

Together with the system (2.4) for the composite-grid partitioning T_h, we consider the FE approximation to the problem (2.4) on the coarse-grid partitioning $\tilde{T}_h$; this leads to the system

$$\tilde{A}\tilde{P} = \tilde{F}, \tag{3.1}$$

where the vector $\tilde{P}$ corresponds to the unknown values of the pressure at the nodal points of the coarse grid $\tilde{T}_h$.

Let us partition the vector P (and similarly $\tilde{P}$) in such a way that the unknowns in the refined region Ω_1 are in the first group and the remaining unknowns are in the second group, i.e.

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{matrix} \} \text{ in } \Omega_1 \\ \} \text{ in } \Omega \backslash \Omega_1 \end{matrix} , \quad \tilde{P} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix} \begin{matrix} \} \text{ in } \Omega_1 \\ \} \text{ in } \Omega \backslash \Omega_1 \end{matrix} , \quad (\text{note, } P_2 = \tilde{P}_2). \tag{3.2}$$

Then the matrices A and $\tilde{A}$ admit the following domain decomposition block (factorized) forms

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}, \; S = A_{22} - A_{21}A_{11}^{-1}A_{12}\,,$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{S} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}, \; \tilde{S} = \tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}. \quad (3.3)$$

Then the two-grid preconditioner B of Bramble, Ewing, Pasciak and Schatz [7] derived algebraically in [5] is defined for the system (2.4) as follows:

$$B = \begin{bmatrix} A_{11} & 0 \\ A_{21} & \tilde{S} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}, \quad (3.4)$$

i.e. in the factorized form of A, we have replaced the Schur complement S by $\tilde{S}$. From the representation (3.4) of B, we see that we have to invert A_{11} (which corresponds to the stiffness matrix in the refined region Ω_1) twice. Inverting $\tilde{S}$ is equivalent to solving with the coarse-grid matrix $\tilde{A}$ and taking $\tilde{P}_2$ of the resulting vector $\tilde{P}$. This is seen from the representation

$$\tilde{A}^{-1} = \begin{bmatrix} * & * \\ * & \tilde{S}^{-1} \end{bmatrix} \begin{matrix} \} & \Omega_1 \\ \} & \Omega\backslash\Omega_1 \end{matrix} \;.$$

Thus, solving the system with the preconditioning matrix B reduces to solving two problems with the pivot block A_{11} and one problem with the coarse-grid matirx $\tilde{A}$. The practical importance of the preconditioner B is based on the following main result.

Theorem 1 *The preconditioner B is spectrally equivalent to the matrix A with constants which do not depend on the mesh sizes h_f and h_c (but possibly depend on the ratio h_f/h_c, on the local jump of the coefficient α in one coarse element and in the degree of the polynominals r).*

Proof. Since

$$B^{-1}A = U^{-1} \begin{bmatrix} I & 0 \\ 0 & \tilde{S}^{-1}S \end{bmatrix} U, \text{ where } U = \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

the eigenvalues of $B^{-1}A$ are either 1 or equal to the eigenvalues of $\tilde{S}^{-1}S$. Then the spectral equivalence of B and A reduces to the spectral equivalence of S and $\tilde{S}$. Using the property of the Schur complement that

$$P_2^T S P_2 = \inf_{P_1} P^T A P, \quad P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \; \} \; \Omega_1$$

and similarly for $\tilde{S}$, the problem reduces to finding constants γ_1 and γ_2 such that

$$\gamma_1 \inf_{\tilde{P}_1} \tilde{P}^T \tilde{A} \tilde{P} \le \inf_{\tilde{P}_1} P^T A P \le \gamma_2 \inf_{\tilde{P}_1} \tilde{P}^T \tilde{A} \tilde{P} .$$

This is the most difficult and technical part of the proof. It is based on the possibility of obtaining the stiffness matrix A employing an element by element assembling procedure and the local analysis technique based on the two-level ordering of the unknowns. The full proof will appear in the Proceedings of the Domain Decomposition Conference, published by the Soviet Academy of Sciences.

4. Numerical Examples

We consider the following model problems for (2.1):

Problem 1 (with smooth coefficient $\alpha(x)$ and localized solution $u(x)$):

$$\alpha(x) = 1/\left[1 + 10(x_1^2 + x_2^2)\right], \quad u(x) = \phi(x_1)\phi(x_2),$$

$$\phi(t) = \begin{cases} \sin^2 \pi \dfrac{t - 0.875}{0.125} & , \quad t \in (0.875, 1), \\ 0 & , \quad t \in (0, 0.875). \end{cases}$$

Problem 2 (with piecewise constant coefficient $\alpha(x)$ and piecewise smooth solution $u(x)$):

$$\alpha(x) = \begin{cases} 100, & x_1, x_2 > 0.875, \\ 1, & \text{otherwise}, \end{cases}$$

$$u(x) = \frac{(x_1 - 0.875)(x_2 - 0.875)}{\alpha(x)} \sin \frac{\pi}{2} x_1 \sin \frac{\pi}{x} x_2$$

We solve these problems on a grid with n_c^2 initial coarse-grid points with a regular local refinement in the subdomain $\Omega_1 = \{(x_1, x_2), x_1, x_2 > 0.75\} \subset \Omega$ with $h_c/h_f = 3, 5$ and 7. We apply the preconditioned conjugate gradient method using the BEPS preconditioner described above. The stopping criterion is

$$R^T R < \epsilon, \quad \text{with } \epsilon = 10^{-18} ,$$

where $R = F - AP$ is the residual vector and P is the current iterate. As an initial guess, we choose a piecewise constant interpolant of the coarse-grid solution $\tilde{P} = \tilde{A}^{-1}\tilde{F}$.

In Table 1 we present the numerical experiments for the two problems formulated above for the lowest-order Raviart-Thomas finite elements, $r = 0$. We report the number of unknowns N in the reduced system for P, the aspect ratio h_c/h_f, the number of iterations *iter* for achieving the required accuracy ϵ, and the average reduction factor

$$\rho = (\Delta/\Delta_0)^{1/iter}, \Delta = (R^T R)^{1/2}, \Delta_0 = (R_0^T R_0)^{1/2} ,$$

where R_0 is the residual of the initial guess and R is the residual of the last iterate that satisfies the stopping criterion.

Table 1

n_c	h_c/h_f	Problem 1		Problem 2		N
		iter	ρ	iter	ρ	
12	3	3	0.10×10^{-2}	4	0.18×10^{-2}	272
	5	3	0.12×10^{-2}	4	0.15×10^{-2}	528
	7	3	0.12×10^{-2}	4	0.22×10^{-2}	912
24	3	4	0.39×10^{-2}	4	0.69×10^{-2}	864
	5	4	0.44×10^{-2}	4	0.74×10^{-2}	1440
	7	4	0.46×10^{-2}	4	0.73×10^{-2}	2304
48	3	4	0.41×10^{-2}	4	0.95×10^{-2}	3456
	5	4	0.45×10^{-2}	5	0.11×10^{-2}	5760
	7	4	0.45×10^{-2}	5	0.11×10^{-2}	9216

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