CHAPTER 4

On Some Modern Approaches to Constructing Spectrally Equivalent Grid Operators

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Abstract. A finite element grid system $L_h u_h = f_h$ ($L_h = L_h^* > 0$) arises as a result of discretization of the mixed boundary value problem in a bounded d-dimensional domain Ω ($d \geqslant 2$) for the steady-state equation of heat conduction with the piecewise-constant thermal conductivity. The considered p-level triangulation $T^{(p)}(\Omega)$ is obtained due to the recurrent refinement of the zero-level triangulation $T^{(0)}(\Omega)$. We suggest constructions of the multigrid preconditioner B_h spectrally equivalent to the operator L_h and such that the solution to the system of type $B_h v_h = g_h$ can be obtained in, at most, KN_h arithmetic operations, N_h is the order of the system to be solved. It is especially valuable that the equivalence estimates $\delta_1 \geqslant \delta_0 > 0$ from the inequalities

$$\delta_0 B_h \leqslant L_h \leqslant \delta_1 B_h$$

lead to the condition number $\delta \equiv \delta_1 \delta_0^{-1}$ close to unity even for the case of strongly varying thermal conductivity ($\delta = 1.5$ is typical of $d = 3$ and the cubic grid). We use ideas of the grid domain decomposition and the splitting of the finite element space into a direct sum of subspaces. Possible generalizations are indicated including the case of $L_h \geqslant 0$, $B_h \geqslant 0$, and also the case of boundary value problems on two-dimensional manifolds with complex geometry (for example, on the surface of closed polyhedron in the three-dimensional space).

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1. Introduction. Modern constructions of preconditioners $B_h \equiv B = B^* > 0$ [1-3] lead for d-dimensional problems with $d = 2$ and $d = 3$ to the estimates

$$\delta_0 B \leqslant L \leqslant \delta_1 B \tag{1.1}$$

($L_h \equiv L = L^* > 0$ is the original grid elliptic operator) with condition numbers $\delta \equiv \delta_1 \delta_0^{-1}$ independent of the grid and close to unity even for problems with strongly varying coefficients. The approach to be developed below gives required constructions of operators B also for the general case $d \geqslant 2$ due to combining grid domain decomposition (finite element space splittings) and the theory of two-stage iterative methods [2].

Essential are also geometrical aspects of partitioning of the d-dimensional simplex into 2^d congruent constituent simplexes with edges reduced by half (one of them can match the other by an isometric mapping).

The following notation is introduced here: H is an Euclidean space; $\mathscr{L}(H)$ is a linear normalized space of linear operators mapping H into H; $\mathscr{L}^+(H) \equiv \{B \mid B \in \mathscr{L}(H), B = B^* > 0\}$; $H(B)$ is an Euclidean space with the scalar product $(u,v)_B \equiv (Bu,v)$, $\|u\|_B \equiv (u,u)_B^{1/2}$, $\|A\|_B \equiv \max_{\|u\|_B = 1}\{\|Au\|_B\}$, I is an identity operator.

2. Original grid problem. Let Ω be a bounded domain from $\mathbb{R}^d$ with the given triangulation $T^{(0)}(\Omega)$ composed of a finite number of simplexes $T_0 \in T^{(0)}(\Omega)$; Γ is the boundary of Ω; $[\Omega] \equiv \Omega \cup \Gamma$, $\Gamma_0 = [\Gamma_0]$ consists of a finite number of $(d-1)$-dimensional faces of simplexes $T_0 \in T^{(0)}(\Omega)$; $\Gamma_0 \subset \Gamma$; $x \equiv [x_1,\ldots,x_d]$, $(u,v)_{0,\Omega} \equiv \int_\Omega u(x)v(x)\,d\Omega$. Let us consider the Hilbert spaces $V_1 \equiv W_2^1(\Omega;\Gamma_0)$ and G [2]:

$$(u,v)_{1,\Omega} \equiv (1,\nabla u \cdot \nabla v)_{0,\Omega}, \qquad (u,v)_G \equiv (a(x),\nabla u \cdot \nabla v)_{0,\Omega} \tag{2.1}$$

$a(x) = a(T_0)$ for $x \in T_0$, $a(T_0)$ is a constant, $\|u\|_G \equiv \|u\|$.

The $(l+1)$-level triangulation $T^{(l+1)}(\Omega)$ is constructed by using $T^{(l)}(\Omega)$ due to the refinement of $T^{(l)}(\Omega)$ by half [the simplex $T_l \in T^{(l)}(\Omega)$ is partitioned into 2^d simpexes $T_{l+1} \in T^{(l)}(\Omega)$, $l = 0,\ldots,p-1$].

Let $\Omega^{(l)}$ be a set of vertices $P_i^{(l)}$ of simplexes T_l, which do not belong to Γ_0 and each vertex (node) $P_i^{(l)}$ be in correspondence with the standard basis continuous piecewise-linear function $\hat{\psi}_i^{(l)}(x)$: $\hat{\psi}_i^{(l)}(P_i^{(l)}) = 1$, $\hat{\psi}_i^{(l)} = 0$ in the remaining nodes of $\Omega^{(l)}$, $\hat{\psi}_i^{(l)}(x)$ is linear on each $T_l \in T^{(l)}(\Omega)$. Let

$$\hat{G}^{(l)} \equiv \left\{ \hat{u}(x) \,\middle|\, \hat{u}(x) = \sum_{P_i^{(l)} \in \Omega^{(l)}} u_i \hat{\psi}_i^{(l)}(x) \right\}, \qquad l = 0,\ldots,p \tag{2.2}$$

N_{l+1} is the number of nodes in $\Omega^{(l+1)}$, $N_{l+1} = N_l + N_l^{(1)}$, $\mathbb{R}^{N_{l+1}} \equiv H^{(l+1)} = H_1^{(l+1)} \times H_2^{(l+1)}$, $H_2^{(l+1)} = H^{(l)}$, $\underline{u}_{l+1} = \{u_i\} \in H^{(l+1)}$, $\underline{u}_{l+1} = [\underline{u}_1^{(l+1)}, \underline{u}_2^{(l+1)}]^{\mathrm{T}}$,

$\underline{u}_i^{(l+1)} \in H_i^{(l+1)}$, $i = 1,2$. Along with the basis $\{\hat{\psi}_i^{(l+1)}(x)\}$ for $\hat{G}^{(l+1)}$ consider the basis $\{\bar{\psi}_i^{(l+1)}(x)\}$ with $\bar{\psi}_i^{(l+1)} = \hat{\psi}_i^{(l+1)}$ for $P_i^{(l+1)} \in \Omega^{(l+1)} \backslash \Omega^{(l)}$ and $\bar{\psi}_i^{(l+1)} = \hat{\psi}_i^{(l)}$ for $P_i^{(l+1)} \in \Omega^{(l)}$ assuming the numbers of nodes from $\Omega^{(l+1)} \backslash \Omega^{(l)}$ to be less than those from $\Omega^{(l)}$. The indicated choice of the basis leads to the splitting

$$\hat{G}^{(l+1)} = \hat{G}_1^{(l+1)} \oplus \hat{G}_2^{(l+1)} \subset G, \quad l = 0,\dots,p-1 \tag{2.3}$$

$$\hat{G}_2^{(l+1)} = \hat{G}^{(l)}, \quad \hat{G}_1^{(l+1)} \equiv \{\hat{u} \mid \hat{u} \in \hat{G}^{(l+1)}, \hat{u}(P_i^{(l)}) = 0 \ \forall P_i^{(l)} \in \Omega^{(l)}\}. \tag{2.4}$$

The Gramm matrices for these bases take the form

$$L^{(l+1)} = \begin{bmatrix} L_{11}^{(l+1)} & L_{12}^{(l+1)} \\ L_{21}^{(l+1)} & L_{22}^{(l+1)} \end{bmatrix}, \quad \bar{L}^{(l+1)} = \begin{bmatrix} \bar{L}_{11}^{(l+1)} & \bar{L}_{12}^{(l+1)} \\ \bar{L}_{21}^{(l+1)} & \bar{L}_{22}^{(l+1)} \end{bmatrix} \tag{2.5}$$

$$\bar{L}_{11}^{(l+1)} = L_{11}^{(l+1)}, \quad \bar{L}_{22}^{(l+1)} = L^{(l)}.$$

Note that

$$(L^{(l+1)}\underline{u}_{l+1}, \underline{u}_{l+1}) = \|\hat{u}^{(l+1)}\|^2 = (\bar{L}^{(l+1)}\underline{v}_{l+1}, \underline{v}_{l+1}), \quad \hat{u}^{(l+1)} \in \hat{G}^{(l+1)}. \tag{2.6}$$

3. Schur matrices and angles of subspaces. Let $S_2(L^{(l+1)}) \equiv L_{22}^{(l+1)} - L_{21}^{(l+1)} \times (L_{11}^{(l+1)})^{-1} L_{12}^{(l+1)}$ be the Schur matrix for $L^{(l+1)}$; the angle $\alpha_{l+1} \in [0,\pi/2]$ between $\hat{G}_1^{(l+1)}$ and $\hat{G}_2^{(l+1)}$ from (2.3) is determined by the inequality [2]

$$|(\hat{u}_1^{(l+1)}, \hat{u}_2^{(l+1)})| \leqslant \cos\alpha_{l+1} \|\hat{u}_1^{(l+1)}\| \, \|\hat{u}_2^{(l+1)}\|, \quad \hat{u}_i^{(l+1)} \in \hat{G}_i^{(l+1)}, \quad i = 1,2 \tag{3.1}$$

if $d = 2$, then $\cos^2\alpha_{l+1} < 3/4$ [1]. In 1978 the author [2] published

Lemma 3.1. The Schur matrix $S_2(\bar{L}^{(l+1)})$ is the Gramm matrix for elements $\varphi_1,\dots,\varphi_{N_l}$ with $\varphi_i \equiv \bar{\psi}_i^{(l+1)} - P_1\bar{\psi}_i^{(l+1)}$, where P_1 is the orthoprojector in G onto $\hat{G}_1^{(l+1)}$.

Lemma 3.1 (see also [1]) implies that $S_2(L^{(l+1)}) = S_2(\bar{L}^{(l+1)})$;

$$(S_2(L^{(l+1)})\underline{u}_2^{(l+1)}, \underline{u}_2^{(l+1)}) = \|(I - P_1)\hat{u}_2^{(l+1)}\|^2 \equiv X$$

$$\hat{u}_2^{(l+1)} \equiv \sum_{P_i^{(l)} \in \Omega^{(l)}} u_{2,i}^{(l+1)} \hat{\psi}_i^{(l)} \in \hat{G}_2^{(l+1)} \tag{3.2}$$

$$s^2 \|\hat{u}_2^{(l+1)}\|^2 \leqslant X \leqslant \|\hat{u}_2^{(l+1)}\|^2, \quad s^2 \leqslant \sin^2\alpha_{l+1}$$

$$s^2 L^{(l)} \leqslant S_2(L^{(l+1)}) = S_2(\bar{L}^{(l+1)}) \leqslant L^{(l)}.$$

Theorem 3.1. Let all simplexes T_0 from $T^{(0)}(\Omega)$ be standard parts of the partitions of some cubes into $d!$ congruent simplexes [2]. Then the angles α_{l+1} [see (3.1)] satisfy the estimates

$$\cos \alpha_{l+1} \leqslant \gamma \equiv (d2^{-d})^{1/2}, \quad s^2 = 1 - d2^{-d} > 0.$$

4. Two-stage iterative methods. The following theorem [2] is well known:

Theorem 4.1. Let $\Lambda \in \mathscr{L}^+(H)$ and an iterative method be known for solving systems $\Lambda \underline{v} = \underline{g}$ which leads in k iterations to the relation $\underline{v}^k - \underline{v} = Z_k(\underline{v}^0 - \underline{v})$ with the operator of reduction of the error $Z_k = Z$, which is symmetric in $\mathscr{L}(H(\Lambda))$ and such that $\|Z\|_\Lambda \leqslant q < 1$. Then,

$$B \equiv \Lambda(I - Z)^{-1} \in \mathscr{L}^+(H), \quad (1 - q)B \leqslant \Lambda \leqslant (1 + q)B \tag{4.1}$$

if also $Z \geqslant 0$ in $\mathscr{L}(H(\Lambda))$, then $\Lambda \leqslant B$; the solution of the system $B\underline{w} = \underline{g}$ coincides with the kth iteration $\underline{v}^k$ in the indicated iterative method for $\underline{v}^0 = 0$.

Theorem 4.2 [1,2]. Let the iterative method

$$\Lambda(\underline{v}^{n+1} - \underline{v}^n) = -\tau_n(\Lambda \underline{v}^n - \underline{g}), \quad n = 0,\ldots,k-1 \tag{4.2}$$

be considered where $A \in \mathscr{L}^+(H)$, $\Lambda \in \mathscr{L}^+(H)$, $\sigma_0 A \leqslant \Lambda \leqslant \sigma_1 A$, $\sigma_0 > 0$, $\sigma \equiv \sigma_1\sigma_0^{-1}$, $\{\tau_n^{-1}\} = \{t_i\}$ or $\{\tau_n^{-1}\} = \{t_i^+\}$, $i = 0,\ldots,k-1$, $t_i \equiv \varphi(\cos \pi(2i+1)/(2k))$, $t_i^+ \equiv \varphi(\cos \pi(2i+1)/k)$, $\varphi(t) \equiv \frac{1}{2}[\sigma_1 + \sigma_0 + (\sigma_1 - \sigma_0)t]$. Then the operator $Z \equiv (I - \tau_0 A^{-1}\Lambda)\ldots(I - \tau_{k-1}A^{-1}\Lambda)$ is symmetric in $\mathscr{L}(H(\Lambda))$, $\|Z\|_\Lambda \leqslant q_k$ (in case of $\{t_i\}$), $0 \leqslant Z \leqslant q_k^+ I$ (in case of $\{t_i^+\}$), $q_k \equiv [T_k((\sigma + 1)/(\sigma - 1))]^{-1}$, $q_k^+ \equiv 2q_k/(1 + q_k)$, $(1 + q_k)/(1 - q_k) = 1/(1 - q_k^+)$, $T_k(\lambda)$ is the Tchebyshev polynomial of the kth degree.

5. Construction of multigrid operator B. The following lemma is valid:

Lemma 5.1. There exist constants $\sigma_{11} \geqslant \sigma_{01} > 0$ defined only by the geometry of the simplexes T_0 such that for the level $l + 1 = 1,\ldots,p$ there exists a diagonal matrix $A_{11}^{(l+1)} \in \mathscr{L}(H_1^{(l+1)})$, and additionally $\sigma_{01}A_{11}^{(l+1)} \leqslant L_{11}^{(l+1)} \leqslant \sigma_{11}A_{11}^{(l+1)}$.

Making use of Theorem 5.1 for iterative method (4.2) with $A = A_{11}^{(l+1)}$, $\Lambda = L_{11}^{(l+1)}$, $\sigma_0 = \sigma_{01}$, $\sigma_1 = \sigma_{11}$, $k_1 = k_1$ and $\{t_i^+\}$, we obtain the operator

$$B_{11}^{(l+1)} \equiv L_{11}^{(l+1)}(I_1 - Z_1^+)^{-1}, \; 0 \leqslant Z_1^+ \equiv Z_{1,k_1} \equiv Z \leqslant q_{k_1}^+ I_1 \text{ in } H_1^{(l+1)}(L_{11}^{(l+1)})$$

where $q_{k_1}^+$ is small for large k_1. Note that the indicated iterations permit obvious parallelizing; a more complicated choice of $A_{11}^{(l+1)}$ is possible especially for square grids which increases the convergence rate.

Lemma 5.2. Let $B^{(l)} \in \mathscr{L}^+(H^{(l)})$, $\sigma_0^{(l)} B^{(l)} \leqslant L^{(l)} \leqslant \sigma_1^{(l)} B^{(l)}$, $0 < \sigma_0^{(l)} \leqslant 1 \leqslant \sigma_1^{(l)}$, $\sigma^{(l)} \equiv \sigma_1^{(l)} (\sigma_0^{(l)})^{-1}$ and let $\sin^2 \alpha_{l+1} \geqslant s^2 > 0$. Then,

$$s^2 \sigma_0^{(l)} B^{(l)} \leqslant L_{22}^{(l+1)} - L_{21}^{(l+1)} (B_{11}^{(l+1)})^{-1} L_{12}^{(l+1)} \equiv S_2(R^{(l+1)}) \leqslant \sigma_1^{(l)} B^{(l)}.$$

Consider iterative method (4.2) with $A = S_2(R^{(l+1)})$, $A = B^{(l)}$, $\sigma_0 s^2 \sigma_0^{(l)} \leqslant 1$, $\sigma_1 = \sigma_1^{(l)} \geqslant 1$, $k = k_2$ and parameters $\{t_i\}$. By Theorem 4.2 we have $\|Z_2\|_A \leqslant q_{k_2}$, $Z_2 \equiv Z_{2,k_2} \equiv Z$. Therefore,

$$B^{(l+1)} = \begin{bmatrix} B_{11}^{(l+1)} & L_{12}^{(l+1)} \\ L_{21}^{(l+1)} & S_2(R^{(l+1)})(I_2 - Z_2)^{-1} + L_{21}^{(l+1)} (B_{11}^{(l+1)})^{-1} L_{12}^{(l+1)} \end{bmatrix} \tag{5.1}$$

$$B^{(l+1)} \in \mathscr{L}^+(H^{(l+1)}).$$

Theorem 5.1. If the hypotheses of Lemma 5.2 are satisfied, we have

$$\sigma_0^{(l+1)} B^{(l+1)} \leqslant L^{(l+1)} \leqslant \sigma_1^{(l+1)} B^{(l+1)}$$

$$\sigma_0^{(l+1)} \equiv (1 - q_{k_2})(1 + \xi_1)^{-1}, \quad \xi \equiv \frac{1}{1 - \cos^2 \alpha_{l+1}} \cdot \frac{q_{k_1}^+}{1 - q_{k_1}^+}, \quad \sigma_1^{(l+1)} \equiv 1 + q_{k_2}$$

$$\sigma^{(l+1)} = f(\sigma^{(l)}), \quad f(t) \equiv (1 + \xi_1) \left[1 + 2T_{k_2} \left[\frac{t + s^2}{t - s^2} \right] \right]^{-1}, \quad t \geqslant 1. \tag{5.2}$$

Theorem 5.2. Let $s^2 > 0$, the equation $t = f(t)$ [13] have the solution $t^* > 1$ and the operator $B \equiv B^{(p)}$ be constructed by using (5.1) for $l = 0, \ldots, p-1$; $0 < \sigma_0^{(0)} \leqslant 1 \leqslant \sigma_1^{(0)}$, $\sigma^{(0)} \leqslant t^*$. Then inequalities (1.1) with $\delta_1/\delta_0 = t^* = \delta$ are valid for the operators $L^{(p)} \equiv L$ and B.

It is obvious that t^* exists for large k_2. The case $k_2 \leqslant 2^d - 1$ where the determination of $(B^{(l)})^{-1} g_l$ requires $K_0 N_l$ or less arithmetic operations is important. If $s^2 \geqslant 5/8$, then $k_2 = 2$ and $k_1 = 8$ yields $\delta = 1.4$ (similar $B^{(l)}$ from [3] for $d = 3$ and the cubic grid require $k_2 = 4$, $\delta = 6.6$). For $d = 2$ the general case $s^2 > 1/4$ is related to $k_2 = 2$; if the grid is square, then $\delta = 1.074$ for $k_2 = 3$, $k_1 = 3$ and $\delta = 1.4$ for $k_2 = 2$, $k_1 = 3$.

6. Possible generalizations. For local grid refinement the only difference is related to the fact that the vertices $P_i^{(l+1)}$ lying on the common boundary of the domains with the old and new grids must not belong to $\Omega^{(l+1)}$. We have obtained generalizations for the case of $\Gamma_0 = \varnothing$ as well, and also for non-local boundary conditions. Generalizations are considered for problems on two-dimensional manifolds with the complex geometry (for example, for a finite number of polygons lying on different planes which do not have any common sides) and also for systems of elasticity theory.

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