CHAPTER 31

On a Parallel Schwarz Algorithm for Symmetric Strongly Elliptic Integral Equations

F. K. Hebeker*

Abstract: In this note we introduce and analyze theoretically a parallel Schwarz alternating algorithm to treat symmetric strongly elliptic integral equations. The range of applications as well as the extension to a multilevel version is briefly sketched.

1. INTRODUCTION

The principle of domain decomposition has shown up as a versatile and efficient tool to create parallelism for solving mathematical problems of science and engineering on parallel computers. But in spite of the fact that integral equations form a particularly advantageous approach to use modern computer architectures, the mathematical analysis of parallel versions is just at its beginning ([2]; concerning vectorized integral equation codes see [7] , [6] , for example). As Wendland [14] pointed out, the Poincare-Steklov operator as one of the crucial problems of domain-decomposition methods for certain partial differential equations in fact is represented just in terms of boundary integral equations and, consequently, may be evaluated using the boundary element method. In any case, the problem of solving efficiently integral equations by parallel algorithms appears and will be addressed to by the present note. We intend to parallelize the

*Institute of Supercomputing and Applied Mathematics, IBM Scientific Center, Tiergartenstraße 15, D-6900 Heidelberg.

solution procedure of integral equations "at its roots", which means that parallelism is created on the continuous rather than on linear algebra level.

We investigate a parallel Schwarz method for symmetric integral equations on manifolds with or without boundary. A corresponding algorithm has been introduced by Lions [9] in case of partial differential equations. Note that for integral equations, in contrast to differential equations, parallelism can not easily be generated by the common "red-black-ordering" [10] , due to the nonlocal character of integral operators! The boundary integral equations dealt with in this paper are furthermore assumed as strongly elliptic, so that Galerkin type boundary element methods are quasioptimal, as pointed out by Hsiao and Wendland [8] and investigated by them in a series of papers (see [13] , e.g.). In case of manifolds with boundaries, see [15] and [11] , for instance. Many integral equations appearing in the applications fall under the present analysis.

On the other hand, Lions [9] developed a mathematical framework to analyze the Schwarz method for partial differential equations. In the present note we extend this approach to integral equations, boundary elements, and a new multilevel version.

This paper is organized as follows: in Sect. 2, boundary integral operators are introduced as specific pseudodifferential operators, and their well known basic analytical properties are briefly summarized. We will study boundary elements and their numerical properties in Sect. 3. Sect. 4 then is devoted to define a basic two-level Schwarz algorithm for integral equations and to state a convergence theorem, the proof of it is given in Sect. 5. The paper is finished with some concluding remarks concerning applications and possible extensions of the present analysis, particularly to a multilevel version which is parallel of high degree.

2. BOUNDARY INTEGRAL EQUATIONS

Let $\Omega \subset \mathbb{R}^3$ be a smoothly bounded finite open domain. The solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega , \quad u = g \text{ on } \partial\Omega \qquad (2.1)$$

may be obtained in terms of a simple layer potential

$$u(x) = (S\psi)(x) = \int_{\partial\Omega} \frac{1}{4\pi\,|x-y|}\,\psi(y)\mathrm{d}o_y \,, \tag{2.2}$$

the surface source of which solves the boundary integral equation

$$S\psi = g \quad \text{on } \partial\Omega \,. \tag{2.3}$$

In this paper we consider more general boundary integral equations

$$A\psi = g \quad \text{on } M \,, \tag{2.4}$$

where

$$(A\psi)(x) = \int_M k(x,y)\psi(y)\mathrm{d}o_y \,. \tag{2.5}$$

Here denotes M a sufficiently smooth closed or open manifold of dimension *d*.

Assumption 1.: ψ, g are real p-vectors, k is a symmetric $p \times p$ -matrix, $p \geq 1$, leading to a *symmetric* system of (real) pseudodifferential operators $A : V \to V'$, where V' is the dual space of V (and $<.,.>$ the duality pairing between V and V') . A is further assumed as *strongly elliptic* of order $2\alpha \in \mathbb{R}$ (in the sense of Hsiao and Wendland [8]), which means that there exists a matrix $\theta(x)$ and a positive constant $\gamma > 0$ such that Garding's inequality

$$<\theta A\phi, \phi> \;\geq\; \gamma\|\phi\|_V^2 - |\kappa(\phi,\phi)| \,, \quad \phi \in V \,, \tag{2.6}$$

holds, with a compact bilinearform κ on the "energy space" V of the operator system A .

In practice, $V = H^\alpha(M)$ or $V = \tilde{H}^\alpha(M)$ in case of M closed or open manifold, resp. Here denotes $H^\alpha(M)$ the usual Sobolev-Slobodetski spaces, and

$$\tilde{H}^\alpha(M) = \{\phi \in H^\alpha(M) \colon \mathit{supp}\ \phi \subset M\}.$$

In the context of boundary integral equations, the $\tilde{H}$ – spaces have been used by [15] and further for screen problems by Stephan (e.g. [11]). The strong ellipticity implies

$$A : V \to V' \quad \text{continuously,} \tag{2.7}$$

where $V' = H^{-\alpha}(M)$. Consequently, A is Fredholm of index zero. We meet further the

Assumption 2.: a) $\theta = 1$;
b) A is definite (i.e. $< A\phi, \phi > = 0$ iff $\phi = 0$), so that Garding's inequality is even strengthened to coerciveness:

$$< A\phi, \phi > \;\geq\; \gamma \|\phi\|_V^2, \quad \text{all } \phi \in V. \tag{2.8}$$

Consequently, $A: V \to V'$ isomorphically. Symmetry of A then implies that

$$a(\phi, \psi) \;=\; < A\phi, \psi > \tag{2.9}$$

is a scalar product on V, inducing the energy norm

$$|\,\phi\,| \;=\; \sqrt{a(\phi, \phi)}\,, \tag{2.10}$$

which by (2.7), (2.8) is equivalent to the norm $\|.\|_V$. For instance, the simple layer operator S is a definite and symmetric strongly elliptic pseudodifferential operator or order -1 , hence $V = H^{-1/2}(M)$ or $V = \tilde{H}^{-1/2}(M)$ (the latter if M is open). Further integral operators satisfying the hypotheses of this section will be treated in Sect. 6.

3. BOUNDARY ELEMENT METHOD

Let $V_h = S_h^{l,m}(M)$ denote a finite element system in the sense of Babuska and Aziz [1] , transplanted to the manifold M [13] . Choose $l > m$ such that

$$V_h \subset H^m(M) \cap \tilde{H}^a(M) \subset V\,,$$

where $a = [\alpha] + 1$, and $h > 0$ denotes the mesh size of an appropriate partition of M. Usually, this family V_h of boundary elements spaces is assumed to possess the "convergence and inverse properties", (e.g. [8]) as well, but this aspect is not stressed in the present paper. Rather we require an extension theorem of boundary elements which follows immediately from a corresponding theorem for finite elements due to Widlund [16] .

Extension Theorem: Let $M' \subset\subset M'' \subset M$ denote open submanifolds, and $V_h' \subset H^\alpha(M')$, $V_h'' \subset \tilde{H}^\alpha(M'')$ denote the corresponding boundary element spaces of the same degree as V_h. Then, for sufficiently small $h > 0$, there exist linear extension operators

$$E_h : V_h' \to V_h'',$$

which are uniformly bounded w.r.t. h in the norm of $H^\alpha(M)$.

Operators of this kind are constructed (following Widlund) by composing Calderon's extension theorem, mollifying, interpolating, patching together the element constructed so far and the original one, and finally interpolating again. The original result from [16] is formulated in terms of the spaces $H^m(\Omega)$, m nonnegative integer, but is easily transplanted to the boundary and generalized to the negative-order spaces and, by use of interpolation theory, to the intermediate spaces as well.

The finite-dimensional spaces V_h serve to establish a Galerkin-type boundary element method to solve (2.4) numerically: find $\psi_h \in V_h$ such that

$$a(\psi_h, \chi) = \langle g, \chi \rangle \quad \text{for all } \chi \in V_h. \tag{3.2}$$

ψ_h is called the Galerkin approximate of ψ. Cea's lemma then implies quasioptimality of this boundary element method:

$$| \psi_h - \psi | \leq const. \inf_{\chi \in V_h} | \chi - \psi | . \tag{3.3}$$

This error term has been further estimated by [8] or [11] in case of manifolds without or with boundary, resp. Note that, in the latter case, due to the severe edge singularity, the present approach has to be further improved if satisfactory convergence properties were expected [11] .

4. SCHWARZ ALTERNATING METHOD

The major task to parallelize the algorithm (3.2) is solving the linear system, whereas the evaluation of the integrals is parallelized straightforward. In the present paper we intend to parallelize (3.2) "at its roots", namely on the level of integral equations rather than on the level of linear algebra.

In this section, we investigate the case of covering by two sets. Covering by more than two sets will be postponed to Sect. 6. Let

$$M = M_1 \cup M_2 \tag{4.1}$$

be an open covering of M , moreover, we further meet the

Assumption 3.: M_1 and M_2 overlap *uniformly,* that means

$$\begin{gathered} |x_1 - x_2| \geq \delta_0 > 0 \\ \text{for all } x_1 \in \partial M_1 - \partial M,\ x_2 \in \partial M_2 - \partial M. \end{gathered} \tag{4.2}$$

Let ζ_1, ζ_2 denote a partition of unity on M w.r.t. M_1, M_2, with this specific property:

$$\begin{gathered} |x_i - y_i| \geq \delta_1 > 0 \\ \text{for all } x_i \in \partial(supp\, \zeta_i) - \partial M,\ y_i \in \partial M_i - \partial M. \end{gathered} \tag{4.3}$$

This partition of unity serves to prove the decomposition

$$V = V_1 + V_2, \tag{4.4}$$

where $V_i = \tilde{H}^\alpha(M_i)$. Here are functions of V_i tacitly assumed to be extended to M by zero, which is consistently possible by definition of the $\tilde{H}$-spaces.

An analogous decomposition holds on the discrete level:

$$V_h = V_{h1} + V_{h2}, \tag{4.5}$$

with

$$V_{hi} = V_h \cap \tilde{H}^\alpha(M_i), \quad i = 1,2. \tag{4.6}$$

Schwarz algorithm is defined now as follows:

1. Start with any $\psi_{h1}^{(0)}, \psi_{h2}^{(0)} \in V_h$. Set $k = 1$.
   Solve now four discrete screen problems:

2. Find $\tilde{\eta}_{h1} \in V_{h1}$ so that for all $\chi_{h1} \in V_{h1}$:

$$a(\tilde{\eta}_{h1}, \chi_{h1}) \;=\; <g, \chi_{h1}> - \,a(\psi_{h2}^{(k-1)}, \chi_{h1}).$$

   Set $\tilde{\psi}_{h1} = \tilde{\eta}_{h1} + \psi_{h2}^{(k-1)}$ on M_1, and $\tilde{\psi}_{h1} = \psi_{h2}^{(k-1)}$ on $M - M_1$.

3. Find $\tilde{\eta}_{h2} \in V_{h2}$ so that for all $\chi_{h2} \in V_{h2}$:

$$a(\tilde{\eta}_{h2}, \chi_{h2}) \;=\; <g, \chi_{h2}> - \,a(\psi_{h1}^{(k-1)}, \chi_{h2}).$$

   Set $\tilde{\psi}_{h2} = \tilde{\eta}_{h2} + \psi_{h1}^{(k-1)}$ on M_2, and $\tilde{\psi}_{h2} = \psi_{h1}^{(k-1)}$ on $M - M_2$.

4. Find $\eta_{h1}^{(k)} \in V_{h1}$ so that for all $\chi_{h1} \in V_{h1}$:

$$a(\eta_{h1}^{(k)}, \chi_{h1}) \;=\; <g, \chi_{h1}> - \,a(\tilde{\psi}_{h2}, \chi_{h1}).$$

   Set $\psi_{h1}^{(k)} = \eta_{h1}^{(k)} + \tilde{\psi}_{h2}$ on M_1, and $\psi_{h1}^{(k)} = \tilde{\psi}_{h2}$ on $M - M_1$.

5. Find $\eta_{h2}^{(k)} \in V_{h2}$ so that for all $\chi_{h2} \in V_{h2}$:

$$a(\eta_{h2}^{(k)}, \chi_{h2}) \;=\; <g, \chi_{h2}> - \,a(\tilde{\psi}_{h1}, \chi_{h2}).$$

   Set $\psi_{h2}^{(k)} = \eta_{h2}^{(k)} + \tilde{\psi}_{h1}$ on M_2, and $\psi_{h2}^{(k)} = \tilde{\psi}_{h1}$ on $M - M_2$.

6. Set $k = k + 1$. Return to Step 2.

Hence Schwarz algorithm reduces solving (3.2) on V_h to computing in parallel two (alternating) independent sequences $(\psi_{h1}^{(k)})$, $(\psi_{h2}^{(k)})$ of smaller problems on V_{h1} or V_{h2}. This basic algorithm using two subdomains will be generalized up to more subdomains in Sect. 6. We remark that the integrals on the right hand side $a(\psi_{h2}^{(k-1)}, \chi_{h1}), \ldots$ may be rapidly evaluated by use of the panel clustering method by Hackbusch and Novak [3] . For a variant, based on weighted quadrature formulae, cf. Volk [12].

Concerning this algorithm we have the

Theorem: *For sufficiently small fixed h the method converges with convergence rate* ρ_h,

$$\rho_h \le \bar{\rho} < 1\,, \tag{4.7}$$

$\bar{\rho}$ *not depending on h. Consequently, the error estimate*

$$\|\psi_{hi}^{(k)} - \psi\|_\alpha \le const.\left(\bar{\rho}^k \|\psi_{hi}^{(0)} - \psi_h\|_\alpha + \inf_{\chi \in V_h} \|\chi - \psi\|_\alpha\right) \tag{4.8}$$

holds, for $i = 1,2$.

We will prove this result over the subsequent section by extending some ideas by Lions [9] to integral equations and boundary elements. It suffices to show that each of the alternating sequences defined above converges.

5. PROOF OF THE THEOREM

As it is easily seen,

$$\begin{aligned} \tilde{\psi}_{h1} - \psi_{h1}^{(k-1)} &= P_{h1}(\psi_h - \psi_{h1}^{(k-1)}) \\ \psi_{h1}^{(k)} - \tilde{\psi}_{h1} &= P_{h2}(\psi_h - \tilde{\psi}_{h1}) \end{aligned} \tag{5.1}$$

where $P_{hi}: V_h \to V_{hi}$ denotes the orthogonal projection w.r.t. the scalar product $a(.,.)$. Consequently, with

$$Q_{hi} = I - P_{hi}\,, \tag{5.2}$$

we obtain

$$\begin{aligned} \psi_h - \tilde{\psi}_{h1} &= Q_{h1}(\psi_h - \psi_{h1}^{(k-1)}) \\ \psi_h - \psi_{h1}^{(k)} &= Q_{h2}(\psi_h - \tilde{\psi}_{h1})\,, \end{aligned} \tag{5.3}$$

hence the error $\psi_h - \psi_{h1}^{(k)}$ propagates with $Q_{h2}Q_{h1}$. Analogously for $\psi_h - \psi_{h2}^{(k)}$. Hence we have to estimate the projections.

Lemma: For any sufficiently small h and for any $\phi_h \in V_h$ there correspond $\phi_{h1} \in V_{h1}$ and $\phi_{h2} \in V_{h2}$ with $\phi_h \equiv \phi_{h1} + \phi_{h2}$ on M and

$$|\phi_{h1}|^2 + |\phi_{h2}|^2 \leq c_0^2 |\phi_h|^2 , \tag{5.4}$$

where $c_0 > 0$ does not depend on $h > 0$ and ϕ_h.

Proof: Set $\phi_{h1} = \phi_h$ on $M - M_2$, and use the Extension Theorem of Sect. 3 to extend ϕ_{h1} to an element of V_{h1}. Now set

$$\phi_{h2} = \begin{cases} \phi_h - \phi_{h1} : & \text{on } M_2 \\ 0 : & \textit{elsewhere} \end{cases} \tag{5.5}$$

Consequently, with $\phi_{h2} \in V_{h2}$,

$$\phi_{h1} + \phi_{h2} \equiv \phi_h \tag{5.6}$$

and

$$|\phi_{hi}| \leq \textit{const.} |\phi_h| , \quad (i = 1,2) \tag{5.7}$$

hold, the constant not depending on h. This implies the Lemma.

The Theorem now results from this

Lemma: The operators Q_{h1} or Q_{h2} are contractions on the sets $Q_{h2}V_h$ or $Q_{h1}V_h$, respectively.

Proof: First, with ϕ_{h1}, ϕ_{h2} from the previous Lemma, from

$$\begin{aligned} |\phi_h|^2 &= a(\phi_h, \phi_{h1} + \phi_{h2}) \\ &= a(P_{h1}\phi_h, \phi_{h1}) + a(P_{h2}\phi_h, \phi_{h2}) \\ &\leq \left(|P_{h1}\phi_h|^2 + |P_{h2}\phi_h|^2\right)^{1/2}\left(|\phi_{h1}|^2 + |\phi_{h2}|^2\right)^{1/2} \\ &\leq c_0 |\phi_h| \left(|P_{h1}\phi_h|^2 + |P_{h2}\phi_h|^2\right)^{1/2} \end{aligned}$$

we conclude

$$|\phi_h| \le c_0\left(|P_{h1}\phi_h|^2 + |P_{h2}\phi_h|^2\right)^{1/2}. \tag{5.8}$$

Hence,

$$\begin{aligned}|Q_{h1}\phi_h|^2 &= |Q_{h2}Q_{h1}\phi_h|^2 + |P_{h2}Q_{h1}\phi_h|^2 \\ &\ge |Q_{h2}Q_{h1}\phi_h|^2 + \frac{1}{c_0^2}|Q_{h1}\phi_h|^2.\end{aligned}$$

Assuming $c_0 > 1$ without loss of generality,

$$|Q_{h2}Q_{h1}\phi_h| \le \sqrt{1-\frac{1}{c_0^2}}\,|Q_{h1}\phi_h| \tag{5.9}$$

follows. Estimate (5.9) holds in case of indices (1,2) replaced by (2,1), too. Whence the Lemma.

(5.9) implies

$$|\psi_h - \psi_{h1}^{(k)}| \le \bar{\rho}\,|\psi_h - \psi_{h1}^{(k-1)}| \tag{5.10}$$

with $\bar{\rho} := (1 - c_0^{-2})^{1/2}$, and the same estimate holds for $|\psi_h - \psi_{h2}^{(k)}|$. Hence both series $(\psi_{h1}^{(k)})$, $(\psi_{h2}^{(k)})$ converge to ψ_h. Combined with (3.3) this implies (4.7) and (4.8) as well, since $|\,.\,|$ and $\|.\|_\alpha$ are equivalent norms. The Theorem is proved.

6. APPLICATIONS AND EXTENSIONS

Since one sequence of the classical Schwarz algorithm has been replaced here by a total of two (independent) sequences, an advantage over the classical algorithm is obtained by performing a multilevel variant by use of a multiprocessor system only.

For this, we proceed as follows. Let (M_{11}, M_{12}) and (M_{21}, M_{22}) be an open covering of M_1 and M_2, resp., where we assume uniform overlapping again. Now, each iterative step on M_1 (or M_2) may be split into two parallel sequences of subproblems on M_{11} and M_{12} (or M_{21}, M_{22}). This kind of splitting may be continued. Consequently, by way of this "inner-outer" iteration, one obtains a degree of parallelism as high as desired w.r.t. the computer architecture one has in mind, at the cost of solving a doubled set of boundary value problems. For instance, if M is split into $p = 2^k$ $(k \in \mathbb{N})$ subdomains, then the optimum degree of parallelism is $\frac{p}{2}$.

Quite a lot of (symmetric) integral equations appearing in practice may be treated by means of the present algorithm, particularly all of the first-kind boundary integral equations arising from the linear problems of mathematical physics (e.g. Lame, Stokes, bipotential equations). In addition to (2.3), we mention explicitly the hypersingular equation (treated by [4] , [8] , [5] , for instance)

$$D\psi(x) := -\frac{\partial}{\partial n_x} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{1}{4\pi \mid x-y \mid} \psi(y) do_y + \int_{\partial\Omega} \psi(y) do_y = g(x) \quad (6.1)$$

which is obtained when the solution of the Neumann problem

$$\Delta u = 0 \quad in \quad \mathbb{R}^3 - \overline{\Omega}, \quad \frac{\partial u}{\partial n} = g \quad on \quad \partial\Omega \quad (6.2)$$

is looked for in terms of the double layer potential

$$u(x) = -\int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{1}{4\pi \mid x-y \mid} \psi(y) do_y. \quad (6.3)$$

In fact, D is a definite and symmetric strongly elliptic pseudodifferential operator of order $+1$.

REFERENCES

1. Babuska I., and Aziz A.K., Survey lectures on the mathematical foundation of the finite element method. In: Aziz A.K. (ed.), The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations, pp. 3-359, New York 1972.

2. Davies A.J., Quadratic isoparametric boundary elements. In: Brebbia C.A. (ed.), Boundary Elements X, Vol. 3, pp. 657-666, Berlin 1988.

3. Hackbusch W., and Novak Z., On the fast matrix multiplication in the boundary element method by panel clustering. Numer. Math. **54** (1989), 463-491.

4. Giroire J., and Nedelec J.C., Numerical solution of an exterior Neumann problem using a double layer potential. Math. Comp. **32** (1978), 973-990.

5. Hebeker F.K., On multigrid methods of the first kind for symmetric boundary integral equations of nonnegative order. In: Hackbusch W. (ed.), Robust Multigrid Methods, pp. 128-138, Braunschweig 1988.

6. Hebeker F.K., Experiences with the vectorization of a boundary element code for the Neumann problem of the Laplacean. In: Brebbia C.A., and Peters A. (eds.), Applications of Supercomputers in Engineering: Fluid Flow and Stress Analysis Applications, pp.159-167, Amsterdam 1989.

7. Hodous M.F., Katnik, Bozek, and Kline, Vector processing applied to boundary element algorithms on the CDC Cyber 205. In: Vector and Parallel Computing in Scientific Applications, Paris 1983.

8. Hsiao G.C., and Wendland W.L., The Aubin-Nitsche lemma for integral equations. J. Integr. Equ. **3** (1981), 299-315.

9. Lions P.L., On the Schwarz alternating method I. In: Glowinski et al. (eds.), First International Symposium on Domain Decomposition Methods for Partial Differential Equations, pp. 1-42, Philadelphia 1988.

10. Meurant G., Domain decomposition methods for partial differential equations on parallel computers. In: Dongarra J. et al. (eds.), Vector and Parallel Computing, pp. 241-253, Chichester 1989.

11. Stephan E., Boundary integral equations for screen problems in $\mathbb{R}^3$. Integr Equ. Oper. Th. **10** (1987), 236-257.

12. Volk K., On numerical quadrature of boundary integral equations. 19 pp., IBM Heidelberg Scientific Center, Techn. Rep. 1990.

13. Wendland W.L., On some mathematical aspects of boundary element methods for elliptic problems. In: Whiteman J.R. (ed.), The Mathematics of Finite Elements and Applications, Vol. 5, pp. 193-227, London 1985.

14. Wendland W.L., Domain decomposition in boundary element methods. Lecture presented at the Fourth International Symposium on Domain Decomposition Methods, Moscow 21-25 May, 1990.

15. Wendland W.L., Stephan E., and Hsiao G.C., On the integral equation method for the plane mixed boundary value problem of the Laplacean. Math. Meth. Appl. Sci. **1** (1979) 265-321.

16. Widlund O.B., An extension theorem for finite element spaces with three applications. In: Hackbusch W. et al. (eds.), Numerical Techniques in Continuum Mechanics, pp. 110-122, Braunschweig 1987.