CHAPTER 21

Overlapping Domain Decomposition Methods for FE-Problems with Elliptic Singular Perturbed Operators

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Abstract. This paper provides a theoretical justification to a domain decomposition method with overlapping applied to the solution of non-symmetric singularly perturbed elliptic problems. Such problems originate mainly from implicit discretization schemes for parabolic equations. The method is based on the use of the rapid exponential decrease property of grid Green's function when the distance from the point of location to the source function increases. For simplicity one considers in this paper a model problem with a number of non significant restrictions.

1. **Introduction.** Domain decomposition methods constitute an important actively developed approach to approximate realization of implicit schemes for unsteady problems. Information about the present state of this methodology is available in the Proceedings of International Symposiums on Domain Decomposition Methods ([5-7]).

In recent years a new approach has been suggested ([9, 12]) for constructing and justifying domain decomposition methods for numerical realization of implicit schemes for unsteady diffusion equations. This approach is based on the property of rapid exponential decrease (of the form $e^{-|x-x_0|/(\Delta t)^{1/2}}$) of the grid Green's function of the singularly perturbed elliptic finite element operator when the spacing from the point x_0 of location of the source function increases.

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This approach was extended (in [11]) to implicit difference schemes for unsteady convection-diffusion equations. The above publication suggested a new technique for investigating the behavior of grid Green's functions of singularly perturbed elliptic operators. The idea was to use the preconditioned Richardson's method, and two versions of the domain decomposition with overlapping subdomains were proposed together with their justification in the case of regular triangulations.

This paper, using as an example a non-symmetric singularly perturbed elliptic model problem, gives a detailed description and justification of one of the versions of the domain decomposition method with overlapping subdomains. Most assumptions and restrictions are introduced here only with the aim of greater simplicity and better illustration of the results to be presented. The most important assumption concerns the local finite element counterpart of the extension theorem. This assumption limits in theory the set of admissible spatial grids since despite numerous publications ([1, 14, 15]) there are still some points to be clarified here. This paper should thus be regarded as a sequel to [11] with a more detailed description of a number of basic problems, although it contains new fundamental results such as, for example, the justification of the method in the case of irregular triangular grids.

2. **Preconditioned Richardson's method.** Let Ω be a unit square with boundary $\partial\Omega$ and let $g \in L_2(\Omega)$, $\|g\| = 1$, be a given function. Let us consider the following variational elliptic problem: find $u \in H^1(\Omega)$ such that

$$a_\tau(u,v) \equiv (u,v) + \tau a(u,v) = \tau(g,v) \ \forall v \in H^1. \tag{2.1}$$

Here, $(\cdot,\cdot)$ is the ordinary scalar product in $L_2(\Omega)$, $\tau \in (0,1]$ is a parameter and

$$a(u,v) = \int_\Omega [\nabla u \cdot \nabla v + (\vec{b} \cdot \nabla u)v] d\Omega, \tag{2.2}$$

where the vector function $\vec{b} = (b_1, b_2)$ is assumed to be smooth in Ω and to satisfy the conditions

$$\operatorname{div} \vec{b} = 0 \text{ in } \Omega,\ b_n = 0 \text{ on } \partial\Omega,\ \max_{i=1,2} \max_{x \in \Omega} |b_i(x)| \le \sqrt{2}. \tag{2.3}$$

Here, b_n denotes the normal component of $\vec{b}$ at the boundary $\partial\Omega$.

Next, we construct in Ω a square grid Ω_d with step size $d = n^{-1}$, where n is a positive integer, and also a triangular grid Ω_h. Assume that the lines of the grid Ω_d constitute a union of sides of triangles from Ω_h, as shown, for example, in Fig. 1.

Denote by H^1_h a standard piecewise-linear finite element subspace of H^1 and consider the finite element problem ([4]): find $u^h \in H^1_h$ such that

$$a_\tau(u^h,v) = \tau(g,v) \ \forall v \in H^1_h. \tag{2.4}$$

This problem leads within the framework of the usual nodal basis to the linear algebraic system

$$A_\tau u \equiv (M + \tau A)u = f, \tag{2.5}$$

with the $N \times N$ mass matrix $M=M^T > 0$ and with the $N \times N$ matrix $A=A_o+A_1$. The matrices A_o and A_1 are determined by using the relations

$$(A_o v,w) = \int_\Omega \nabla v^h \cdot \nabla w^h d\Omega,$$

$$(A_1 v,w) = \int_\Omega (\vec{b} \cdot \nabla v^h) w^h d\Omega,$$

which are assumed to be valid for all v, $w \in \mathbf{R}^N$ or, equivalently, for all v^h, $w^h \in H^1_h$, where v^h, w^h denote piecewise-linear prolongations of v, w. The above-made assumptions imply that $A_o=A_o^T \geq 0$ and $A_1= -A_1^T$.

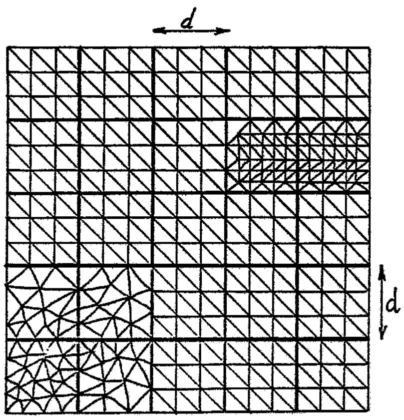

Figure 1. A case of grids Ω_d and Ω_h.

To solve system (2.5) one makes use of the following preconditioned Richardon's iterative method

$$B(u_k - u_{k-1}) = -\alpha(A_\tau u_{k-1} - f), \quad k=1,2,... \tag{2.6}$$

under the condition that $u_o=0$, where α is a relaxation parameter and B is a symmetric positive definite matrix such that

$$c_o(Bv,v) \leq (A_\tau v,v) \equiv ((M + \tau A_o)v,v) \leq c_1(Bv,v) \; \forall v \in \mathbb{R}^N \tag{2.7}$$

where c_o and c_1 are two constants independent of τ and Ω_h. Introduce the matrices

$$R = (M + \tau A_o)^{-1/2} A_1(M + \tau A_o)^{-1/2} = -R^T,$$

$$S_o = B^{-1/2}(M + \tau A_o)B^{-1/2} = S_o^T > 0, \tag{2.8}$$

$$S_1 = B^{-1/2} A_1 B^{-1/2} = -S_1^T.$$

It is obvious that in case where $B=M+\tau A_o$, we have $S_o=E$ and $R=S_1$.

To study method (2.6), we need the following

Lemma 2.1. Under the above assumptions, we have

$$\| R \| \equiv \rho(R) \leq \frac{1}{\sqrt{\tau}} \tag{2.9}$$

<u>Proof:</u> Let us consider the eigenvalue problem

$$i\mu\psi = R\psi, \quad \mu \in \mathbb{R}^1, \quad \psi \in \mathbb{C}^N,$$

which we rewrite, taking into account the representation $\psi = \psi_1 + i\psi_2$, ψ_1, $\psi_2 \in \mathbb{R}^N$, into the equivalent form

$$-\mu\psi_2 = R\psi_1,$$

$$\mu\psi_1 = R\psi_2.$$

Making use of the notation $\varphi_i = (M+\tau A_o)^{-1/2}\psi_i$, i=1,2, we rewrite this system into the more convenient following form:

$$-\mu(M + \tau A_o)\varphi_2 = A_1\varphi_1,$$

$$\mu(M + \tau A_o)\varphi_1 = A_1\varphi_2.$$

Hence, we have

$$\mu = \frac{(A_1\varphi_2,\varphi_1) - (A_1\varphi_1\varphi_2)}{\sum\limits_{i=1,2} ((M+\tau A_o)\varphi_i,\varphi_i)} = \frac{\int\limits_\Omega (\vec{b}\cdot\nabla\varphi_2^h)\varphi_1^h d\Omega - \int\limits_\Omega (\vec{b}\cdot\nabla\varphi_1^h)\varphi_2^h d\Omega}{\sum\limits_{i=1,2} \int\limits_\Omega [\,|\varphi_i^h|^2 + \tau|\nabla\varphi_i^h|^2] d\Omega}.$$

Then, making use of the following inequalities

$$2\left|\int\limits_\Omega b_1 \frac{\partial\varphi_1^h}{\partial x_1}\varphi_2^h d\Omega\right| \le \frac{1}{\sqrt{\tau}} \int\limits_\Omega |\varphi_2^h|^2 d\Omega + \sqrt{\tau}\int\limits_\Omega b_1^2\left|\frac{\partial\varphi_1^h}{\partial x_1}\right|^2 d\Omega,$$

$$\int\limits_\Omega b_1^2\left|\frac{\partial\varphi_1^h}{\partial x_1}\right|^2 d\Omega \le \max_{x\in\Omega} |b_1|^2 \int\limits_\Omega \left|\frac{\partial\varphi_1^h}{\partial x_1}\right|^2 d\Omega \le 2\int\limits_\Omega \left|\frac{\partial\varphi_1^h}{\partial x_1}\right|^2 d\Omega,$$

and of the third condition (2.3) we obtain the sought estimate

$$\rho(R) = \max|\mu| \le \tau^{-1/2}. \qquad \square \tag{2.10}$$

Return now to method (2.6). Let $w_k = u_k - A_\tau^{-1}f$ be error vectors. Then,

$$\|w_k\|_B^2 \equiv \|\hat{w}_k\|^2 = \|\hat{w}_{k-1}\|^2 - 2\alpha(S_o\hat{w}_{k-1},\hat{w}_{k-1}) + \alpha^2\|(S_o+\tau S_1)\hat{w}_{k-1}\|^2,$$

where $\hat{w}=B^{1/2}w$. Hence, making use of the inequalities

$$(S_o\hat{w}_{k-1},\hat{w}_{k-1}) \geq c_o\ \|\hat{w}_{k-1}\|^2,$$

$$2\,|(S_o\hat{w}_{k-1},\ S_1\hat{w}_{k-1})| \ \leq \frac{1}{\sqrt{\tau}}\|S_o\hat{w}_{k-1}\|^2 + \sqrt{\tau}\,\|S_1\hat{w}_{k-1}\|^2,$$

$$\|S_o\hat{w}_{k-1}\| \ \leq c_1\|\hat{w}_{k-1}\|,$$

$$\|S_1\hat{w}_{k-1}\| \ \leq c_1\|R\|\cdot\|\hat{w}_{k-1}\| \ \leq \frac{c_1}{\sqrt{\tau}}\ \|\hat{w}_{k-1}\|,$$

we obtain

$$\|\hat{w}_k\|^2 \ \leq [1 - 2\alpha c_o + \alpha^2 c_1^2(1+\sqrt{\tau})^2]\ \|\hat{w}_{k-1}\|^2 \ \leq [1 - 2\alpha c_o + 4\alpha^2 c_1^2]\,\|\hat{w}_{k-1}\|^2.$$

These inequalities imply that for a proper choice of the relaxation parameter α (for example

$$\alpha = \alpha_{opt} = \frac{c_o}{4c_1^2}\ )$$

method (2.6) converges in the norm $\|\cdot\|_B$ as a geometric sequence with the ratio $q = 1-\alpha_{opt} < 1$ which is independent of the grid Ω_h and of the value of the parameter α in the interval (0,1]. It is interesting to note that if the numbers c_o and c_1 converge to one and if

$$\alpha = \alpha_{opt} = \frac{1}{(1+\sqrt{\tau})^2}, \tag{2.11}$$

then the ratio q converges to the quantity $1-\alpha_{opt}=\sqrt{\tau}(2+\sqrt{\tau})/(1+\sqrt{\tau})^2$, i.e. for $\tau \ll 1$ we have $q \sim 2\sqrt{\tau}$ and, hence, method (2.6) converges very rapidly.

3. **Two-level domain decomposition preconditioner.** To construct a two-level domain decomposition preconditioner (DD-preconditioner), we make use of the results obtained in [10, 11]. We consider the finite difference case; using therefore the grid Ω_d we partition the domain Ω into two subdomains Ω_1 and Ω_2, as shown in Fig. 2. It is seen in this figure that Ω_2 is a multiply-connected domain consisting of squares $\Omega_2^{(i)}$, i=1, ...,p, where p is a positive integer of order n^2, the side length of these squares being equal to d. According to the theory of DD-methods with alternating Neumann-Dirichlet boundary conditions ([2,13]), the matrix $M+\tau A_o$ has the following block form:

$$M + \tau A_o = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.1}$$

and the corresponding first-level DD-preconditioner is of the form ([3.10])

$$B_1 = \begin{bmatrix} B_{11} + A_{12}\ A_{22}^{-1}\ A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{3.2}$$

where the submatrix B_{11} is determined by using the relation

$$(B_{11}v_1,w_1) = \int_{\Omega_1} [v_1^h w_1^h + \tau \nabla v_1^h \cdot \nabla w_1^h] d\Omega \ \forall v_1,w_1 \in \mathbf{R}^{N_1}. \tag{3.3}$$

As it is generally know, we have $M+\tau A_o \geq B_1$ by construction and, hence, to estimate the condition number of the matrix $B_1^{-1}(M+\tau A_o)$ in the B_1-norm, it is sufficient to estimate its maximal eigenvalue. If some specific assumptions on the grid Ω_h hold, this eigenvalue can be bounded by a constant independent of Ω_h and of the parameter τ.

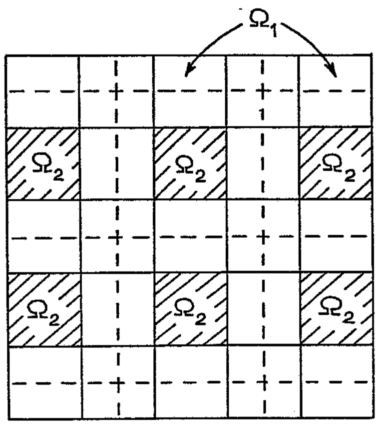

Figure 2. The first level of DD-procedure

Let us construct in Ω an auxiliary grid Ω_{2d}, as shown in Fig. 2 (with dotted lines). This corresponds to the partitioning of Ω into internal squares with side length 2d and near-boundary rectangles which we denote by $\Omega_{2d}^{(i)}$, $i=1,\dots,\tilde{n}^2$, where $\tilde{n} \sim n$. Then, as it is generally known,

$$\rho(B_1^{-1}(M+\tau A_o)) = \max_{v \in U} \frac{\hat{a}(v^h)}{\hat{b}(v^h)} \leq \max_{1 \leq i \leq \tilde{n}} \max_{v \in U_i} \frac{\hat{a}_i(v^h)}{\hat{b}_i(v^h)} \leq \max_{v \in U_e} \frac{\hat{a}_e(v^h)}{\hat{b}_e(v^h)}$$

where

$$U = \{v: v \in \mathbf{R}^N, A_{21}v_1 + A_{22}v_2 = 0\},$$

$$\hat{a}(v^h) = \int_{\Omega} [\,|v^h|^2 + \tau\,|\nabla v^h|^2]\, d\Omega,$$

where $\hat{b}$ is a restriction of $\hat{a}$ onto Ω_1, $\hat{a}_i$ and $\hat{b}_i$ are restrictions of $\hat{a}$ and $\hat{b}$ onto the subdomains $\Omega_{2d}^{(i)}$ and $\Omega_{2d}^{(i)} \cap \Omega_1$, respectively, and e is the value of index i at which the first maximum is attained. Note also that by U_i we denote the restriction of U corresponding to the transition from the domain Ω to its subdomain $\Omega_{2d}^{(i)}$.

Without loss of generality assume that $\Omega_{2d}^{(e)}$ is an internal subdomain of Ω and using the transformation $\tilde{x}_i=x_i/(2d)$, i=1,2, map it onto the unit square $G \equiv \tilde{\Omega}_{2d}^{(e)}$. In addition, the subdomains $\Omega_1 \cap \Omega_{2d}^{(e)}$ and $\Omega_2 \cap \Omega_{2d}^{(e)}$ are mapped onto G_1 and G_2, respectively. As a result, we obtain

$$\rho(B_1^{-1}(M + \tau A_o)) \leq \max_{v \in \tilde{U}_e} \frac{\tilde{a}_e(v^h)}{\tilde{b}_e(v^h)},$$

where

$$\tilde{a}_e(v^h) = \int_G [\,|v^h|^2 + \frac{\tau}{4d^2}\,|\tilde{\nabla} v^h|^2]dG,$$

and $\tilde{b}_e$ is a restriction of $\tilde{a}_e$ onto the subdomain G_1. Now we will make two assumptions. First, we choose $d=\sqrt{\tau}$. Second, we assume that the finite element counterpart of the theorem of norm-preserving extension from $H_{\tilde{h}}^1(G_1)$ to $H_{\tilde{h}}^1(G_2)$ is valid for the grids Ω_h and the partitionings of domain Ω into subdomains. This means that there exists a positive constant c_2 independent of the grids Ω_h, such that for any function $v \in H_{\tilde{h}}^1(G_1)$ there exists such a function $v \in H_{\tilde{h}}^1(G)$ which is identically equal to v_1 in G_1 and obeys the inequality

$$\tilde{a}_e(v) \leq c_2 \tilde{b}_e(v).$$

It is obvious that we assume the constant c_2 to be also independent of the value of index e, i.e. of the choice of the subdomain $\Omega_{2d}^{(e)}$. The last statement is valid, for example, for regular triangular grids Ω_h([1,14,15]) and also for many other grids which are constructed by using various refinement procedures.

The theory of domain decomposition methods with alternating Neumann-Dirichlet boundary conditions implies that, under the assumptions which have been made, the maximal eigenvalue of the matrix $B_1^{-1}(M+\tau A_o)$ and, hence, its condition number are bounded by the same constant c_2 which, by virtue of the assumption made on the extension theorem, is independent of the grids Ω_h and, by virtue of the technique of construction of the matrix B_1, is also independent of the value of the parameter τ.

At the second stage of construction of the two-level DD-preconditioner we partition the domain Ω_1 into two subdomains Ω_{11} and Ω_{12}, as shown in Fig. 3, and construct the corresponding block partitioning of the matrix B_{11} considering it as the stiffness matrix for the domain Ω_1:

$$B_{11} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{3.4}$$

and the corresponding DD-preconditioner

$$B_2 = \begin{bmatrix} \hat{B}_{11} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{3.5}$$

where the submatrix B_{11} is determined by using the relation

$$(\hat{B}_{11}\hat{v}_1, \hat{w}_1) = \int_{\Omega_{11}} [\hat{v}_1^h \hat{w}_1^h + \tau \nabla \hat{v}_1^h \cdot \nabla \hat{w}_1^h] d\Omega \;\; \forall \hat{v}_1, \hat{w}_1 \in \mathbf{R}^{N_{11}}.$$

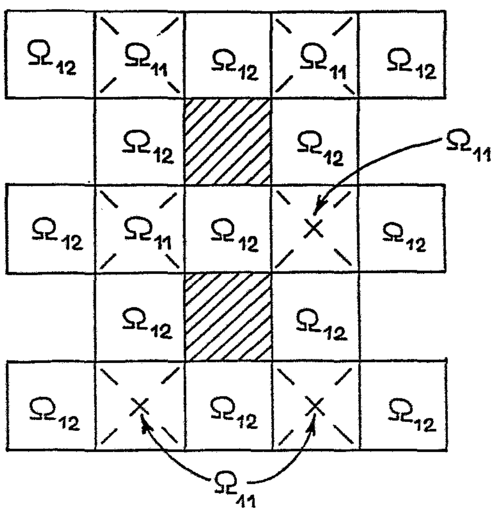

Figure 3. Second level of DD-procedure

As in the first stage, the eigenvalues of the matrix $B_2^{-1}B_{11}$ are larger than or equal to unity. To estimate its maximal eigenvalue, we make use of the same approach but with other partitionings into superelements whose boundaries are shown by the dotted lines in Fig. 3. Under the same assumptions on the value of d and on the grids Ω_h concerning the finite element counterpart of the extension theorem, it is not difficult to show that the maximal eigenvalue of the matrix $B_2^{-1}B_{11}$ is bounded from above by a constant c_3 independent of the value of $\tau \in (0,1]$ and of the grids Ω_h.

Define the resulting two-level DD-preconditioner by the following formula ([10]):

$$B = \begin{bmatrix} \hat{B}_2 + A_{12}\ A_{22}^{-1}\ A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.6}$$

Under the assumptions made above, it follows from [10,11], and from the facts previously proved, that we have the following

Proposition 3.1. The eigenvalues of the matrix $B^{-1}(M+\tau A_o)$ belong to the interval [1,c], where $c=c_2c_3$ is a positive constant independent of the value of the parameter $\tau \in (0,1]$ and of the grids Ω_h.

4. **DD-method with overlapping.** This section consists of two mutually connected parts. First, we apply the iterative method

$$B(u_k - u_{k-1}) = -\alpha_{opt}(A_\tau u_{k-1} - f),\ k=1, ..., s, \tag{4.1}$$

with the initial guess $u_o=0$, the matrix B from (3.6) and the relaxation parameter $\alpha_{opt}=(4c^2)^{-1}$ in order to study the properties of the solution of system (2.5) with the special right-hand side f and also to obtain certain auxiliary estimates. Then we use the obtained results to construct and justfy completely a new DD-method with overlapping subdomains. During the discussion we shall assume that the parameter τ is as small as it is necessary for carrying out new assumptions.

Section 2 implies that to solve system (2.5) by the method (4.1) with accuracy $\epsilon_1 < 1$ (in the sense that the following inequality holds:

$$\| w_s \|_B \leq \epsilon_1 \| w_o \|_B \tag{4.2}$$

where $w_s = u_s - u$, $w_o = u$ and u is the exact solution to system (2.5)), it is sufficient to choose the value of s by the formula

$$s = \left[\frac{\ln \epsilon_1}{\ln q}\right] + 1, \tag{4.3}$$

where [y] means the integral part of y, and the quantity $q = 1 - \alpha_{opt}$ is independent of Ω_h, and of $\tau \in (0,1]$.

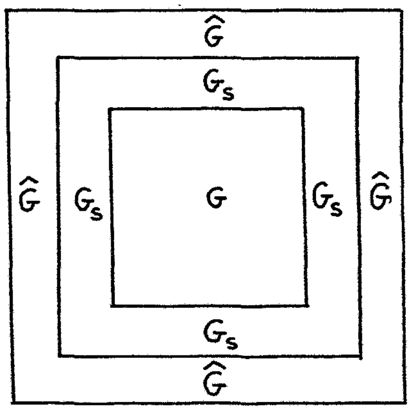

Figure 4. Subdomains G, B_s and $\hat{G}$.

Since we have by definition, $(f,v) = \tau \int_\Omega g v d\Omega \;\; \forall v \in \mathbf{R}^N$, then taking into account the assumption that $\| g \| = 1$ it follows from

$$(A_\tau u, u) = ((M + \tau A_o)u, u) = (f, u)$$

and from the inequality $(f,u) \leq \tau \| u \|$ that we have the estimates

$$\| u \|_B \leq \tau, \quad \| u \|_{M+A_o} = \| u^h \|_{H^1} \leq \sqrt{\tau}. \tag{4.4}$$

From (4.4) and (4.2) we also have the estimates

$$\| w_s \|_B \leq \epsilon_1 \tau, \tag{4.5}$$

$$\| w_s^h \|_{H^1} \leq \epsilon_1 \sqrt{\tau}, \tag{4.6}$$

which we will need below.

In addition to the special property of the right hand side f, assume that

$$\text{supp } g \in G, \tag{4.7}$$

where $G \subset \Omega$ is a square with side lenght r which does not exceed, for example, $\tau^{1/3}$. We assume that $c_4\tau^{1/3} \leq r \leq \tau^{1/3}$, where c_4 is a positive constant independent of τ. The last assumption on the side length of the square G is made mainly for simplifying the formulae and estimates to be given below. We assume that G is the union of cells of the grid Ω_d.

The structure of the employed DD-preconditioner B implies that the supports of the finite element functions u_k^h of method (4.1) belong to the squares G_k with side length $\tau^{1/3}+k\tau^{1/2}$, whose centers coincide with the center of the square G, as shown in Fig. 4. We thus arrive at two conclusions. First, the error function w_s^h of method (4.1) obeys inequalities (4.5) and (4.6). Second, the function u_s^h is zero in the domain $\Omega \backslash G_s$. Hence, for the solution u to system (2.5) we obtain the estimates

$$\int_{\Omega \backslash G_s} [\,| u^h |^2 + \tau \,| \nabla u^h |^2] d\Omega \leq \epsilon_1^2 \tau^2, \text{ and} \tag{4.8}$$

$$\| u^h \|_{H^1(\Omega \backslash G_s)} \leq \epsilon_1 \sqrt{\tau}. \tag{4.9}$$

Now we make use of the assumption that the quantity τ is sufficiently small. Moreover, assume that the values of τ and ϵ_1 are such that the square G_s belongs to the square $\hat{G}$ with side length $2\tau^{1/3}$ as shown in Fig. 4. Such assumption on ϵ_1 is quite natural if we set, for example, $\epsilon_1 = \tau^\beta$, where β is a positive constant. This is the situation which arises with numerical methods for the solution of parabolic problems by implicit schemes which is our main motivation, here. We assume that $\hat{G}$ is also a union of cells of the grid Ω_d.

Bounding again the set of grids Ω_h with the assumption that the finite element counterpart of the local extension theorems holds, and making use of the technique of the previous section, it is easy to prove the following statement: for any function $u^h \in H^1(\Omega\backslash\hat{G})$ there exists a function $v^h \in H^1(\Omega)$ such that $v^h \equiv u^h$ in $\Omega\backslash\hat{G}$ and the inequality

$$\int_{\hat{G}} [|v^h|^2 + \tau^{2/3}|\nabla v^h|^2]dG \le c_5^2 \int_{\Omega\backslash\hat{G}} [|u^h|^2 + \tau^{2/3}|\nabla u^h|^2]d\Omega \quad (4.10)$$

is valid with a positive constant c_5 which can be chosen independently of the grids Ω_h, of the value of the parameter τ and of the location of the square G. It is obvious that according to (4.8) such function v^h obeys the inequality

$$\|v^h\|_{H^1(\hat{G})} \le c_5\epsilon_1\sqrt{\tau}. \quad (4.11)$$

Making use of the already obtained estimates for the solution function u^h and for its extension v^h from $H^1(\Omega\backslash\hat{G})$ to $H^1(\hat{G})$, we consider one of the possible approaches to construct a new method of solution of system (2.5).

Our aim is to solve numerically system (2.5) with an accuracy $\epsilon < 1$, i.e. to find a function $u_\epsilon^h \in H^1$ obeying the inequality

$$\|u_\epsilon^h - u^h\|_{H^1} \le \epsilon. \quad (4.12)$$

The construction of such function u_ϵ^h will be carried out as follows. From the square $\hat{G}$ given above define the space

$$H^1(\Omega;\Omega\backslash\hat{G}) = \{v{:}v \in H^1(\Omega),\ v{=}0 \text{ in } \Omega\backslash\hat{G}\}. \quad (4.13)$$

Then consider the finite element problem: find $\hat{u}^h \in H^1(\Omega;\Omega\backslash\hat{G})$ such that

$$a_\tau(\hat{u}^h,v) = \tau(g,v) \quad \forall v \in H^1(\Omega;\Omega\backslash\hat{G}). \tag{4.14}$$

This problem leads to the following system

$$\hat{A}_\tau\hat{u} = \hat{f}, \quad \hat{f} \in \mathbf{R}^{\hat{N}}, \tag{4.15}$$

where the $\hat{N} \times \hat{N}$ matrix $\hat{A}_\tau = \hat{M} + \tau(\hat{A}_0 + \hat{A}_1)$ is a diagonal block of the matrix A_τ. The matrix $\hat{A}_\tau$ is also the stiffness matrix of the FE-method employed for the differential problem

$$\hat{u} - \tau[\Delta\hat{u} - (\vec{b}\cdot\nabla)\hat{u}] = \tau\hat{g} \text{ in } \hat{G}, \frac{\partial\hat{u}}{\partial n} = 0 \text{ on } \partial\Omega, \hat{u} = 0 \text{ on } \partial G \cap \Omega. \tag{4.16}$$

Suppose that we have found a vector $\hat{u}_\epsilon$ which approximates the solution $\hat{u}$ of system (4.15) with accuracy $\epsilon/2$ in the sense that the inequality

$$\|\hat{u}^h_\epsilon - \hat{u}^h\|_{H^1(\Omega)} \leq \epsilon/2, \tag{4.17}$$

where $\hat{u}^h_\epsilon \equiv 0$ in $\Omega\backslash\hat{G}$, is valid. then arises the question whether the following estimate holds

$$\|\hat{u}^h - u^h\|_{H^1(\Omega)} \leq \epsilon/2. \tag{4,18}$$

If this estimate holds, the considered function $\hat{u}^h_\epsilon$, obviously, obeys inequality (4.12) and, hence, is the sought ϵ-approximation of the function u^h. Let us show that inequality (4.12) can hold if we impose additional constraints on the quantity ϵ_1 and the parameter τ.

The function $\hat{w}^h = \hat{u}^h - u^h$ can be described as follows. In the subdomain $\Omega\backslash\hat{G}$ it identically coincides with u^h and, according to (4.9), obeys the inequality

$$\|\hat{w}^h\|_{H^1(\Omega\backslash\hat{G})} \leq \epsilon_1\sqrt{\tau}. \tag{4.19}$$

Hence, ϵ_1 is surely less than $\epsilon/(2\sqrt{\tau})$. In the subdomain $\hat{G}$ we write the function $\hat{w}^h$ as the sum of two functions:

$$\hat{w}^h = \psi^h + v^h \tag{4.20}$$

where for v^h we choose an extension of $\hat{w}^h$ from the subdomain $\Omega\backslash\hat{G}$ (recall that $\hat{w}^h \equiv u^h$ in $\Omega\backslash\hat{G}$), which obeys inequality (4.11). Then the function ψ^h belongs to $H^1(\hat{G})$, provided that $\psi^h=0$ on $\partial G \cap \Omega$. This implies that the vector $\psi \in R^{\hat{N}}$ is the solution of the linear system

$$\hat{A}_\tau \psi = F, \tag{4.21}$$

where $F=-\hat{A}_\tau v$.

Taking the scalar product of both sides of system (4.21) with the vector ψ we obtain

$$(\hat{A}_\tau \psi,\psi) \equiv \|\psi\|^2_{\hat{M}+\tau\hat{A}_0} = -(A_\tau v,\psi). \tag{4.22}$$

Making use of the third assumption (2.3) concerning the vector function $\vec{b}$, we can readily derive the estimate

$$(A_\tau v,\psi) \leq 2\|v\|_{\hat{M}+\tau\hat{A}_0} = \|\psi\|_{\hat{M}+\tau\hat{A}_0} . \tag{4.23}$$

This estimate and (4.11) imply that

$$(\hat{A}_\tau \psi,\psi)^{1/2} \equiv \|\psi\|_{\hat{M}+\tau\hat{A}_0} \leq 2\|v^h\|_{H^1(\hat{G})} \leq 2c_5\epsilon_1\sqrt{\tau}.$$

We have thus proved the following

Lemma 4.1. The function $\hat{w}^h$ obeys the estimate

$$\|\hat{w}^h\|_{H^1} \leq (3c_5 + 1)\epsilon_1\sqrt{\tau}.$$

This lemma and the previous arguments imply that estimate (4.12) will hold if ϵ_1 obeys the inequality

$$\epsilon_1 \leq \frac{\epsilon}{2(3c_5 + 1)\sqrt{\tau}} . \tag{4.24}$$

There remains one more question concerning the assumption that $G_s \subset \hat{G}$. Indeed, the values of the constants c_1 and c_5, and also of τ and ϵ may happen to be such that this inclusion does not hold. An analysis of this situation will be restricted to the case where $\epsilon=\tau^\beta$, where β is a positive constant. In real problems related to the numerical realization of implicit schemes for parabolic equations it is usually assumed that $\epsilon \sim \tau^\beta$, where β is equal to two (for first-order accurate schemes) or three (for second-order accurate schemes of the Crank-Nicholson type).

Under the assumptions made on ϵ and from the requirement that $G_s \subset \hat{G}$, we obtain the following inequality for the admissible values of τ:

$$s\sqrt{\tau} \le \frac{1}{2}\tau^{1/3} \tag{4.25}$$

or, according to (4.3) and (4.24),

$$\tau^{1/6} \frac{\ln \frac{\tau^{\beta-1/2}}{2(4c_5+1)}}{\ln q} \le 1. \tag{4.26}$$

It is obvious that there exists $\hat{\tau} \in (0,1]$ such that inequality (4.26) holds for all $\tau \in (0,\hat{\tau}]$. Thus the inclusion $G_s \subset \hat{G}$ holds for all sufficiently small τ.

In conclusion, we outline the obvious application of the described algorithm within the framework of DD-methods with overlapping subdomains to solving system (2.5) with an arbitrary right-hand side f. Write down the function f^h as the sum

$$f^h = \sum_{i=1}^{m} f_i^h, \tag{4.27}$$

so that the following condition is satisfied:

$$\text{supp}\, f_i^h \subset G^{(i)}, \quad i=1,...,m. \tag{4.28}$$

Here, m is an integer, and the $G^{(i)}$ are, for example, squares with the same side lengths. Then the solution of system (2.5) reduces to the approximate solution of the system

$$A_\tau u_i = f_i \tag{4.29}$$

with an accuracy $\epsilon^{(i)}$, i=1,...,m, by the algorithm described above, and then to the calculation of the sum of the approximations obtained. The choice $\epsilon^{(i)}=\epsilon\tau^{2/3}$ is, for example, the simplest one and at the same time the cheapest one. Note that for sufficiently small values of τ it is sufficient to choose $\epsilon^{(i)}=\epsilon/4$.

5. **Conclusion.** It is obvious for the reader that most assumptions have been made only to simplify the description of the considered algorithm and of the corresponding arguments. In our view, those DD-methods with overlapping subdomains discussed in this article are highly efficient from the points of view of their arithmetic cost and their implementation on parallel computers. They apply particulary well to the discrete problems obtained from implicit difference schemes for solving unsteady problems. We have considered above an approach for constructing DD-methods with overlapping subdomains which is based on the superposition idea. Another approach was announced in [11]; it can be applied to both linear and non-linear problems ([8]). Its theoretical investigation will take place in the near future.

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