CHAPTER 14

Preconditioning and Boundary Conditions: L_2 and H_1 Theory*

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1. Introduction

Let Ω be a bounded open region in R^2. Let A be an invertible uniformly elliptic operator defined on Ω. That is, in Ω

$$Au = -[(a_{11}u_x)_x + (a_{12}u_y)_x + (a_{12}u_x)_y + (a_{22}u_y)_y] + a_1u_x + a_2u_y + a_0u, \tag{1.1}$$

with boundary conditions

$$u = 0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \gamma_a} = \alpha_0\mu + \alpha_1(\alpha)\frac{\partial u}{\partial \sigma} \text{ on } \Gamma_1 \tag{1.2}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\frac{\partial \cdot}{\partial \gamma_a}$ denote the co-normal derivative. Consider a boundary value problem

$$Au = f \in L_2(\Omega), \tag{1.3}$$

and a finite element discretization

$$A_hU_h = f_h, \quad U_h \in S_h \tag{1.4}$$

Much of the literature on preconditioning for A_h is concerned with the cases where A is symmetric and positive definite and/or $\Gamma_0(A) = \partial\Omega$, i.e. the boundary conditions are Dirichlet conditions on the entire boundary. In this work we will focus our attention on methods which can deal with the case where

$$A \neq A^* \text{ and } \Gamma_0(A) \neq \partial\Omega.$$

Let B be another invertible uniformly elliptic operator defined on Ω. Thus

$$Bv = -[(b_{11}v_x)_x + (b_{12}v_y)_x + (b_{12}v_x)_y + (b_{22}v_y)_y] + b_1v_x + b_2v_y + b_0v. \tag{1.5}$$

Let B_h be a discretization of B acting on the same space, S_h, as A_h. This report is concerned with the preconditioned operators $R_h = A_hB_h^{-1}, L_h = B_h^{-1}A_h$.

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The basic questions addressed are:

(i) Can one find an elliptic operator B so that

$$C_{H_1}(B_h^{-1}A_h) = ||B_h^{-1}A_h||_{H_1}||A_h^{-1}B_h||_{H_1} \leq K_H \quad ?$$

(ii) Can one find an elliptic operator B so that

$$C_{L_2}(B_h^{-1}A_h) = ||B_h^{-1}A_h||_{L_2}||A_h^{-1}B_h||_{L_2} \leq K_L \quad ? \tag{1.6}$$

(iii) Can one find an elliptic operator B so that

$$C_{l_2}(A_hB_h^{-1}) = ||A_hB_h^{-1}||_{L_2}||B_hA_h^{-1}||_{L_2} \leq K_R \tag{1.7}$$

(iv) Given operators B_L and B_R so that (1.6) and (1.7) hold respectively, what can one say about the distribution of the singular values of L_h and R_h?

The interest in such estimtes stems from the well known estimates for the convergence of the Conjugate Gradient methods. That is, if $\mathcal{E}^s = \mathsf{U}_h - \mathsf{U}_h^s$ is the error in the sth Conjugate Gradient iterate U_h^s, then

$$||\mathcal{E}^s|| \leq 2(\frac{c-1}{c+1}^s)||\mathcal{E}^0|| \tag{1.8}$$

where c denotes the appropriate Condition number.

However, the optimality theorem of Conjugate Gradient method implies that the estimate (1.8) may be a serious overestimate when the singular values of $B_h^{-1}A_h$ or $A_hB_h^{-1}$. (depending on the implementation) cluster about a few values.

Note: In practice one uses $\hat{B}_h^{-1}$, an approximate inverse of B_h, e.g., a single multigrid sweep.

2. Basic Results: H_2 Regularity

These topics have been discussed in detail in [FMP], [MP], [JMPW], [GMP]. The basic results are:

Theorem 1 [FMP, MP]: Let A, B be invertible. Let A_h, B_h be families of finite-element discretizations.

(a) Suppose $A_h^{-1} \to A^{-1}$, $B_h^{-1} \to B^{-1}$ pointwise in L_2. Assume there exists a $K_L > 0$, independent of h, $0 < h \leq h_0$ such that

$$||B_h^{-1}A_h||_{L_2} \leq K_L. \tag{2.1a}$$

Then, there exists a K_L^1 and

$$||B^{-1}A||_{L_2} \leq K_L^1. \tag{2.1b}$$

(b) Suppose $A_h^{-1} \to A^{-1}$, $B_h^{-1} \to B^{-1}$ pointwise in H_1. Assume there exist a $K > 0$, independent of h, $0 < h \leq h_0$, such that

$$||B_h^{-1}A_h||_{H_1} \leq K. \tag{2.2a}$$

Then, there exists a K^1 and

$$||B^{-1}A||_{H_1} \leq K^1. \tag{2.2b}$$

(c) Suppose $(A_h^*)^{-1} \to (A^*)^{-1}, (B_h^*)^{-1} \to (B^*)^{-1}$ pointwise in L_2. Assume there exists a K_R, independent of h, $0 < h \leq h_0$, such that

$$||A_hB_h^{-1}||_{L_2} \leq K_R. \tag{2.3a}$$

Then, there exists a K_R^1 and

$$||AB^{-1}||_{L_2} \leq K_R^1. \quad \blacksquare \tag{2.3b}$$

We deal with the H_1 estimate, (1.9b) first. The result is elegant and complete.

Theorem 2 [MP]: Let A and B be invertible, then (1.9b) holds if and only if

$$\Gamma_0(A) = \Gamma_0(B). \tag{2.4}$$

That is, if and only if the partition of the boundary $\partial\Omega$ into $\Gamma_0 \cup \Gamma_1$ is the same for both operators. ■

Theorem 3[MP]: Suppose the discretizations A_h, B_h are obtained as direct Galerkin schemes, i.e., the operators A_h and B_h are obtained by simply restricting the usual weak form (bilinear forms $a(u,v), b(u,v)$) to the subspace S_h. Suppose (2.4) holds, then (2.2a) holds. ■

While there is much to be done to obtain such results (as in theorem 3) for other discretizations, theorems 2 and 3 complete our discussion of the H_1 case. We now turn to the L_2 case. Our first results are for the case where both A and B are H_2 regular. That is: there exists $K_1(A), K_1(B)$ such that, for every $f \in L_2$, $Au = Bv = f$ implies $u, v, \in H_2$ and

$$||u||_{H_2} \leq K_1(A)||f||_{L_2}, \tag{2.5a}$$

$$||v||_{H_2} \leq K_1(B)||f||_{L_2} \tag{2.5b}$$

Theorem 4[MP]: Suppose A, B are invertible and (2.5a), (2.5b) hold. Then

(a) AB^{-1} is a bounded operator mapping L_2 into L_2 with

$$||AB^{-1}||_{L_2} \leq K < \infty$$

if the domain of A equals the domain of B. That is, if A and B have the same boundary conditions.

(b.) $B^{-1}A$ (which is originally defined on the domain of A) can be extended to a bounded operator mapping L_2 into L_2 with

$$||B^{-1}A||_{L_2} \leq K < \infty$$

if the domain of A^* equals domain of B^*. That is, if A^* and B^* have the same boundary conditions.

Proof: The proof of (a) is immediate. Since (2.5b) holds, $B^{-1} : L_2 \to H_2 \cap D(B) = D(A)$, boundedly. And, of course, for $\phi \in D(A)$, hence in H_2

$$||A\phi||_{L_2} \leq K_2(A)||\phi||_{H_2}.$$

Hence

$$||AB^{-1}f||_{L_2} \leq K_2(A)K_1(B)||f||_{L_2}$$

The proof of (b) follows from (a) and the relationship

$$||B^{-1}A||_{L_2} = ||A^*(B^*)^{-1}||_{L_2} \quad \blacksquare$$

Theorem 5: Suppose A, B are invertible and, not only (2.5a), (2.5b) hold, but all invertible second order elliptic operators E of the form (1.1), (1.2) with smooth (say C^∞) coefficients and boundary conditions which use the same decomposition of $\partial\Omega = \Gamma_0 \cup \Gamma_1$ as either A or B also are H_2 regular. Note: this condition is satisfied whenever (1) $\partial\Omega$ is smooth and (2) distance $(\Gamma_0(A), \Gamma_1(A)) > 0$. And, (1.12a) and (1.12b) are extremely unlikely when (2) is not satisfied — see [G]. Then, the sufficient conditions of Theorem 4 are also necessary.

Proof: The proof of this theorem given in [MP] depends on a constuction and is somewhat technical. Hence, we omit it. ■

In this context the results for the discrete operators A_h, B_h depend on two conditions:

Condition Op: The family A_h satisfies Condition Op if there exists a constant $M_1(A)$, depending on A, but not on h, such that; for every $f \in L_2$ we have

$$||A_h^{-1}f - A^{-1}f||_{L_2} \leq h^2 M_1(A)||f||_{L_2}.$$

Remark: When A is H_2 regular it is reasonable to expect that Condition Op holds [Ci]. Conversely, if Condition Op holds then A is H_2 regular [W].

Condition INV: The family A_h satisfies Condition INV, if there exists a constant $M_2(A)$, depending on A but not on h such that; for every $u^h \in S_h$ we have

$$||A_h u^h||_{L_2} \leq M_2(A)h^{-2}||u^h||_{L_2}.$$

Theorem 6: Let A and B be two invertible uniformly elliptic operators which are H_2 regular. Let the families of discretizations A_h, B_h satisfy both Condition Op and Condition INV. Then

(a) Let the Boundary Conditions for A be the same as the Boundary Conditions for B. Then there is a constant K_R, independent of h, such that

$$||A_h B_h^{-1}||_{L_2} + ||B_h A_h^{-1}||_{L_2} \leq K_R.$$

(b) Let the Boundary Conditions for A^* be the Boundary Conditions for B^*. Then there is a constant K_L, independant of h, such that

$$||B_h^{-1} A_h||_{L_2} + ||A_h^{-1} B_h||_{L_2} \leq K_L.$$

Proof: See [MP]. The proof of (b) without the assumption on boundary conditions but with the equivalent assumption that $A^{-1}B$ and $B^{-1}A$ could be defined as bounded operartors in L_2 was given in [BP]. Unfortunately, the authors of [BP] were unaware of theorems 4 and 5 and hence made an error in the example they discussed. ∎

Theorem 7 [MP]: Let A and B be invertible, uniformly elliptic operators which satisfy

$$D(A^*) \neq D(B^*).$$

Let A_h, B_h be families of discretizations which satisfy conditions OP. Then there is a constant $K > 0$ such that

$$||B_h^{-1} A_2||_{L_2} \geq K h^{-1/2}$$

$$||A_h^{-1} B_h||_{L_2} \geq K h^{-1/2}. \blacksquare$$

Before we discuss L_2 estimates without H_2 regularity we digress to discuss some computational results.

3. One Dimensional Computational Results

Let

$$Av = -(a(x)v')' + a_1v' + av, 0 < x < 1 \tag{3.1a}$$

with boundary conditions

$$v(0) = 0, v'(1) + \alpha v(1) = 0; \tag{3.1b}$$

while

$$Bv = -(b(x)v')' + b_1v' + b_0v, 0 < x < 1 \tag{3.2a}$$

with boundary conditions

$$v(0) = 0, v'(1) + \beta v(1) = 0.$$

We assume $a(x)$, $b(x)$ are smooth, positive, and bounded away from zero. The discrete operators are obtained by simple central differences (Note: finite difference equations, not finite element equations) See [JMPW] for a more detailed discussion of the experimental study. In this report we present a few typical examples which illuminate the later discussion.

Computation 1

$$Av = -v'' + 8v' \quad v(0) = 0, v'(1) = 0 \tag{3.3a}$$

$$Bv = -v'' \qquad v(0) = 0, v'(1) + 8v(1) = 0. \tag{3.3b}$$

In this case we expect

$$C_h(B_h^{-1}A_h) = ||A_h^{-1}B_h||_h||B_h^{-1}A_h||_h \leq K \tag{3.4}$$

where

$$||v||_h = (h\Sigma|v_k|^2)^{1/2} \tag{3.5}$$

The results are summarized in Table 1

TABLE 1

Table 4.1 Singular Values of $(B_h)^{-1}A_h$				
N	$C((B_h)^{-1}A_h)$	$\sigma(N)$	$\sigma(N-1)$	$\sigma(1)$
40	6.1493	0.4430	0.9189	2.7239
121	6.3406	0.4339	0.8935	2.7514
364	6.3488	0.4345	0.8901	2.7587
769	6.3438	0.4351	0.8897	2.7604

Computation 2:

$$Av = -v'' + 8v' \quad v(0) = v'(0) = 0 \tag{3.6a}$$

$$Bv = -v'', \quad v(0) = v'(0) = 0 \tag{3.6b}$$

The results are summarized in Table 2.

TABLE 2

Table 4.2 Singular Values of $(B_h)^{-1}A_h$				
N	$C((B_h)^{-1}A_h)$	$\sigma(N)$	$\sigma(N-1)$	$\sigma(1)$
40	72.416	0.4138	0.6798	29.967
121	158.70	0.3231	0.5218	51.274
364	434.78	0.2033	0.4894	88.397
769	900.73	0.1424	0.4842	128.27

These computations are consistent with results of [MP]; that is

$$C_h((B_h^{-1})A_h) \quad \geq \quad Kh^{-1}$$

Nevertheless, these results raised additional questions. The fact is: The Conjugate Gradient Iterations based on the normal equations converged much faster that one would expect from (1.8). Therefore, we undertook further computations exploring the distibution of the singular values.

Computation 3:

$$Av = -v'' + 8v' \quad 0 < x < 1 \tag{3.9a}$$

$$v(0) = 0, \quad v'(1) = 0 \tag{3.9b}$$

$$Bu = -u'' \tag{3.10a}$$

$$u(0) = 0, \quad u'(1) + 8u(1) = 0 \tag{3.10b}$$

In this case we expect (3.4a) to hold. Actually, B_h^{-1} was replaced by $\hat{B}_h^{-1}$, a multigrid sweep for the solution of B. Figure 3 shows the distribution of the singular value of $\hat{B}_h^{-1}$ for 4 different calculations. In this figure, μ denotes the number of unknowns. The numbers "j; num" on the right of the lines are to be read as follows:

j = number of singular values > 2
num = value of the largest singular value

Observe the "clustering" of these singular values about $x = 1$. In fact, the clustering is actually stronger. The printer could not handle the large number of values very close to "1."

Computation 4:

$$Av = -(a(x)v')' + 8v', \quad 0 < x < 1 \tag{3.11a}$$

$$v(0) = 0 \quad v'(1) = 0 \tag{3.11b}$$

$$Bu = -u'', \tag{3.12a}$$

$$u(0) = 0, \quad u'(1) + 8u(1) = 0 \tag{3.12b}$$

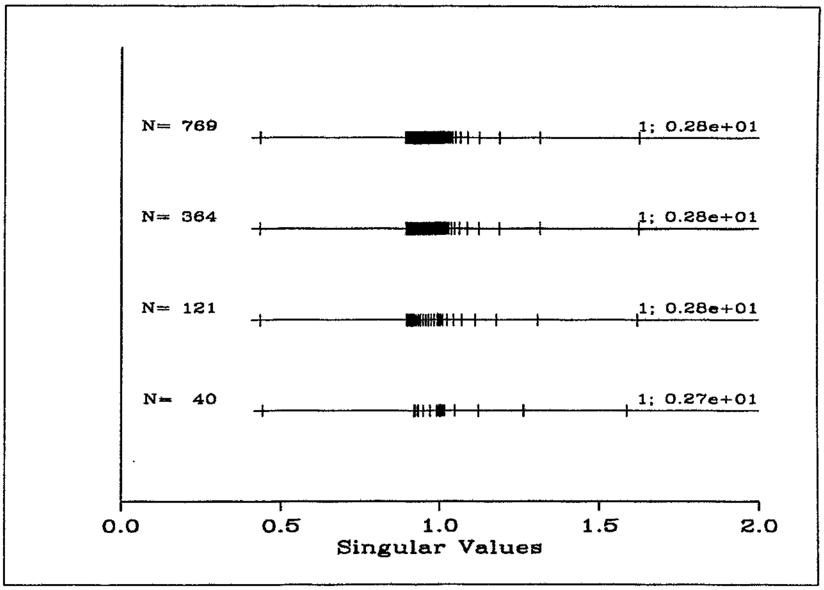
with

$$a(x) = 1 + 1/2 \sin \pi x$$

In this case we again expect (3.4) to hold.

Figure 3

Figure 4.3 Singular Value Distribution of $(B_h^{(1)})^{-1}A_h$



Observe the clustering of the singular values in he interval [1/3, 3/2], the range of the function $a(x)$. Observe also that these singular values actually "fill in" that interval.

4. Results Without H_2 Estimates

The computational results, and the theoretical explanation of them found in [JMPW] are special cases of the result in [GMP].

We no longer assume H_2 regularity. We no longer assume Condition OP. We no longer assume Condition INV. However, we do assume

$$a_{ij}(x,y) = \mu(x,y)b_{ij}(x,y) \tag{4.1a}$$

$$0 < \mu_0 \leq \mu(x,y) \leq \mu_1 \tag{4.1b}$$

Because we are unable to do the complete "integration by parts" or applications of the "divergence theorem" necessary to obtain A^*, B^*; we deal with $A^{\#}$ and $B^{\#}$ the operators we would have obtained (as adjoints) if such procedures were correct.

Theorem 4.1 [GMP]: Let A, B and B^* be invertible. Let

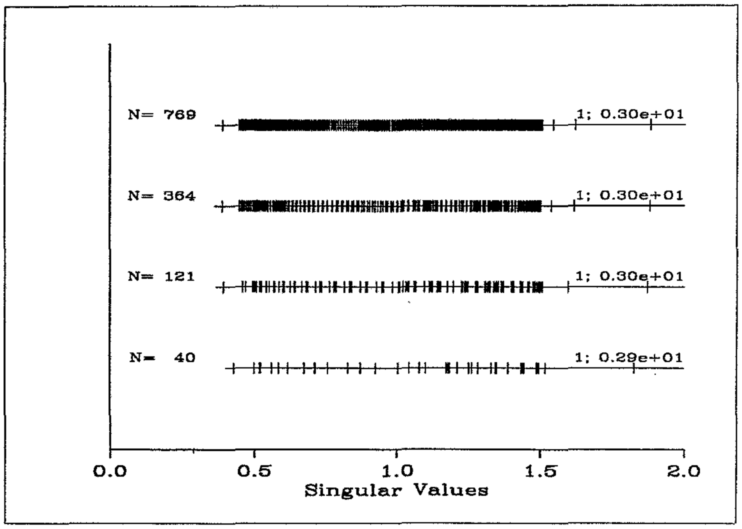
$$\Gamma_0(A) = \Gamma_0(B) \tag{4.2}$$

Let

$$L = B^{-1}A, \quad Q = L - \mu I. \tag{4.3}$$

Figure 4

Figure 4.7 Singular Value Distribution of $(B_h^{(1)})^{-1}A_h$



Then L and Q are bounded operators on $L_2(\Omega)$ if and only if: the boundary conditions for $A^{\#}$ are the same as the boundary conditions for $B^{\#}$.

Moreover, in that case, $\exists C$ such that

$$||Qu||_{H_1} \leq C||u||_{L_2} \tag{4.4}$$

That is, Q is a compact operator on $L_2(\Omega)$ ∎.

Theorem 4.2: Let A and B be invertible. Let (4.2) hold. Let

$$R = AB^{-1}, \quad \hat{Q} = R - \mu I. \tag{4.5}$$

Then R and $\hat{Q}$ are bounded operators on $L_2(\Omega)$ if and only if: the boundary conditions for A are the same as the boundary conditions for B.Moreover, in that case there is a $C^1 > 0$ such that

$$||\hat{Q}u||_{H_{1/2}} \leq C'||u||_{L_2} \tag{4.6}$$

That is $\hat{Q}$ is a compact operator on $L_2(\Omega)$ ∎.

Theorem 4.3: Let A_h and B_h be discretizations of A and B obtained by simply restricting the weak form to S_h. (I) Assume A_h^* and B_h^* are invertible. In particular, there are constants β, α independant of h, such that

$$||(B_h^*)^{-1}v^h||_{H_1} \leq \beta||v^h||_{L_2}, \quad ||(A_h^*)^{-1}v^h||_{H_1} \leq \alpha||v^h||_{L_2}. \tag{4.7}$$

(II) Assume the boundary conditions of $A^{\#}$ are the same as the boundary conditions of $B^{\#}$. Then, there is a constant K, independant of h such that

$$||L_h||_{L_2} = ||B_h^{-1}A_h||_{L_2} \leq K \tag{4.8}$$

Further, under reasonable hypothesis on $B_h^{-1}, (B_h^*)^{-1}, (A_h^{-1})$ and $(A_h^*)^{-1}$ we have:

Let $\sigma^j(h) \geq \sigma^{j+1}(h) \geq 0$ be the singular values of $L_h = B_h^{-1}A_h$. Then

(A.) For every $\epsilon > 0, \exists J = J(\epsilon)$ and $h_0 > 0$ such that for all $h, 0 < h \leq h_0$, there are at most $J(\epsilon)$ such singular values outside the interval

$$[\mu_0 - \epsilon, \mu_1 + \epsilon]$$

(B) The singular values of L_h "fill in" the interval $[\mu_0, \mu_1]$. ∎

A similar theorem holds for $R_h = A_h B_h^{-1}$. ∎

References

[FMP] Faber, V., T.A. Manteuffel, S.V. Parter, "On the Theory of Equivalent Operators and Applications to the Numerical Solution of Uniformly Elliptic Partial Differential Equations," *Advances in Applied Mathematics 11*, pp 109-163 (1990).

[G] Grisvard, P., *Elliptic Problems in Nonsmooth Domain*, Pitman, Boston, 1985.

[GMP] Goldstein, C.J., T.A. Manteuffel, S.V. Parter, "Preconditioning and Boundary Conditions Without H_2 Estimates: L_2 Conditions Numbers and the Distribution of the Singular Values," Los Alamos National Laboratory Report LA-UR-90-1856.

[JMPW] Joubert, W.D., T.A. Manteuffel, S.V. Parter, and S-P Wong, "Preconditioning Second Order Elliptic Operators: Experiment and Theory," Los Alamos National Laboratory Report, LA-UR-90-1615.

[MP] Manteuffel, T.A. and S.V. Parter, "Preconditioning and Boundary Conditions," *SIAM Journal of Numerical Analysis 27*, pp 656-694 (1990).