CHAPTER 47

A Domain Decomposition Method for Elliptic Boundary Value Problems: Application to Unsteady Incompressible Fluid Flow[1]

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Abstract. A consistent variational discretization of the time-dependent Navier-Stokes equations typically leads to a Poisson-like equation for the pressure which needs to be solved at each time step. The fact that the pressure is in L^2 makes this operator more difficult to invert than the standard (H^1) Laplace operator. In this paper we propose a new pressure solver with the following key ingredients: domain decomposition; a global coarse system to take care of long-range interactions (solved directly); a global fine system to solve for the *difference* between the final solution and the coarse "skeleton" solution (solved iteratively); a *local* decoupled preconditioning system (solved directly for each subdomain). Numerical results suggest that the number of iterations is independent of the number of subdomains.

1 Introduction

In this paper we discuss a hybrid direct-iterative method for inverting the consistent pressure Poisson operator arising from discretization of the time-dependent incompressible Navier-Stokes equations based on variational forms (spatial) (Brezzi, 1974; Girault and Raviart, 1986; Bernardi, Maday and Métivet, 1987), and operator splitting techniques (temporal) (Maday, Patera, and Rønquist, 1991). Many domain decompostion methods are based on the idea of combining the solution of independent systems for subdomains of the original domain with a global iterative procedure to properly propagate information between these subdomains (Dryja and Widlund, 1989; Mandel, 1989). The method we propose here combines these ideas with ideas from multigrid methods (Brandt, 1977; Hackbusch and Trottenberg, 1982; Bank and Douglas, 1985; Rønquist and Patera, 1987; Maday and Muñoz,

[1]This work was supported by Nektonics, Inc. and by NASA's materials processing and micro-gravity programs under contracts NAS1-19102 and NAS3-26132

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1988), however, we believe the way these are combined is new, and suggests a new way of inverting other elliptic operators as well.

The fact that the consistent pressure Poisson operator is an L^2 operator makes it harder to invert than the standard (H^1) Laplace operator. A domain decomposition approach yields a pressure operator which enforces no continuity conditions for the pressure across subdomain interfaces, however, there still exists couplings between the subdomains via the incompressibility enforcement. In the context of non-overlapping subdomains, the pressure operator does not even have any shared interface nodes. Hence, there exists no natural interface operator similar to the Schur complement associated with the standard discrete Laplace operator, which again excludes an iterative substructuring approach (Bjørstad and Widlund, 1986), and the use of some of the recently developed interface preconditioners (Chan and Keyes, 1989).

To illustrate the method we start with the pressure system derived using a domain decomposition approach. As a particular example we use a spatial discretization based on the spectral element method (Patera, 1984; Maday and Patera, 1989; Rønquist, 1988), and we shall in the following treat each element as one subdomain. The original pressure system is then decomposed into two new pressure systems. The first set of equations is a consistent global coarse system which serves the purpose of computing a global "skeleton" for the solution, in our case, the constant pressure levels inside each element or subdomain. The dimension of this system is small (equal to the number of subdomains), and is solved directly. The second system is a consistent global set of equations for the *difference* between the final solution and the global "skeleton". In this respect the method is almost like a *reversed* multigrid procedure in which the difference is computed from the coarse system rather than the fine system (coarse grid correction). This second pressure system is solved iteratively using block preconditioned conjugate gradient iteration (Golub and Van Loan, 1983).

A consequence of this type of decompostion is that the global coarse system inherits the original boundary conditions, however, in the case of the the consistent pressure Poisson operator there is no pressure boundary conditions; the imposition of Dirichlet velocity boundary conditions implicitly results in a Neumann operator for the pressure. Furthermore, since the second set of equations is associated with computing *differences* in the nodal solution values, it naturally suggests the use of a *local* decoupled preconditioner, resulting in no communication cost associated with a parallel implementation.

The method described in this paper is quite general, and applies to other elliptic operators as well, e.g., the standard (H^1) Laplace operator. Another application where this method should prove useful is for non-conforming finite element methods (Strang, 1973; Anagnostou, Maday, Mavriplis, and Patera, 1989), in which case methods based on interface operators become more complicated.

The paper is organized as follows. In Section 2 we briefly review the consistent spatial discretization of the Stokes system based on variational forms, and explain the necessary notation. In Section 3 we describe the method as applied to the discrete consistent pressure Poisson operator, and in Section 4 and 5 we give some numerical results for the one- and

two-dimensional case, which suggest that the number of iterations is independent of the number of subdomains.

2 The Stokes and Navier-Stokes problems

We first consider the steady Stokes problem in d space dimensions: Find a velocity $\mathbf{u}$ and a pressure p in a domain $\Omega \in \mathcal{R}^d$ such that

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in\ \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad in\ \Omega, \tag{2}$$

subject to homogeneous Dirichlet velocity boundary conditions, $\mathbf{u} = \mathbf{0}$, on the domain boundary $\partial\Omega$. Here $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{f}$ is the prescribed force and ν is the kinematic viscosity.

The equivalent variational formulation of (1)-(2) is: Find $(\mathbf{u}, p)$ in (X^d, M) such that

$$\mu(\nabla\mathbf{u}, \nabla\mathbf{w}) - (p, \nabla \cdot \mathbf{w}) = (\mathbf{f}, \mathbf{w}) \qquad \forall \mathbf{w} \in X^d, \tag{3}$$

$$(\nabla \cdot \mathbf{u}, q) = 0 \qquad \forall q \in M, \tag{4}$$

where the proper spaces for $\mathbf{u}$ and p such that (3)-(4) is well posed are (Brezzi, 1974; Girault and Raviart, 1986)

$$X = H_0^1(\Omega) \tag{5}$$

$$M = L_0^2(\Omega) = L^2(\Omega) \cap \{\phi \in L^2(\Omega);\ \int_\Omega \phi\, d\Omega = 0\}. \tag{6}$$

Here $L_0^2(\Omega)$ is the space of all functions which are square integrable over Ω with zero average, while $H_0^1(\Omega)$ is the space of all functions which are square integrable, whose derivatives are also square integrable over Ω, and which satisfy the homogeneous Dirichlet velocity boundary conditions.

We consider here numerical approximations to the Stokes problem based on the variational form (3)-(4): Find $(\mathbf{u}_h, p_h) \in (X_h^d, M_h)$ such that

$$\mu(\nabla\mathbf{u}_h, \nabla\mathbf{w})_h - ((p_h, \nabla \cdot \mathbf{w}))_h = (\mathbf{f}, \mathbf{w})_h \qquad \forall \mathbf{w} \in X_h, \tag{7}$$

$$((\nabla \cdot \mathbf{u}_h, q))_h = 0 \qquad \forall q \in M_h, \tag{8}$$

where for each value of the parameter h, $X_h \subset X$ and $M_h \subset M$ are compatible subspaces of X and M (Brezzi, 1974; Babuska, 1971; Girault and Raviart, 1986; Bernardi, Maday and Métivet, 1987) that approach X and M as the discretization parameter h goes to zero. In (7)-(8) $(\cdot,\cdot)_h$ and $((\cdot,\cdot))_h$ denote evaluation of the continuous inner product $(\cdot,\cdot)$ by numerical quadrature (note however that the $(\cdot,\cdot)_h$ and $((\cdot,\cdot))_h$ may be different).

Choosing appropriate (compatible) discrete spaces X_h and M_h with associated bases, we arrive at a set of algebraic equations given in matrix form as

$$\underline{A}\,\underline{u}_i - \underline{D}_i^T\,\underline{p} = \underline{B}\,\underline{f}_i, \qquad i = 1, \ldots, d, \tag{9}$$

$$\underline{D}_i\,\underline{u}_i = 0, \tag{10}$$

where $\underline{A}$ is the discrete Laplace operator ($\nu = 1$), $\underline{B}$ is the mass matrix, $\underline{\mathbf{D}} = (\underline{D}_1, ..., \underline{D}_d)$ is the discrete gradient operator, and underscore refers to basis coefficients. In (9)-(10) we assume that the homogeneous boundary conditions are imposed by eliminating appropriate rows and columns.

By employing a Uzawa decoupling procedure (block Gaussian elimination) to the saddle Stokes system (9)-(10), we arrive at the following discretely equivalent pressure system

$$\underline{D}_i \underline{A}^{-1} \underline{D}_i^T \underline{p} = -\underline{D}_i \underline{A}^{-1} \underline{f}_i \ , \tag{11}$$

which is well conditioned and can be solved efficiently by employing a global (nested) iterative procedure (Maday, Meiron, Patera, and Rønquist; 1991).

The extension to solve the full unsteady Navier-Stokes equations is done most efficiently by combining the above spatial discretization with an operator splitting approach in time (Maday, Patera, and Rønquist, 1991). Such an approach requires, at each time step, the solution of the following pressure system,

$$\underline{D}_i \underline{B}^{-1} \underline{D}_i^T \underline{p} = \underline{g}, \tag{12}$$

where $\underline{g}$ is a vector representing the known (nodal) inhomogeneities. Unlike the steady Stokes pressure operator in (11), the unsteady Stokes/Navier-Stokes pressure operator in (12),

$$\underline{E} \equiv \underline{D}_i \underline{B}^{-1} \underline{D}_i^T, \tag{13}$$

is ill-conditioned. The operator $\underline{E}$ is, in fact, the consistent discrete pressure Poisson operator with a spectrum which scales roughly like the standard discrete Laplacian, $\underline{A}$.

2.1 Spectral Element Discretization

In the following we limit our discussion to the conforming spectral element method (Patera 1984; Maday and Patera 1989; Rønquist 1988), in which the domain Ω is broken up into K disjoint subdomains,

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}^k. \tag{14}$$

The approximation spaces (subspaces) can be expressed as

$$X_h = X \cap P_{N,K}(\Omega) \tag{15}$$

$$M_h = M \cap P_{N-2,K}(\Omega), \tag{16}$$

where

$$P_{\mathcal{N},K}(\Omega) = \{\Phi \in L^2(\Omega); \Phi_{|\Omega_k} \in P_{\mathcal{N}}(\Omega_k), k = 1, ..., K\}, \tag{17}$$

and $P_{\mathcal{N}}(\Omega_k)$ denotes the space of all polynomials of degree less than or equal to $\mathcal{N}$ in each spatial direction. We refer to Maday, Patera and Rønquist (1987), and Bernardi, Maday and Métivet (1987) for a justification of the choice of discrete spaces. By choosing different polynomial degrees for the velociy and pressure, the main conclusions from the theoretical analysis can be summarized as: (i) the discrete solution is unique, that is, there exist no

spurious pressure modes, and (ii) spectral convergence is obtained as the polynomial degree, N, is increased for fixed number of elements (or subdomains), K.

In order to arrive at (9)-(10) we also need to choose appropriate bases for X_h and M_h in (7)-(8). Within each element Ω^k we express the velocity and pressure in terms of high-order Lagrangian interpolants through the tensor-product Gauss-Lobatto and Gauss points, respectively. The inner-products in (7)-(8) are evaluated using Gauss numerical quadrature (Davis and Rabinowitz, 1985), Gauss Legendre for $((\cdot,\cdot))_h$ and Gauss-Lobatto Legendre for $(\cdot,\cdot)_h$. Choosing appropriate test functions we arrive at a set of algebraic equations of the form (9)-(10).

3 Domain decompostion pressure solver

The discrete pseudo Laplace operator $\underline{E}$ is more difficult to invert iteratively than the standard discrete Laplace operator $\underline{A}$. This is mainly due to the fact that $\underline{E}$ is a discrete version of a mixed L^2/H^1-operator in the sense that the pressure $p \in L_0^2(\Omega)$, while the velocity $\mathbf{u} \in (H_0^1(\Omega))^d$, and that both the compatible subspaces X_h and M_h are needed in order to define $\underline{E}$. A pressure $p \in L_0^2(\Omega)$ also implies that no pressure boundary conditions are imposed; the imposition of homogeneous Dirichlet velocity boundary conditions implicitly results in a Neumann operator for the pressure. A consequence of this is that the discrete system (12) is singular (the pressure is determined only up to a constant), and hence, for solvability, the right-hand-side $\underline{g}$ must be orthogonal with respect to the constant vector ($\underline{1}^T \underline{g} = 0$).

The fact that $p \in L_0^2(\Omega)$ also implies the complete lack of a natural interface operator similar to the Schur complement associated with the standard (H^1) discrete Laplace operator, $\underline{A}$, which excludes an iterative substructuring approach (Bjørstad and Widlund, 1986), and the use of some of the recently developed interface preconditioners (Chan and Keyes, 1989). Note however that, although the pressure $p \in L^2$, the resulting discrete pressure system (12) still couples subdomains via the incompressibility enforcement.

We will now describe a method to invert $\underline{E}$ which employs the following key ingredients:

- domain decomposition
- a global coarse system to take care of long-range interactions (solved directly).
- a global fine system to solve for the *difference* between the final solution and the global "skeleton" solution (solved iteratively).
- a *local* decoupled preconditioning system (solved directly and independently on each subdomain).

Although the technique here is applied to the discrete (L^2) pressure operator, the idea is more general and can be applied to other elliptic (L^2 or H^1) operators as well. A more complete description and analysis is given in Rønquist (1991).

The objective here is to develop a solver for the system

$$\underline{E}\underline{p} = \underline{g} \tag{18}$$

where $\underline{g}$ is a known inhomogeneity; this solver can then be used to solve (12) at each time step. We take a function decomposition approach in which we decompose the finite-dimensional pressure space M_h into two disjoint parts, i.e., we search for a solution

$$p_h = p_{h,0} + p_{h,N} \in M_h = M_{h,0} \oplus M_{h,N} \tag{19}$$

where

$$M_{h,0} = L_0^2(\Omega) \cap \mathcal{P}_{0,K}(\Omega) \tag{20}$$

$$M_{h,N} = L_{0,K}^2(\Omega) \cap \mathcal{P}_{N-2,K}(\Omega) \tag{21}$$

and

$$L_{0,K}^2(\Omega) = \{\phi \in L_0^2(\Omega_k), k = 1, ..., K\}. \tag{22}$$

With this decomposition we can develop an efficient solver which has two main parts. One part is used to find a $p_{h,0} \in M_{h,0}$ in order to establish a *global* "skeleton" for the solution, consisting of finding the constant pressure levels inside all the elements or subdomains. The other part is used to compute a *local* variation $p_{h,N} \in M_{h,N}$ around this skeleton, where local means on an elemental or subdomain level. The final solution $p_h = p_{h,0} + p_{h,N}$ is then found by an interaction of these two parts, involving both iterative and direct solution techniques.

We proceed by writing (18) as

$$\underline{E}(\underline{p}_N + \underline{I}\underline{p}_0) = \underline{g}, \tag{23}$$

where $\underline{p}_N$ are the nodal values associated with $p_{h,N}$, and $\underline{p}_0$ is a vector containing the pressure levels inside all the elements. The operator $\underline{I}$ is an interpolation operator which expresses these constant pressure levels in terms of the basis for $\underline{M}_h$. Note that

$$dim(\underline{M}_h) = K(N-1)^d - 1 \tag{24}$$

$$dim(\underline{M}_{h,N}) = K(N-1)^d - K \tag{25}$$

while

$$dim(\underline{M}_{h,0}) = K - 1 \quad . \tag{26}$$

We can now derive the set of equations for the pressure levels $\underline{p}_0$ by multiplying (23) from the left by $\underline{I}^T$, to arrive at

$$\underline{E}_0\underline{p}_0 = \underline{g}_0 \quad , \tag{27}$$

where

$$\underline{E}_0 = \underline{I}^T\underline{E}\underline{I} \tag{28}$$

and

$$\underline{g}_0 = \underline{I}^T(\underline{g} - \underline{E}\underline{p}_N). \tag{29}$$

Note that in deriving the coarse system (27) we still keep $\mathbf{u}_h \in X_h^d$. In Rønquist (1991) it is shown how the the matrix $\underline{E}_0$ can be derived in a much more efficient way than is done in (28).

By formally solving (27) for $\underline{p}_0$ and substituting the result into (23), we can also derive the set of equations for $\underline{p}_N$,

$$\underline{E}_N \underline{p}_N = \underline{g}_N \quad , \tag{30}$$

where

$$\underline{E}_N = \underline{E} - \underline{E}\,\underline{I}\,\underline{E}_0^{-1}\underline{I}^T\underline{E}, \tag{31}$$

and

$$\underline{g}_N = \underline{g} - \underline{E}\,\underline{I}\,\underline{E}_0^{-1}\underline{I}^T\underline{g}. \tag{32}$$

The algorithm now works as follows. First, we form the right-hand-side $\underline{g}_N$ in (32), involving a single inversion of of the coarse pressure operator $\underline{E}_0$. We then proceed with solving (30) for $\underline{p}_N$ using preconditioned conjugate gradient iteration (see below). From (31) we note that each conjugate gradient iteration also requires the inversion of the coarse pressure operator. Once the solution $\underline{p}_N$ has been computed, the right-hand-side $\underline{g}_0$ for the coarse pressure system is computed from (29), and $\underline{p}_0$ is found by solving (27) using a direct solver.

3.1 Solvability

As mentioned earlier the original pressure operator $\underline{E}$ in (18) has one zero eigenvalue corresponding to the hydrostatic mode (constant eigenvector). After the decompostion (19) this mode will be inherited by the coarse pressure operator $\underline{E}_0$ in (27), which can easily be verified by multiplying (26) from the left by $\underline{1}^T$ and using the symmetry property of $\underline{E}$.

Due to the decompostion (19) the dimension of the nullspace for $\underline{E}_N$ will be K, the number of subdomains. This can be verified by multiplying (30) from the left by $\underline{I}^T$. Hence, the corresponding eigenvectors are the vectors which are unity within one subdomain and are zero in all the other subdomains. In order to ensure convergence in the conjugate gradient iteration we must make sure that the residual stays orthogonal to this nullspace.

3.2 Additional remarks

In practice, the nullspace of $\underline{E}_N$ also allows us to construct a *block*($\underline{E}$) preconditioner consisting of the elemental *local* $\underline{E}$-matrices, effectively corresponding to imposing homogenous Dirichlet velocity boundary conditions on all the external *and* internal elemental (or subdomain) interfaces. Each of the local $\underline{E}$-matrices will therefore be a local Neumann operator, and

$$ker(block(\underline{E})) = ker(\underline{E}_N)\,. \tag{33}$$

We now make several comments regarding this approach. First, the fact that we are able to use a decoupled *block*($\underline{E}$) preconditioner, suggests the use of a direct solver (or any fast elemental solver) for the preconditioner, implying that (for fixed geometry) only

elemental back-substitutions are necessary in order to perform a preconditioning step (one per conjugate gradient iteration). This preconditioning step is also readily parallelizable as it involves *no* communication between the subdomains. Due to the decomposition (19) the *block*($\underline{E}$) preconditioner is spectrally close to $\underline{E}_N$, and preliminary numerical results suggests that the convergence rate for the system (30) is independent of the number of elements $\underline{K}$.

Second, in solving (30), the computational cost per conjugate gradient iteration is essentially the cost to compute a matrix-vector product $\underline{E}_N \underline{p}_N$, which is roughly twice the cost to perform a matrix-vector product $\underline{E}\underline{p}$, see (31). Hence, using tensor-product sum-factorization techniques (Orszag, 1980), the cost per iteration scales like KN^{d+1} in $\mathcal{R}^d$.

Third, the inversion of $\underline{E}_0$ in (27) and (31)-(32) can be done efficiently by emplying a direct solver due to the relatively small dimension of this system ($\underline{E}_0 \in \mathcal{R}^{K\times K}$). For a fixed geometry this inversion is done during the initialization, and only back substitutions are required during the transient simulation.

Finally, although we here only consider a domain decompostion resulting from a spectral element discretization, the approach is equally applicable to other types of discretizations and other elliptic operators as well.

4 One-dimensional results

We consider here the spectrum and condition number of the discrete pressure operators $\underline{E}$, $\underline{E}_N$, and $\underline{E}_0$ in the one-dimensional case. The domain is $\Omega =]-1, 1[$, which is broken up into K spectral elements (or subdomains), each of order N. Homogeneous velocity boundary conditions, $u = 0$, are imposed at $x = \pm 1$.

We first compute the eigenvalues of the pressure operators with respect to the mass matrix $\underline{\tilde{B}}$ asscociated with the pressure mesh (unpreconditioned case),

$$\underline{E}\underline{\chi}_i = \lambda_i \underline{\tilde{B}}\underline{\chi}_i \quad , i = 1, ..., K(N-2) , \tag{34}$$

$$\underline{E}_N \underline{\chi}_j = \lambda_j \underline{\tilde{B}}\underline{\chi}_j \quad , j = 1, ..., K(N-2) , \tag{35}$$

$$\underline{E}_0 \underline{\chi}_k = \lambda_k \underline{\tilde{B}}_0 \underline{\chi}_k \quad , k = 1, ..., K . \tag{36}$$

In Fig. 1 we plot all the *non-zero* eigenvalues for a spectral element discretization $K = 10$, $N = 6$, recalling that $\underline{E}$ and $\underline{E}_0$ each has an additional zero eigenvalue (hydrostatic mode), and $\underline{E}_N$ has an additional K zero eigenvalues. We note that the part of the spectrum of $\underline{E}$ which is associated with the low wavenumbers roughly coincides with the spectrum of the coarse pressure operator $\underline{E}_0$, while the spectrum of $\underline{E}_N$ is clustered with a condition number which is much smaller than for the original pressure operator, $\underline{E}$. Due to the small dimension of the coarse pressure system, the decompostion (19) allows for a very efficient way of removing the low wavenumber error components, and in this respect is similar to a multigrid-like procedure.

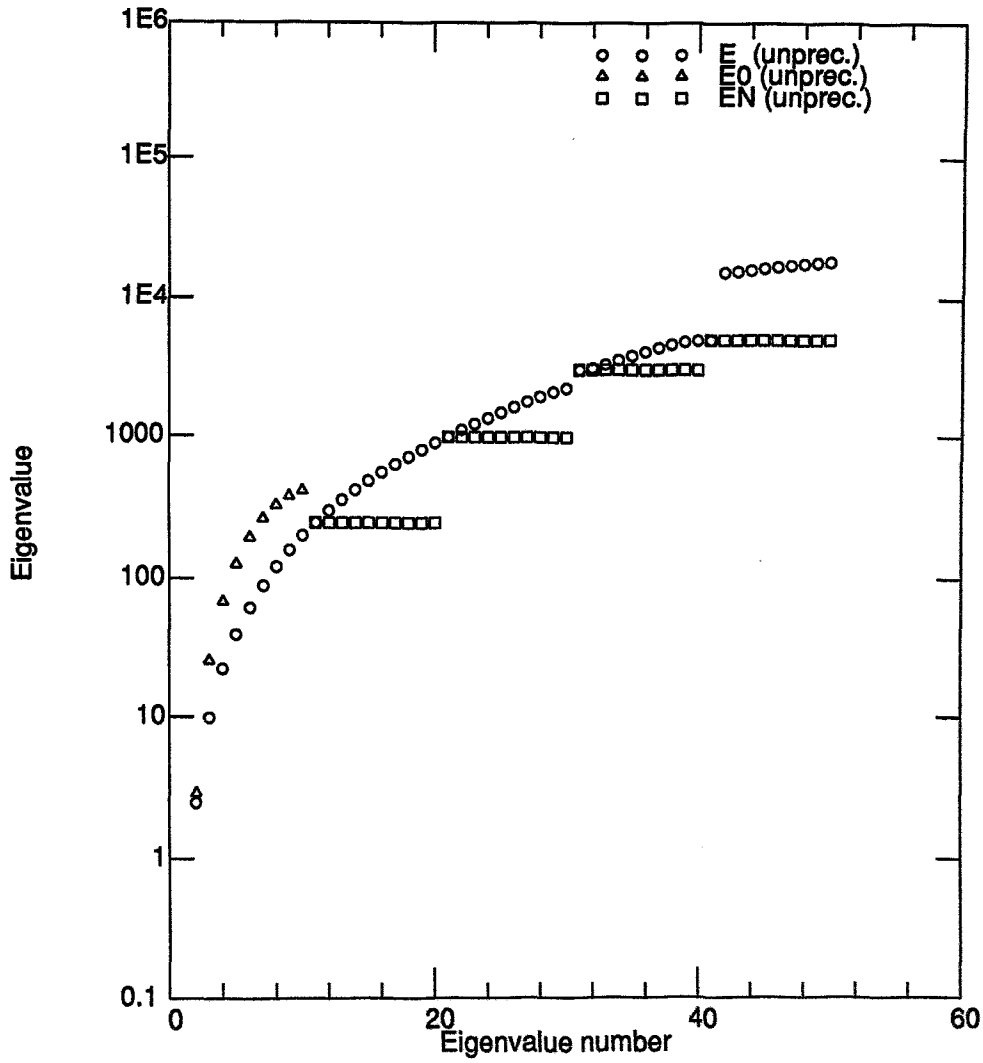

Figure 1: The spectra for the one-dimensional discrete pressure operators $\underline{E}$ (○), $\underline{E}_0$ (△), and $\underline{E}_N$ (□) with respect to the mass matrix (unpreconditioned case), for a spectral element discretization $K = 10$ and $N = 6$.

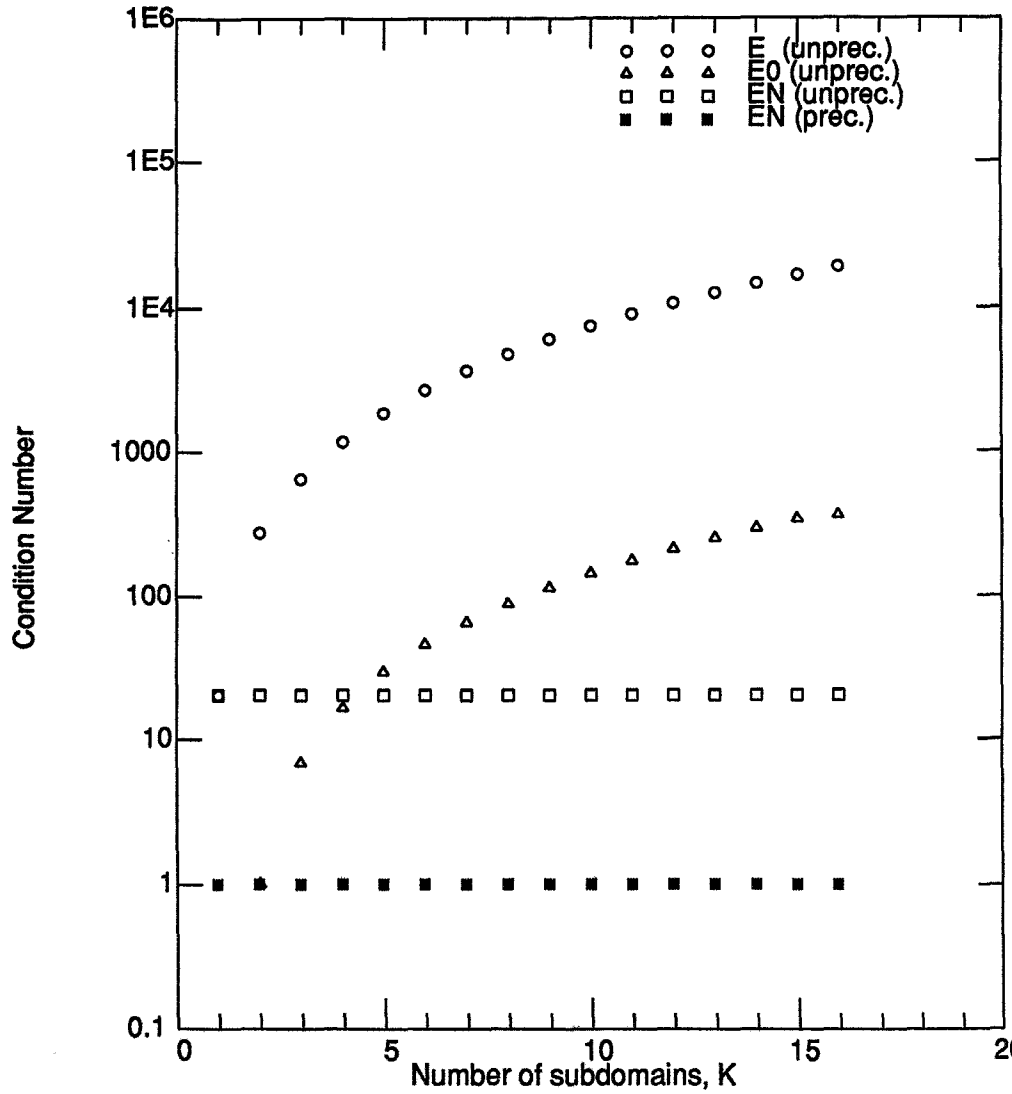

Figure 2: The condition number for the one-dimensional discrete pressure operators $\underline{E}$ (○), $\underline{E}_0$ (△), and $\underline{E}_N$ (□) with respect to the mass matrix (unpreconditioned case) as a function of the number of spectral elements, K, each of fixed order $N = 6$. The condition number for $\underline{E}_N$ preconditioned with *block*($\underline{E}$) is also plotted (■).

In Fig. 2 we plot the condition number of $\underline{E}$, $\underline{E}_N$, and $\underline{E}_0$ as a function of the number of subdomains, K. Similar to the standard (H^1) discrete Laplace operator, the condition number of $\underline{E}$ and $\underline{E}_0$ with respect to the mass matrix (unpreconditioned case) scales like K^2. Note, however, that the condition number of $\underline{E}_N$ with respect to the mass matrix (unpreconditioned case) is a constant which depends only on the order N of each element. We also plot the condition number of $\underline{E}_N$ with respect to *block*($\underline{E}$) (preconditioned case) in order to demonstrate that the *local* decoupled preconditioner *block*($\underline{E}$) suggested by our domain decompostion approach is sufficient to reduce this condition number to unity, and thus allow for perfect preconditioning in the one-dimensional case.

5 Multi-dimensional results

We now consider an unsteady Stokes calculation in a two-dimensional domain $\Omega =]-1,1[^2$ with homogeneous Dirichlet velocity boundary conditions, kinematic viscosity $\nu = 0.1$, and a body force $\mathbf{f} = (f_x, f_y)^T = (-0.6y, 0)^T$, which mimic a natural convection problem in a square cavity. The domain is broken up into K similar quadrilateral elements, each of order N. The time step is $\Delta t = 0.1$. All the two-dimensional calculations were performed on an Intel i860 hypercube.

In Table 1 we show the number of preconditioned conjugate gradient iterations required in order to reduce the initial pressure residual with 5 orders of magnitude when solving the system (30) (the first time step). We give the number of iterations for different number of subdomains K, each of fixed order $N = 7$. For completeness we also give the total number of pressure degrees-of-freedom. The numerical results suggest that the convergence rate is independent of K.

Table 1

K	No. iterations	No. pressure d.o.f.
4	25	100
16	25	400
64	28	1600
144	28	3600

Next, we repeat the last experiment, but now keeping the number of subdomains (or spectral elements) fixed to $K = 16$, while varying the order of the elements, N. From Table 2 we see that the number of preconditioned conjugate gradient iterations increases with N, and the growth rate is approximately linear. This dependence is due to the simple form of our preconditioner which consists of independent local Neumann operators, and thus does not correctly account for the modes along inter-elemental boundaries.

Table 2

N	No. iterations	No. iterations/N
5	17	3.4
7	25	3.6
9	31	3.4
11	35	3.2
13	40	3.1

ACKNOWLEDGEMENTS

The author would like to thank Prof. Anthony T. Patera for helpful comments and discussions. This work was supported by Nektonics, Inc. and by NASA's material processing and micro-gravity programs under contracts NAS1-19102 and NAS3-26132.

REFERENCES

Anagnostou, G., Maday, Y., Mavriplis, C. and Patera, A.T. (1989). On the mortar element method; generalizations and implementation, in *3rd Int. Symp. on Domain Decomposition Methods*, eds. T.F. Chan, R. Glowinski, J. Periaux and O.B. Widlund, SIAM.

Bank, R.E. and Douglas, C.C. (1985). Sharp estimates for multigrid rates of convergence with general smoothing and acceleration, *SIAM J. Numer. Anal.*, **22**, pp. 617-633.

Bernardi, C., Maday, Y. and Metivet, B. (1987). Spectral approximation of the periodic-nonperiodic Navier-Stokes equations, *Numer. Math.*, **51**, pp. 655-700.

Bjørstad, P. and Widlund, O.B. (1986). Iterative methods for the solution of elliptic problems on regions partitioned into substructures, *SIAM J. Numer. Anal.*, **23**, pp. 1097-1120.

Brandt, A. (1977). Multi-level adaptive solutions to boundary value problems, *Math. Comp.*, **31**, pp. 330-390.

Brezzi, F. (1974). On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *Rairo Anal. Numer.*, **8**, R2, p. 129.

Chan, T.F. and Keyes, D.E. (1989). Interface preconditioners for domain-decomposed convection-diffusion operators, in *3rd Int. Symp. on Domain Decomposition Methods*, eds. T.F. Chan, R. Glowinski, J. Periaux and O.B. Widlund, SIAM, pp. 245-262.

Davis, P.J. and Rabinowitz (1985). *Methods of Numerical Integration*, Academic Press.

Dryja, M. and Widlund, O.B. (1989). Towards a unified theory of domain decomposition algorithms for elliptic problems, in *3rd Int. Symp. on Domain Decomposition Methods*, eds. T.F. Chan, R. Glowinski, J. Periaux and O.B. Widlund, SIAM, pp. 3-21.

Girault, V. and Raviart, P.A. (1986). *Finite Element Approximation of the Navier-Stokes Equations*, Springer.

Golub, G.H. and Van Loan, C.F. (1983) *Matrix Computations*, John Hopkins University Press, Baltimore, Maryland.

Maday, Y. and Muñoz (1988). Spectral element multigrid. II. Theoretical justification, *J. Sci. Comput.*, **3**, 4.

Maday, Y. and Patera, A.T. (1989). Spectral element methods for the Navier-Stokes equations, in *State of the Art Surveys in Computational Mechanics* (Edited by A.K. Noor),

ASME, New York, pp. 71-143.

Maday, Y., Patera. A.T., and Rønquist, E.M. (1991). An operator-integration-factor splitting method for time-dependent problems: application to incompressible fluid flow. *J. Sci. Comput.*, to appear.

Maday, Y., Meiron, D., Patera, A.T. and Rønquist, E.M., Analysis of iterative methods for the steady and unsteady Stokes problem: Application to spectral element discretizations, *SIAM J. Sci. Stat. Comput.*, submitted.

Mandel, J (1989). Hierarchical preconditioning and partial orthogonalization for the p-version finite element method, in *3rd Int. Symp. on Domain Decomposition Methods*, eds. T.F. Chan, R. Glowinski, J. Periaux and O.B. Widlund, SIAM, pp. 141-156.

Patera, A.T. (1984). A spectral element method for fluid dynamics; Laminar flow in a channel expansion, *J. Comput. Phys.*, **54**, p.468.

Orszag, S.A. (1980). Spectral methods for problems in complex geometries, *J. Comput. Phys.*, **37**, p. 70-92.

Rønquist, E.M. (1988). *Optimal Spectral Element Methods for the Unsteady Three-dimensional Incompressible Navier-Stokes Equations*, Ph.D. Thesis, Massachusetts Institute of Technology.

Rønquist, E.M. and Patera, A.T. (1987). Spectral element multigrid. I. Formulation and numerical results, *J. Sci. Comput*, **2**, 4, p. 389.

Rønquist, E.M. (1991). A domain decompostion method for elliptic boundary value problems, in progress.

Strang, G. and Fix, G. (1973) *An Analysis of the Finite Element Method*, Prentice-Hall.