CHAPTER 53

A Multidomain Spectral Collocation Solver for the Elasticity Problem†

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Abstract
We present two different domain decomposition algorithms for the spectral collocation approximation to the two-dimensional system of linear elasticity. Single and multidomain iterative procedures are effectively combined with finite element preconditioning.

1. Introduction.

In recent years there has been a considerable development in domain decomposition methods for the numerical approximation of partial differential equations in the framework of finite differences, finite elements and spectral methods. Several reasons underly the diffused interest in these techniques. Among others, we mention the possibility of defining effective algorithms for parallel implementations. Moreover, these methods are very convenient in order to handle complex geometries, non smooth solutions, or equations containing different parameters in different regions of the physical domain.These methods have concerned various fields of applications: structural mechanics, fluid-dynamics, aerodynamics. Here we consider a domain decomposition method for spectral collocation approximations to the linear elasticity problem. For the same kind of problem, domain decomposition techniques in the framework of finite element methods have been recently developed, e.g. in [1],[7],[5],[10].

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In section 2 we formulate a multidomain problem in which the physical domain Ω is partitioned into two non-intersecting subdomains, Ω_1 and Ω_2. For the sake of simplicity, we deal with the case of two subdomains, but the discussion can be generalized to the case of several subregions.
In section 3 we introduce the spectral approximation to the multidomain problem. This consists of enforcing the equilibrium equations at the interior Gaussian nodes of each Ω_i. Furthermore the boundary conditions are imposed at the nodes of the physical boundary. Finally the transmission conditions (the continuity of the displacements and of the normal stresses), are asked to be satisfied at the collocation points of the common interface.
In the last two sections we propose two iterative methods to achieve the solution to the multidomain spectral problem.
In section 4 we generalize the Dirichlet/Neumann method to the case of the elasticity problem (e.g. [1], [12]). Each local subproblem in Ω_i is solved by gradient-like inner iterations with a *local* finite element preconditioner defined by the finite element matrix of the elasticity system involving the Gaussian grid in Ω_i.
In section 5 we define a *global* finite element preconditioner, associated to the grid of all collocation nodes in Ω, and define an iterative procedure for the global preconditioner. Each step consists of evaluating the residual of the spectral system in each Ω_i, and then in solving a finite element problem for the global domain. The latter task can be accomplished, e.g., by a finite element domain decomposition approach as, e.g., in [15].
We also present some numerical results concerning the convergence's properties of the two methods. In particular, we show that the rate of convergence of both schemes is independent of the number of collocation nodes in each Ω_i.

2. The linear elasticity problem and its domain decomposition formulation.

Let us consider an elastic body B occupying the domain $\Omega \subset \mathbb{R}^2$, whose boundary $\partial\Omega$ is decomposed into two parts Γ_N and Γ_D. Let $\mathbf{f} = (f_1, f_2)$ be a surface load on Ω and $\mathbf{g} = (g_1, g_2)$ a boundary load on Γ_N. We assume that B is fixed along the boundary Γ_D. We denote by $\mathbf{u} = (u_1, u_2)$ the displacement and by $\underline{\sigma} = (\sigma_{ij})(\mathbf{u})$ the stress tensor. We also introduce the deformation $\underline{\epsilon}(\mathbf{u}) = (\epsilon_{ij}(\mathbf{u}))$ (strain tensor) associated with $\mathbf{u}$

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2 \tag{2.1}$$

The constitutive relation (Hooke's law) holds between the stresses and the deformations:

$$\sigma_{ij}(\mathbf{u}) = \lambda\, div\mathbf{u}\, \delta_{ij} + \mu\, \epsilon_{ij}(\mathbf{u}), \quad i,j = 1,2 \tag{2.2}$$

The positive constants λ and μ can be expressed as:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{1+\nu}$$

where E is the modulus of elasticity (Young modulus) and ν is the contraction ratio (Poisson ratio) of the elastic material of the body B. Furthermore we introduce the equilibrium equations:

$$(L\mathbf{u})_i \equiv -\sum_{j=1}^{2} \frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} = f_i \ in\ \Omega,\ i = 1,2 \tag{2.3}$$

together with the following boundary conditions:

$$\begin{cases} (B\mathbf{u})_i \equiv \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u}) n_j = \mathbf{g}_i & on\ \Gamma_N,\ i = 1,2 \\ \mathbf{u} = \mathbf{0} & on\ \Gamma_D \end{cases} \tag{2.4}$$

Here $n = (n_i)$ denotes the outward unit normal to Γ. We can also give a variational formulation of the elasticity problem:
find $\mathbf{u} \in V$ such that:

$$a(\mathbf{u}, \mathbf{v}) \equiv \int_\Omega (\lambda\ div\mathbf{u}\ div\mathbf{v} + \mu \sum_{i,j=1}^{2} \epsilon_{ij}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}))\ dx = F(\mathbf{v}),\ \forall \mathbf{v} \in V \tag{2.5}$$

where

$$F(\mathbf{v}) = \sum_{i=1}^{2} \int_\Omega f_i v_i\ dx + \sum_{i=1}^{2} \int_{\Gamma_N} g_i v_i\ ds \tag{2.6}$$

and $V = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = \mathbf{0}\ on\ \Gamma_D\}$.
We consider a decomposition of Ω into two non intersecting subdomains Ω_1 and Ω_2 (thus $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$); we denote by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ the common interface between the two subdomains and by n_Γ the unit normal vector on Γ directed from Ω_1 to Ω_2.
If $\mathbf{u}_i, i = 1,2$, is the restriction of $\mathbf{u}$ to the subdomain Ω_i, problem $(2.3) - (2.4)$ can be written in the following multidomain equivalent form:

$$\begin{cases} L\mathbf{u}_i = \mathbf{f} & in\ \Omega_i,\ i = 1,2 \\ B\mathbf{u}_i = \mathbf{g} & on\ \Gamma_N \cap \partial\Omega_i,\ i = 1,2 \\ \mathbf{u}_i = \mathbf{0} & on\ \Gamma_D \cap \partial\Omega_i,\ i = 1,2 \\ \mathbf{u}_1 = \mathbf{u}_2 & on\ \Gamma \\ \sum_{j=1}^{2} \sigma_{kj}(\mathbf{u}_1)(n_\Gamma)_j = \sum_{j=1}^{2} \sigma_{kj}(\mathbf{u}_2)(n_\Gamma)_j & on\ \Gamma,\ k = 1,2 \end{cases} \tag{2.7}$$

Let $V_D^1(\Omega_i)$ be the space of functions of $H^1(\Omega_i)$ vanishing on $\{\Gamma \cup \Gamma_D\} \cap \partial\Omega_i$, and let V_Γ be the set of functions defined on Γ, which are traces of functions of V. The variational counterpart of problem (2.7) reads as:

$$\begin{cases} \mathbf{u}_i \in [H^1(\Omega_i)]^2, \mathbf{u}_i = \mathbf{0}\ on\ \partial\Omega_i \cap \Gamma_D,\ i = 1,2 \\ a_i(\mathbf{u}_i, \mathbf{v}) = \sum_{j=1}^{2} \int_{\Omega_i} f_j v_j\ dx + \sum_{j=1}^{2} \int_{\Gamma_N \cap \partial\Omega_i} g_j v_j\ ds,\ \forall \mathbf{v} \in [V_D^1(\Omega_i)]^2 \\ \mathbf{u}_1 = \mathbf{u}_2\ on\ \Gamma \\ \sum_{i=1}^{2} a_i(\mathbf{u}_i, \mathbf{w}_i^*) = \sum_{i=1}^{2} \int_{\Omega_i} \sum_{j=1}^{2} f_j(\mathbf{w}_i^*)_j\ dx,\ \forall \mathbf{w} \in V_\Gamma \end{cases} \tag{2.8}$$

where $a_i(\cdot,\cdot)$ is the restriction of $a(\cdot,\cdot)$ to Ω_i, , $\mathbf{w}^* \in V$ is any continuous extension of $\mathbf{w}$ to Ω such that $B\mathbf{w} = \mathbf{0}$ on Γ_N, and $\mathbf{w}_i^*$ is the restriction of $\mathbf{w}^*$ to Ω_i.

3. Spectral collocation approximation.

A numerical approximation by the spectral collocation method consists of imposing the equilibrium equations at Gaussian points internal to Ω_i and enforcing the interface and boundary conditions at some selected points of $\partial\Omega_i$. We confine ourselves to the case of a plane rectangular domain Ω, partitioned into two rectangles Ω_i. Let D be the reference domain $(-1,1)^2$ and let (N_x, N_y) be a couple of natural numbers. Hereafter we denote by N the common value of N_x and N_y.The Legendre-Gauss-Lobatto collocation points in D are the roots $\{\zeta_{km}, 0 \le k,m \le N\}$ of the polynomial

$$(1-x^2)(1-y^2)\frac{\partial}{\partial x}L_N(x)\frac{\partial}{\partial y}L_N(y)$$

where $L_k(t)$ is the k-th Legendre polynomial in $[-1,1]$. Moreover we denote by

$$C_i = \{(x_k^i, y_m), 0 \le k,m \le N\} \ for\ i = 1,2 \tag{3.1}$$

the set of collocation points of Ω_i which are the images of the Legendre points in D, through the linear transformation that maps D into Ω_i.
For the sake of simplicity we also set:

$$C_i^{int} = C_i \cap \Omega_i,\ C_i^b = C_i \cap \partial\Omega,\ C^\Gamma = C_i \backslash (C_i^{int} \cup C_i^b),\ i = 1,2. \tag{3.2}$$

We denote by $\mathbb{P}_N(\Omega_i)$ the space of polynomials of degree N in each direction, and by $\mathbb{P}_N(\Gamma)$ the space of polynomials of degree N on Γ, vanishing at the endpoints of Γ. Finally $\{(\omega_k^i, \omega_m),\ 0 \le k,m \le N\}$ are the Legendre-Gauss-Lobatto discrete weights in Ω_i (e.g,[4],[2]).
We can now introduce the spectral collocation approximation to problem (2.7). For simplicity of notation we assume $\Gamma_N = \emptyset$. The spectral solutions $\mathbf{u}_{i,N} \in \mathbb{P}_N(\Omega_i) \otimes \mathbb{P}_N(\Omega_i)$, $i = 1,2$ verify:

$$\begin{cases} L\mathbf{u}_{i,N} = \mathbf{f} & in\ C_i^{int}\ i = 1,2 \\ \mathbf{u}_{i,N} = \mathbf{0} & in\ C_i^b\ \ i = 1,2 \\ \mathbf{u}_{1,N} = \mathbf{u}_{2,N} & in\ C^\Gamma \\ B\mathbf{u}_{1,N} - B\mathbf{u}_{2,N} = -(L\mathbf{u}_{1,N} - \mathbf{f})\omega^- - (L\mathbf{u}_{2,N} - \mathbf{f})\omega^+ & in\ C^\Gamma \end{cases} \tag{3.3}$$

Here we have set $\omega^- = \omega_N^1$, $\omega^+ = \omega_0^2$.
Obvious generalizations apply for the case of a nonhomogeneous Dirichlet data and/or when $\Gamma_N \neq \emptyset$.

Remark 3.1 Note that at each interface collocation point the jump of the normal stresses balances a suitable linear combination of the residual of the equilibrium equations from both sides. This matching condition makes the above collocation

scheme equivalent to a Galerkin approximation to (2.8) with Gauss-Lobatto quadratures, i.e.,for which the integrals are replaced by Gauss-Lobatto quadrature formulas and the bilinear forms $a_i(\cdot,\cdot)$ are substituted by discrete forms, in which the integrals are still approximated by the same quadrature formulas.
It can be proven that there exists a unique solution to the multidomain problem which is stable, in the energy norm, with respect to N. Furthermore, by generalizing a proof carried out for Laplace equation, it is possible to give a convergence estimate (see [2], Ch. XIII). Precisely, it can be proven that the error $\mathbf{u}_{i,N} - \mathbf{u}_i$ can be bounded in the energy norm by CN^{-s}, where C is a constant independent of N, provided that $\mathbf{u}_i \in H^{s+1}(\Omega_i)$ for some $s > 1$, $i = 1, 2$.

We report now some numerical results showing the spectral accuracy of approximation (3.3). The computational domain is $\Omega = (0,2) \times (0,2)$ and the boundary $\partial\Omega$ is split up into two pieces: $\Gamma_D = [0,2] \times \{0,2\}$ and $\Gamma_N = \{0,2\} \times (0,2)$. The material's parameters E and ν are such that both the constants λ and μ are equal to 1. The data $\mathbf{f}$ and $\mathbf{g}$ of the problem are chosen in order that the two components of the exact solution to (2.3)-(2.4) be:

$$u_1(x,y) = -sin(\pi y) + cos(\pi x)sin(\pi y)$$

$$u_2(x,y) = \begin{cases} y(y-2) & x \leq 1.5 \\ [2(x-1.5)]^\alpha y^2/4 \ + y(y-2) & x > 1.5 \end{cases}$$

with $\alpha \geq 2$, where the larger is α, the smoother is $\mathbf{u}_2$.In figure 3.1 we compare the spectral accuracy of the single domain with that of the two-domain scheme (with interface $\Gamma = \{1.5\} \times (0,2)$) for two choices of the real parameter α. We show the logarithm of $Err(\mathbf{u})$, the relative error between the spectral solution and the exact solution (evaluated in the discrete maximum norm), as a function of the total number of degrees of freedom (d.o.f.) that are used in Ω.

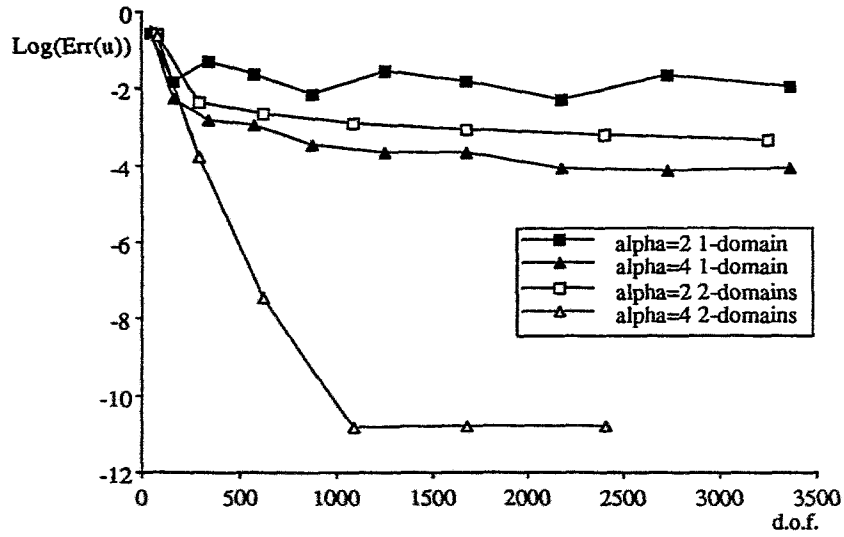

Fig.3.1 *Single versus two-domain spectral accuracy.*

Remark 3.2 In the limit case $\alpha = 2$, the corresponding surface load $\mathbf{f}$ is discontinuous at $x = 1.5$.In this situation, which is of remarkable interest in practical applications, the single domain spectral approximation does not converge monotonically to $\mathbf{u}$ as N goes to infinity. If we instead use a domain decomposition partition in which the interface is set at the discontinuity point $x = 1.5$, numerical results show that the rate of convergence of the error $\mathbf{u}_{i,N} - \mathbf{u}_i$ is now as fast as $1/N^2$.

4. A generalized Dirichlet/Neumann method for the solution of the multidomain spectral problem.

The solution to problem (3.3) can be determined as the limit of a sequence of polynomials $\mathbf{u}^n_{i,N} : \Omega_i \to \mathbb{R}, i = 1, 2$ which are obtained by solving Legendre collocation boundary-value problems in Ω_1 and Ω_2.This iterative procedure is a generalization to the case of the elasticity problem of the Dirichlet/Neumann method formerly introduced for the Poisson equation (e.g. [1], [8], [12]). According to this technique, the transmission conditions at the interface points C^Γ are attributed partly to Ω_1 (the continuity of the solution) and partly to Ω_2 (the continuity of the normal stress in the variational fashion emphasized in Remark 3.1). The iterative procedure reads as follows: for a given $\mathbf{d}^0 \in \mathbb{P}_N(\Gamma) \otimes \mathbb{P}_N(\Gamma)$ we look for a sequence $\mathbf{u}^n_{i,N}, i = 1, 2$ satisfying:

$$\begin{cases} L\mathbf{u}^n_{1,N} = \mathbf{f} & in\ C^{int}_1 \\ \mathbf{u}_{1,N} = \mathbf{0} & in\ C^b_1 \\ \mathbf{u}_{1,N} = \mathbf{d}^n & in\ C^\Gamma \end{cases} \tag{4.1}$$

$$\begin{cases} L\mathbf{u}^n_{2,N} = \mathbf{f} & in\ C^{int}_2 \\ \mathbf{u}_{2,N} = \mathbf{0} & in\ C^b_2 \\ B\mathbf{u}^n_{1,N} - B\mathbf{u}^n_{2,N} = -(L\mathbf{u}^n_{1,N} - \mathbf{f})\omega^- - (L\mathbf{u}^n_{2,N} - \mathbf{f})\omega^+ & in\ C^\Gamma \end{cases} \tag{4.2}$$

where, for $n \geq 2$ $\mathbf{d}^n$ is given by the recursive relation:

$$\mathbf{d}^n = \theta_n \mathbf{u}^{n-1}_{2,N}|_\Gamma + (1 - \theta_n)\mathbf{d}^{n-1} \tag{4.3}$$

Here θ_n is a relaxation parameter which is computed at each step in order to speed up the convergence of the algorithm.

Remark 4.1 The proof of the convergence of the iterative scheme (4.1)-(4.3) can be carried out through a classical technique (e.g., [13]), by checking that the *spectral extension operators* in Ω_1 and Ω_2 are uniformly equivalent one another independently of N. Precisely, for a given $\mathbf{d} \in \mathbb{P}_N(\Gamma)$, the solutions $\mathbf{w}_{i,N}, i = 1, 2$ to the collocation problems

$$\begin{cases} L\mathbf{w}_{i,N} = \mathbf{0} & in\ C^{int}_i \\ \mathbf{w}_{i,N} = \mathbf{0} & in\ C^b_i \\ \mathbf{w}_{i,N} = \mathbf{d} & in\ C^\Gamma \end{cases} \tag{4.4}$$

must have equivalent energy norms. Then convergence can be obtained by following a technique similar to the one introduced in [12] for the multidomain spectral

approximation of the Poisson problem.Moreover, as the equivalence holds with constants independent of N, it follows that it is possible to choose θ_n in a way that the rate of convergence of the iterative procedure is still independent of N.

We present now some numerical examples concerning the convergence of the scheme (4.1)-(4.3). We set

$$E(n) = max\{\frac{\| \mathbf{u}_{1,N}^n - \mathbf{u}_{1,N}^{n-1} \|_{\infty,\Gamma}}{\| \mathbf{u}_{1,N}^n \|_{\infty,\Gamma}}, \frac{\| \mathbf{u}_{2,N}^n - \mathbf{u}_{2,N}^{n-1} \|_{\infty,\Gamma}}{\| \mathbf{u}_{2,N}^n \|_{\infty,\Gamma}}\} \tag{4.4}$$

where $\| \mathbf{v} \|_{\infty,\Gamma} = max\{|\mathbf{v}(P)|, P \in C^\Gamma\}$. In table 4.1 we report the minimum value of n requested in order to satisfy $E(n) < 10^{-8}$, for different choices of the polynomial degree N. These results show that the rate of convergence of the iterative scheme does not deteriorate when the degrees of the polynomial N increase in each subdomain.This theoretical property was emphasized in Remark 4.1.

N	$\alpha = 2$	$\alpha = 4$
8	22	19
16	22	20
24	24	18
32	24	20

Tab.4.1 *Convergence with respect to N, for the multidomain scheme (4.1)-(4.3) .*

5. Local and global finite element preconditioners.

The multidomain formulation allows the reduction of the original problem to a sequence of problems of similar type but with smaller size in every subdomain. The latter can be faced by monodomain spectral solvers. At each step of the Dirichlet/Neumann scheme we must solve two linear systems of the following type:

$$L_{i,N}\mathbf{u}_{i,N}^n = \mathbf{F}_i^n, \ i = 1,2, \ \forall n \geq 0 \tag{5.1}$$

where $L_{i,N}$, $i = 1,2$, is the symmetric pseudospectral matrix including the equilibrium equations and the boundary conditions associated to the subdomain Ω_i. These matrices are ill-conditioned, as their condition number grows as N^4.Therefore a preconditioned iterative procedure is mandatory for solving the linear system. Here we use a finite element preconditioning . For this purpose, we introduce the piecewise bilinear finite elements approximation associated with the grid of Gauss-Lobatto-Legendre points in each subdomain,and we denote by $S_{i,h}$ and $M_{i,h}$, respectively, the Stiffness and Mass matrix. In the case of a pure Dirichlet boundary value problem (e.g., this is the case of problem (5.1) if $\Gamma_N = \emptyset$), the preconditioning finite element matrix is $M_{i,h}^{-1}S_{i,h}$. When Neumann (or mixed) boundary conditions are enforced, the preconditioner needs to be slightly changed as it must also account for one dimensional mass matrices related to the edges. In all cases, the preconditioned

matrix has a condition number independent of N. The preconditioned system is then solved by gradient-like iterative procedures. We refer to [14] for further details and numerical results.

This preconditioning technique suggests to define a new method in order to solve the multidomain spectral system (3.3).This generalizes an idea presented in [3] for Poisson equations. We define a global finite element preconditioner in which the stiffness and mass matrices S_h and M_h are referred to the global grid of collocation nodes $C_1 \cup C_2$. The corresponding iterative procedure reads as follows:

$$S_h(\mathbf{u}_N^{n+1} - \mathbf{u}_N^n) = \omega_n \, M_h \, R_N^n \tag{5.2}$$

where $\mathbf{u}_N^n|_{\Omega_i} = \mathbf{u}_{i,N}^n$, $i = 1, 2$, and

$$R_N^n = \begin{cases} 0 & in\ C_i^b \\ \mathbf{f} - L\mathbf{u}_{i,N} & in\ C_i^{int} \\ (\mathbf{f} - L\mathbf{u}_{1,N})\omega^- + (\mathbf{f} - L\mathbf{u}_{2,N})\omega^+ - (B\mathbf{u}_{1,N} - B\mathbf{u}_{2,N}) & in\ C^\Gamma \end{cases} \tag{5.3}$$

When convergence is achieved, this method provides the solution to (3.3) as well as the first multidomain approach outlined in (4.1)-(4.3). At each step of (5.2) we need to evaluate the spectral residuals in each Ω_i,which can be advantageously accomplished on parallel processors. Furthermore, we need to solve a finite element problem on the physical domain Ω.This can be carried out, for instance, by applying a finite element multidomain technique (e.g. [15]).

We present now an example showing the convergence of the global finite element preconditioner for the same numerical problem described in sect.3. In figure 5.1 we plot the logarithm of the discrete L^2 norm of the preconditioned residual $S_h^{-1} M_h R_N^n$, versus the number of iterations, for different choices of the polynomial degree N,

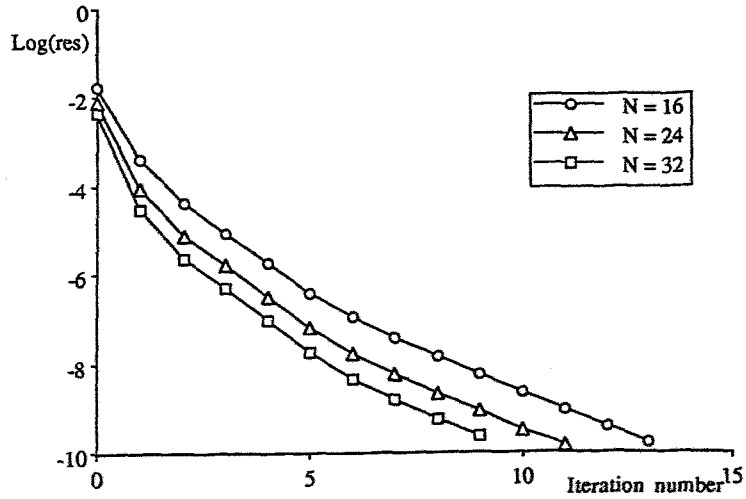

Fig.5.1 *Convergence history for global Richardson iterations.*

fixed $\alpha = 4$. The results show that the rate of convergence of algorithm (5.2) is extremely fast and practically independent of the number of collocation points in each Ω_i.
The last example we present is the application of the two domain decomposition methods to a partition of Ω in which the material parameters E and ν have different values in the two subdomains. We define $E_i = E|_{\Omega_i}$ and $\nu_i = \nu|_{\Omega_i}$. This is a case of practical interest in structural mechanics (e.g., multi-layered problems).
The problem is (2.3)-(2.4), in $\Omega = (0,2)^2$, with interface $\Gamma = \{1\} \times (0,2)$, $\mathbf{f} = \mathbf{0}$; we consider Dirichlet boundary conditions $\mathbf{u} = \mathbf{\Phi}$ on the whole boundary $\partial\Omega$, where the two components of Φ are:

$$\Phi_1(x,y) = sin\ (2\pi x)\ cos\ (4\pi y)$$
$$\Phi_2(x,y) = cos\ (\ \pi x)\ sin\ (6\pi y)$$

In tables 5.1 and 5.2 we report the minimum number of iterations requested in order to obtain convergence for the two schemes, fixed $N_x = N_y = 24$ in each subdomain. Precisely, we require $E(n) < 10^{-8}$ for the Dirichlet/Neumann scheme (4.1)-(4.3) ([DN]), and $\| S_h^{-1} M_h R_N^n \|_{2,d} < 10^{-8}$ for the global finite element preconditioned method (5.2) ([GP]), where $\| \cdot \|_{2,d}$ denotes the discrete L^2-norm.
In table 5.1 we report the rate of convergence for different values of the Young modulus E_1 in Ω_1. We fix $E_2 = 100, \nu_1 = \nu_2 = 0.25$.

E_1 / E_2	.01	1	5	10	30	60	90	100
DN	8	18	28	30	36	37	37	37
GP	12	8	8	9	9	10	10	10

Tab.5.1 *Convergence with respect to* E_1/E_2, *fixed* $E_2 = 100$ *and* $\nu_1 = \nu_2 = 0.25$.

In table 5.2 we report the rate of convergence for different values of the Poisson ratio ν_1 in Ω_1. We fix $\nu_2 = .25, E_1 = E_2 = 100$.

ν_1 / ν_2	1/5	2/5	3/5	4/5	1	6/5	7/5	8/5	9/5
DN	18	17	17	17	18	16	16	17	19
GP	8	8	8	8	8	9	10	12	17

Tab.5.2 *Convergence with respect to* ν_1/ν_2, *fixed* $\nu_2 = 0.25$ *and* $E_1 = E_2 = 100$.

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References

[1] P.E.Bjørstad and O.B.Widlund: *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J.Numer.Anal., 23(1986), pp.1093-1120.
[2] C.Canuto,M.Y.Hussaini,A.Quarteroni and T.A.Zang:Spectral Methods in Fluid Dynamics, Springer-Verlag, New York, 1988.
[3] C.Canuto and P.Pietra: *Boundary and Interface Conditions within a Finite Element Preconditioner for Spectral Methods*, to appear in J.Comput.Phys..
[4] P.J. Davis and P. Rabinowitz: Methods of Numerical Integration, 2nd edn., Academic Press, London, New York, 1984.
[5] Y.H.De Roeck: *A local preconditioner in a domain-decomposition method*, Report TR 89/10, CERFACS (1990).
[6] M.Deville and E.Mund: *Finite element preconditioning for pseudospectral solutions of elliptic problems* ,SIAM J.Stat.Comput. 2(1990), pp.311-342.
[7] M.Dryia and O.B.Widlund: *Towards a Unified Theory of Domain Decomposition Algorithms for Elliptic Problems*, in T.F.Chan et al., Eds., Domain Decomposition Methods for PDEs, SIAM, Philadelphia (1990).
[8] D.Funaro,A.Quarteroni and P.Zanolli: *An iterative procedure with interface relaxation for domain decomposition methods*, SIAM J.Numer.Anal., 25 (1988), pp.1213-1236.
[9] C.Johnson: Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge Univ.Press., Cambridge, 1987.
[10] P.Le Tallec,Y.H.De Roeck and M.Vidrascu: *Domain decomposition methods for large linear elliptic three dimansional problems*, TR/PA/90/28, CERFACS (1990), to appear in Journal of Comp. and Applied Math.
[11] J.L.Lions and E.Magenes: Nonhomogeneous Boundary Value Problems and Applications,I, Springer-Verlag, Berlin, 1972.
[12] A.Quarteroni and G.Sacchi Landriani: *Domain decomposition preconditioners for the spectral collocation methods*, J.Scient.Comput. 3 (1988), pp.45-75.
[13] A.Quarteroni and A.Valli: *Theory and Application of Steklov-Poincaré Operators for Boundary-Value Problems*, in Proc. of the Symp. on "Applied and Industrial Mathematics", R.Spigler ed., Kluwer, Dordrecht (1991), pp.179-203.
[14] A.Quarteroni and E.Zampieri: *Finite element preconditioning for Legendre Spectral Collocation Approximations to Elliptic Equations and Systems*, University of Minnesota Supercomputer Institute Research Report UMSI (1991).
[15] B.F.Smith:*Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity*, Ph.D. thesis, Courant Institute, New York, NY, (1990).