CHAPTER 41

A Submerged Body Moving in a Stratified Medium Via Domain Decomposition Technique

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Abstract

Numerical simulation of a submerged cylinder moving in a density-stratified fluid has been conducted by the isoparametric pseudospectral element method in conjunction with the domain decomposition technique. The computational domain is first decomposed into two blocks with inter-overlapping areas where the overlapped grids are not located at the same places. The solution of the pressure Poisson equation in each block is iteratively solved by the preconditioned method. If the preconditioner is chosen such that it can be expanded in terms of eigenfunctions, the solution of the preconditioner can be reduced to the simplest algebraic problem. The iterative solution between blocks is then updated by the Schwarz alternating procedure (SAP). The effect on the vortex wakes for a moving cylinder in a linearly stratified fluid at the Reynolds number 100 reveals that the vortex shedding, as expected, is gradually inhibited by increasing the BV frequency.

1 Introduction

When a stratified region is disturbed, internal waves are generated and propagate within the medium. The flow field becomes more complicated to analyze for the case of internal waves generated by a submerged body moving in a thermocline because of their interaction with the primary flow field, and the wake patterns around the moving body significantly differ from those in a homogenous fluid. The inhibition of the vertical flow motion causing a complex process can somehow affect the transport of mass, momentum and energy. An understanding toward this nonlinear dynamic behavior mainly depends on the two key parameters: the Reynolds number and the Froude number, $Fr = U/NL$, where U is the body speed; L is the characteristic length; and N is the Brunt-Väisälä (BV) frequency. As expected, when the Reynolds number becomes large, the effect on the primary flow field by the stratified fluid will be minimized, while at the low Reynolds number the Froude number plays a dominant role on the primary flow field. That explains why the patterns of internal waves produced by the submerged part of a ship in the presence of a steep density gradient

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are strongest in the cross-tract section (locally low Reynolds number in the cross direction), and only the momentum wake appears in the moving direction (high Reynolds number), instead. Experimental data [1] will be used to verify the numerical results.

The pseudospectral element method (PSE), which extended the global pseudospectral method to a multi-element scheme, has been applied to the solution of the incompressible Navier-Stokes equations in two- and three-dimensional complex geometries [2, 3]. The definition of complex geometries, however, is limited by their specificity, the cartesian coordinates or the geometries which can be mapped onto the cartesian coordinates by an algebraic or a simplified isoparametric mapping (rectilinear boundary). Thus, how to adapt the isoparametric PSE method to the general curvilinear coordinates is the main object of this paper and needs to be exploited. On the other hand, the SAP iterative scheme has been applied successfully to those configurations where the overlapped grids are located at the same places. But under some circumstances, due to the complexity of the geometrical configuration the overlapped grids can not coincide all together. Therefore, the conjunction of SAP with the isoparametric PSE method should provide great flexibility to deal with complex geometry. As for the boundary conditions for the pressure at the overlapping interface, the continuity equation [4] is still the best approach toward the pressure solution. According to this scheme, the solution of the velocity field, if the convergence of the velocity field in the overlapping area has been met by the SAP, will be exactly the same as that solved by the global (non-decomposed) domain technique. The preconditioned conjugate residual method will be applied to solve the pressure Poisson equation in general curvilinear coordinates.

2 Isoparametric pseudospectral element

Methods which achieve an accurate representation of complex geometries with minimal effort are quite important. We describe one such method here. Let us first define the existence of a mapping function between the physical space (x, y, z) and the computational space (ξ, η, ζ) (a transformed space with non-orthogonal, curvilinear coordinates that are Cartesian-like when viewed with respect to themselves). Once such coordinate relationships are known, shape functions defining geometry can be specified in local coordinates and a one-to-one correspondence between Cartesian and curvilinear coordinates can be established. An isoparametric mapping, same order polynomials interpolating the geometry and the function (any variables), is applied to map a three-dimensional curved geometry (physical space) onto a cube (computational space).

The main objective of the present development was to provide the three-dimensional computational grids around complex geometries in a structured fashion. The pseudospectral element grid generation scheme presented herein utilizes a multiple block structure, namely, the global computational domain based on the geometrical configuration is divided into a few blocks, and each block is then arbitrarily partitioned by the pseudospectral elements. The grid generation is performed in two levels. First, each of these blocks is defined as a parent element, of which the shape function can be defined by a curved isoparametric pseudospectral element. Next, appropriate family elements linearly (or with higher order) interpolating the shape function of their parent elements are allocated within each of these blocks. In other words, a cubic element which contains $N\xi + 1$, $N\eta + 1$ and $N\zeta + 1$ collocation points ($\xi_i = \cos \pi i/N\xi$, $\eta_j = \cos \pi j/N\eta$, $\zeta_k = \cos \pi k/N\zeta$), in the transformed space, $-1 \leq \xi \leq 1, -1 \leq \eta \leq 1, -1 \leq \zeta \leq 1$ (shown in Fig. 1), corresponds to an irregular or regular six-faced (hexahedral) element in the physical space. For an isoparametric mapping, once the collocation points (x, y, z) along the curvy boundaries of each parent element are known, the interior points (including the boundaries of family elements) are interpolated by deforming the (ξ, η, ζ) mesh into its (x, y, z) image using the "trilinear blending function" [5], i.e., the grid points $(x, y, z)_{ijk}$ in the physical space are mapped onto $(\xi = \xi_i, \eta = \eta_j, \zeta = \zeta_k)$ in the transformed space.

Let ϕ be any value of (x, y, z), the interpolation translates the Boolean sum [6] into a form

$$\phi = P_\xi\phi + P_\eta\phi + P_\zeta\phi - P_\xi P_\eta\phi - P_\xi P_\zeta\phi - P_\eta P_\zeta\phi + P_\xi P_\eta P_\zeta\phi \tag{1}$$

where the "projectors" P_ξ, P_η, P_ζ interpolate ϕ between two opposing faces of the six-sided region, the double product projector, $P_\xi P_\eta$, interpolate ϕ in two directions from the four edges along which ξ and η are constant, and the triple product projector, $P_\xi P_\eta P_\zeta$ interpolates ϕ from the eight corners. The detailed description of each projector can be found in [7]. With the linear (or higher order) interpolation functions to constitute each projector, Eq. (1) interpolates the surface boundary exactly.

3 Navier-Stokes equations

Time-dependent incompressible flow can be described as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{S} \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2b}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 \tag{2c}$$

Here $\mathbf{u}$ is the velocity field, p the pressure divided by the reference density ρ_0, ν the kinematic viscosity, ρ the density and S a source term which could be any external force such as gravity, given by $\rho g/\rho_0$, or an electromagnetic force. The Boussinesq approximation assumes that variation of all fluid properties other than density are completely ignored. Density variation is considered in Eq. (2c) only insofar as it affects the source term S. Eqs. (2a, 2b, 2c) describe the momentum, the incompressibility and the mass conservation equations, respectively. For the nonstratified flow the density equation does not need to be considered. The most difficult part of solving the Navier-Stokes equations lies in the fact that the pressure field does not obey an evolution equation, but acts as a Lagrangian constraint which links the continuity equation $\nabla \cdot \mathbf{u} = 0$ to the evolution equations. The best approach to date for the solution of Navier-Stokes equations is Chorin's [8] time-step splitting technique. The first step is to predict the solution to the momentum equation without the pressure term

$$\bar{\mathbf{u}}^{n+1} = \mathbf{u}^n + \Delta t(\nu \nabla^2 \mathbf{u} + \mathbf{S} - \mathbf{u} \cdot \nabla \mathbf{u})^n \tag{3}$$

where the superscript n denotes the nth time step. The second step is to develop the pressure and corrected velocity fields that satisfy the continuity equation by using the relationships

$$\mathbf{u}^{n+1} = \bar{\mathbf{u}}^{n+1} - \Delta t \nabla p \tag{4a}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0. \tag{4b}$$

An equation for the pressure can be obtained by taking the divergence of Eq. (4a). In view of Eq. (4b), it forms

$$\nabla^2 p = \frac{\nabla \cdot \bar{\mathbf{u}}^{n+1}}{\Delta t}. \tag{5}$$

The pressure Poisson equation, Eq. (5), appearing in curvilinear (non-orthogonal) coordinates contains a non-separable operator for which there is no easy way for a direct solution;

this is especially difficult in three-dimensional cases. An iterative solution seems attractive if a good preconditioner (approximate operator to the original one) can be found. A good preconditioner requires (i) less memory and inexpensive effort to invert the resulting matrix and (ii) a fast convergence rate. The second requirement implies that the preconditioner L_{ap} should be close to the original operator L_{sp}, i.e., the condition number of the matrix, $\kappa = \|L_{ap}^{-1}L_{sp}\|$ (the ratio of max and min of the matrix), must not be large. It implies that instead of solving $L_{sp}p = S$, $L_{ap}^{-1}L_{sp}p = L_{ap}^{-1}S$ is solved.

The iterative scheme used to solve the pressure field is the preconditioned conjugate residual method [9], which is valid for a symmetric or asymmetric operator. A certain separable operator L_{ap} [3] is chosen and constructed from the original operator L_{sp}, which appears in Eq. (5) for the pressure solution. The solution of the pressure field $L_{sp}p = S$ is iteratively solved by the preconditioner until the criterion of residual is met.

The iterative procedure using the preconditioned conjugate residual method reads as follows:

Given p^0, compute $r^0 = S - L_{sp}p^0, z^0 = L_{ap}^{-1}r^0, h^0 = z^0$. Then, for $k = 0, 1, 2, \ldots$, until $\| r^k \| < \epsilon$, do

$$p^{k+1} = p^k + \alpha^k h^k \tag{6a}$$

$$r^{k+1} = r^k - \alpha^k L_{sp} h^k \tag{6b}$$

$$z^{k+1} = L_{ap}^{-1} r^{k+1} \tag{6c}$$

$$h^{k+1} = z^{k+1} - \beta^k h^k \tag{6d}$$

where

$$\alpha^k = \frac{(r^k, L_{sp}h^k)}{(L_{sp}h^k, L_{sp}h^k)}, \quad \beta^k = \frac{(L_{sp}z^{k+1}, L_{sp}h^k)}{(L_{sp}h^k, L_{sp}h^k)} \tag{7}$$

Here (,) denotes the inner product.

Let $\mathbf{z}^k$ in Eq. (6c), $k \geq 1$, be expanded in a series of eigenfunctions such that

$$\mathbf{z}^k = \mathbf{E}\xi\, \hat{\mathbf{z}}^k\, \mathbf{E}\eta^T\, \mathbf{E}\zeta^T, \tag{8a}$$

and similarly the residual $\mathbf{r}^k$ is expanded such that

$$\mathbf{r}^k = \mathbf{E}\xi\, \hat{\mathbf{r}}^k\, \mathbf{E}\eta^T\, \mathbf{E}\zeta^T. \tag{8b}$$

Then the three-dimensional preconditioner can be reduced to a simple algebraic equation

$$(\alpha_i + \beta_j + \gamma_k)\hat{z}^k_{i,j,k} = \hat{r}^k_{i,j,k} \tag{9}$$

where α_i, β_j and γ_k are the eigenvalues with respect to separable derivative operators of the preconditioner and $\mathbf{E}\xi, \mathbf{E}\eta, \mathbf{E}\zeta$ are the corresponding eigenvectors associated with each eigenvalue. Note that the overall memory for the pressure solution requires $O(N^3)$, i.e., only the field variables need to be declared.

4 Domain decomposition with Schwarz Alternating Procedure

The solution of flow over a cylinder via the domain decomposition approach consists of first dividing the computational domain into two blocks (or subdomains) with overlapping areas, where the grids inside the overlapping area are not located at the same places. Next implement the Schwarz Alternating procedure (SAP) for exchanging data between different blocks, i.e., solving the problem on each block separately and then updating the

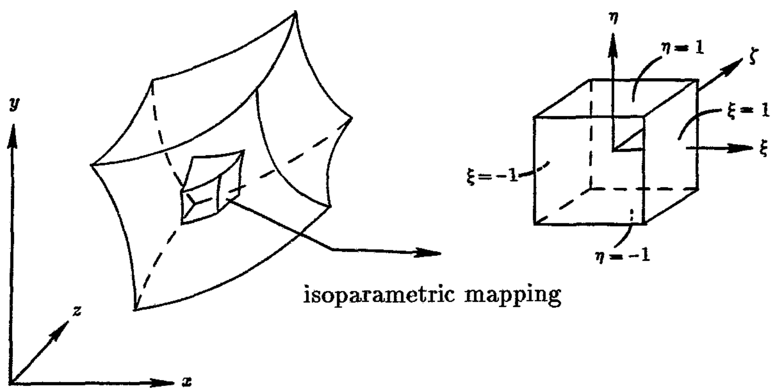

Figure 1. Three-dimensional isoparametric mapping of family elements

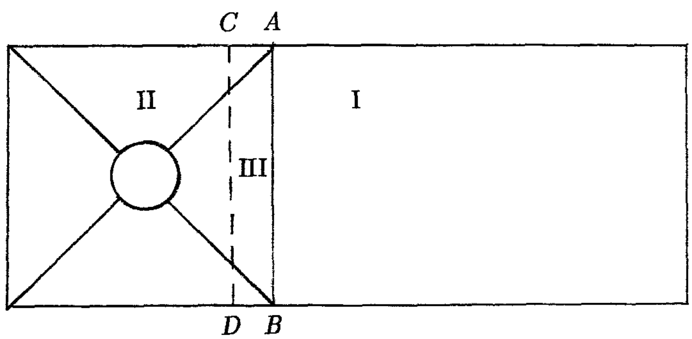

Figure 2. Two-dimensional configuration of domain decomposition

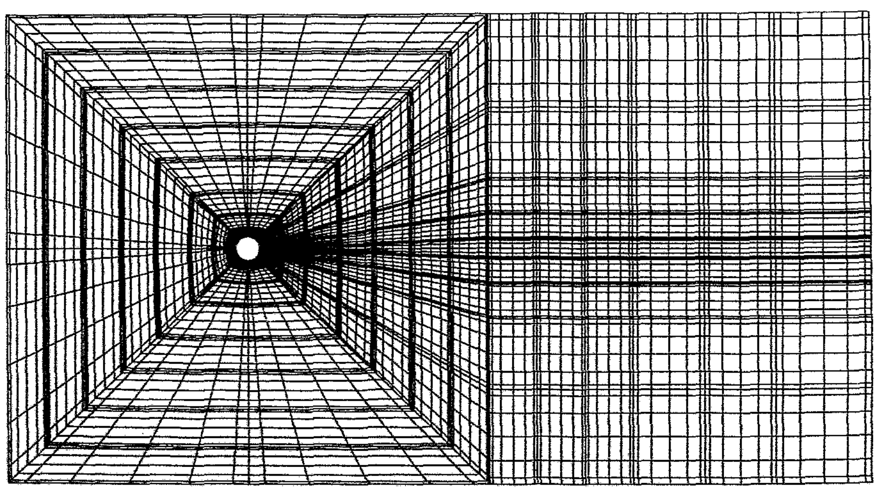

Figure 3. Two-dimensional grids for flow over a cylinder

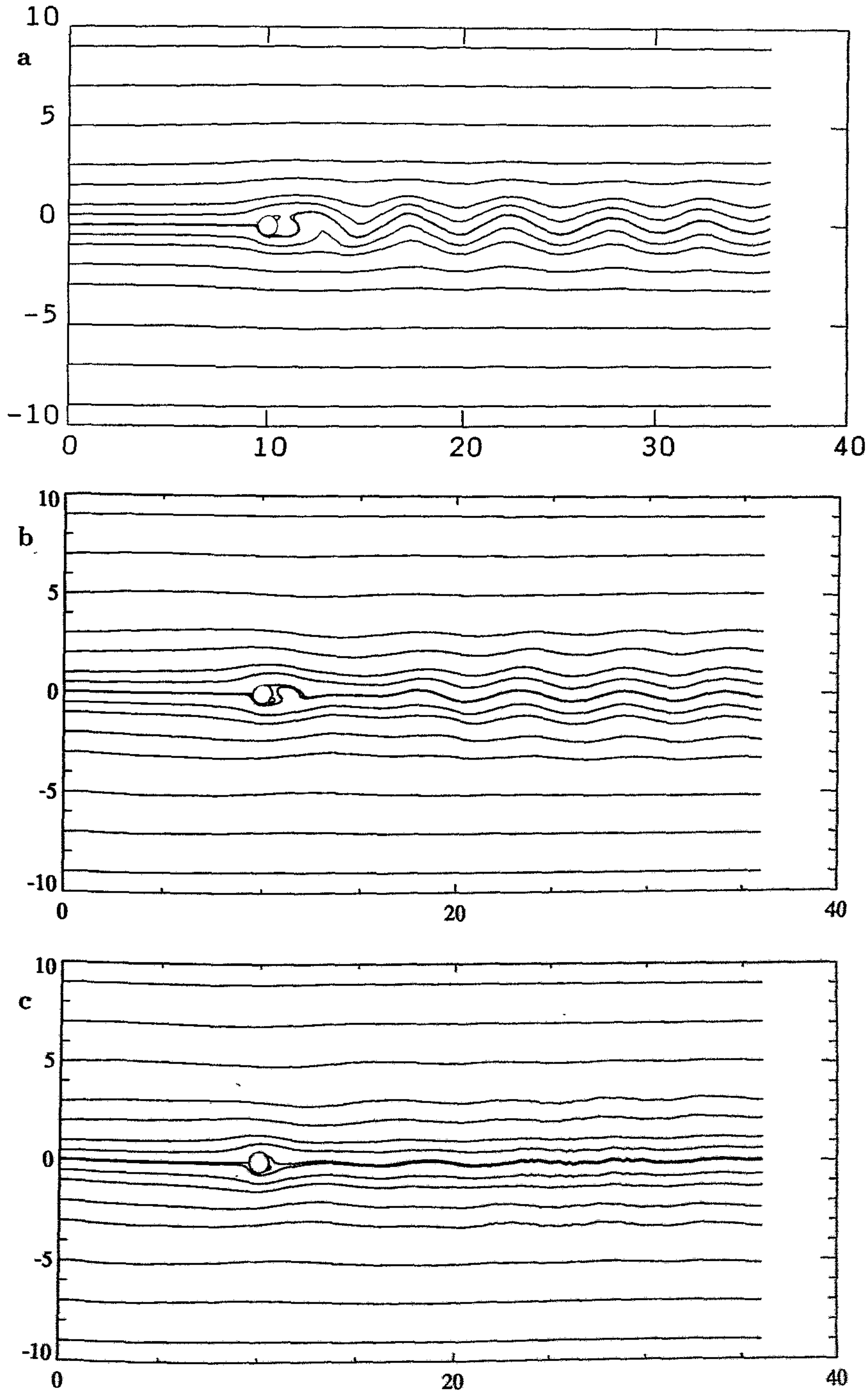

Figure 4. Flow over a moving cylinder at Re = 100 with a) $Fr = \infty$, b) $Fr = 1.15$, c) $Fr = 0.77$

velocity field on the overlapped interfaces. The advantages of this approach include (i) less memory access, local rather than global memory and (ii) easy treatment of complex geometry. The extension of the SAP technique for the solution of incompressible flow in curvilinear coordinates will be straightforward by the proposed preconditioned method.

The SAP iterative solution of the incompressible Navier-Stokes equation in primitive variable form for a two-dimensional stratified flow sketched in Fig. 2 is summarized by the following algorithm:

1. First assume $\mathbf{u}^{n+1}$ on $\overline{AB}$. Usually $\mathbf{u}^n$ will be a good initial guess.
2. Solve domain II employing the boundary conditions derived from the divergence of velocity field on $\overline{AB}$, where the pressure solution is obtained by the preconditioned method.
3. With the interpolated solution of $\mathbf{u}^{n+1}$ on domain III from step (2), solve domain I∪III employing the same type boundary conditions on $\overline{CD}$ to update $\mathbf{u}^{n+1}$ on $\overline{AB}$.
4. Repeat steps (2) & (3) until the velocity $\mathbf{u}^{n+1}$ on $\overline{AB}, \overline{CD}$ does not change.

In order to guarantee that consistent values of velocity (or pressure gradient) be generated in the overlapping domains III, the divergence of velocity field $\nabla \cdot \mathbf{u}$ needs to be actually computed in whichever domain I∪III or II is counted. Since $\mathbf{u}$ on domains III is not known a priori, the divergence of velocity is only set to zero at the first SAP iteration for step (2). According to this approach, the continuity equation is satisfied on domains I and II but not on domain III, since the interpolation procedure somehow will produce the nonzero value of the continuity equation. However, the error index of the continuity equation on domain III will indicate how good the interpolation is.

5 Results and discussion

Fig. 3 provides the generated grids for flow over a cylinder with an aspect ration $H/D = 20$ based on the isoparametric mapping and domain decomposition technique. The radiation boundary condition with a uniform phase speed is applied on the downstream truncated domain to ensure the minimal effect on the convective flow out of the computational domain. Fig. 4a shows the vortex shedding behind a cylinder at the Reynolds number 100 in a homogeneous fluid ($Fr = \infty$), and the secondary separation on the surface still persists. The calculated drag C_D and lift C_L coefficients, $1.38 \leq C_D \leq 1.41$, $-0.27 \leq C_L \leq 0.27$ are in good agreement with published data [10]. As expected, when the stratified effect is gradually increased (decreased Fr), the stratification will distort the wake and tend to supress vortex shedding behind the moving cylinder. In a realistic enviroment, a moving obstacle will be situated in one of the three different vertical regions: above (top mixing layer), below (weak density gradient) or within a thermocline. Here we only focus on the cylinder moving in a linear density-stratified region. With $Fr = 1.15$ the intensity of vortex shedding clearly has been attenuated due to the stratified effect as seen in Fig. 4b. With stronger stratification, $Fr = 0.77$, Fig. 4c shows that the vortex shedding almost disappears, except stationary separation exists. The calculated results compare well with those investigated by the experiment [1].

6 Conclusions

An isoparametric pseudospectral element method combined with the domain decomposition technique has been simulated for flow over a moving cylinder in a density-stratified fluid.

The solution approach is first to divide the computational domain into two blocks with overlapping area where the overlapped grids are not located at the same places. The pressure solution on the "O" domain is iteratively solved by the preconditioned conjugate residual method. The data exchange between two blocks is then implemented by the Schwarz alternating procedure. The numerical results show that the vortex shedding behind a cylinder will be attentuated with increasing BV frequency (or decreasing Fr number).

Acknowledgment

This work was partially supported by the Office of Naval Research under Contract Number N00039 - 89 - C - 0001.

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