CHAPTER 50

Adaptive Multigrid Domain Decomposition Solutions of the Reduced Navier Stokes Equations

Kumar Srinivasan*
Stanley G. Rubin†

Abstract. The presence of disjoint zones, comprising shear layers, separated regions, triple deck structures etc., necessitate use of grid stretching or some type of grid adaptation to optimize the number of grid points. Domain decomposition methods, used in conjunction with an efficient reduced Navier Stokes (RNS) fluid dynamic representation and an adaptive multigrid algorithm, allow for provision of optimal grid resolution. Adaptivity in the direction of refinement is achieved by splitting the local grids, that result from the adaptive multigrid procedure, into several subdomains. This splitting is based on normalized truncation error estimates of key derivatives. Questions in regard to conservation at grid interfaces are addressed. The result is an efficient, composite multigrid procedure. The quality of solutions obtained and gains in computer resources requirements are highlighted.

1) Introduction. Discrete solution procedures that are designed to accurately compute high Reynolds number flows, necessitate the use of local grid refinement, in regions of the flowfield wherein the velocity or pressure gradients can be very large, e.g., boundary or shear layers, shocks waves etc., to efficiently capture these flow gradients. These regions are generally confined to relatively small portions of the overall flow domain and are not known a priori. Therefore an adaptive grid, domain decomposition procedure is most suitable for identifying and resolving such local but globally important phenomena.

Local directional refinement that is driven by specified flow parameters and accuracy limits can be achieved by sequentially splitting the overall flow domain into a variety of subdomains. In the

* Graduate Research Assistant, Department of Aerospace Engineering and Engineering Mechanics, University of Cincinnati, Cincinnati, OH.

† Professor, Department of Aerospace Engineering and Engineeering Mechanics, University of Cincinnati, Cincinnati, OH.

present analysis this domain decomposition strategy is applied, in conjunction with an adaptive multigrid algorithm in order to achieve the appropriate level of grid refinement. In this approach, each grid in the multigrid hierarchy, is of equal or lesser extent than of all the coarser predecessors. The subgrids are split into several multidimensional subdomains that are defined by specified directional and global resolution requirements. A similar approach has been presented in [9], although in this formulation no attempt is made to meet the the disparate need for refinement in the different coordinate directions. In the present investigation this concept has been included by basing the subdomain process on requirements that allow for varying grid resolution in multiple (two) directions and segmentally throughout the flow field.

For the current analysis two dimensional, steady, incompressible, laminar flow examples has been considered. The reduced Navier Stokes (RNS) equations are used to compute the trailing edge flow past a finite flat plate and in the reversed flow region associated with a trough geometry. The use of the RNS system leads to a precise prescription of surface normal boundary conditions on the local subdomain boundaries. This ensures that global mass conservation requirements are satisfied automatically. This is generally not the case with characteristic based, time dependent Navier Stokes schemes where special care has to be taken in order to satisfy global mass conservation. The primitive variable RNS system is also directly applicable on non-staggered grids, unlike other incompressible primitive variable Navier Stokes formulations that require pressure Poisson solver or artificial compressibility concepts.

2) Governing equations and Discretization. The RNS system of equations in sheared cartesian coordinates can be written in non-conservation form as follows.

$$u_\xi + v_\eta = 0 \qquad \text{Continuity.}$$

$$uu_\xi + uy'_b(V + y'_b u)_\xi + V\,[(1+y'^2_b)u_\eta + y'_b V_\eta] + p_\xi = \frac{1}{Re}\,u_{\eta\eta} \qquad \xi\text{-momentum.}$$

$$u(V + y'_b u)_\xi + V(V + y'_b u)_\eta + p_\eta = 0 \qquad \eta\text{-momentum.}$$

$\xi = x$; $\eta = y - y_b(x)$; $V = v - y'_b u$ is the contravariant velocity component in the η or normal direction (for $y'_b(x) << 1$) and $y_b(x)$ is the surface definition. (u,v) are the cartesian velocities in the (x,y) directions. This system of equations is obtained by neglecting the ξ-diffusion terms in the ξ-momentum equation and all diffusion terms in the 'normal' or η-momentum equation. These terms are higher order in the parameter expansion of the full Navier Stokes equations. The resulting RNS system is in effect a composite of the Euler and 2^{nd} order boundary layer equations [3]. The discrete continuity equation is centered at $(i, j-1/2)$. The ξ-momentum is centered at (i,j) and the η-momentum equation at $(i, j+1/2)$. Trapezoidal two point $(i, j\pm1/2)$, or three point central (i,j) differencing, is used for all normal derivatives; all convective axial derivatives are upwind differenced; the p_ξ term in the ξ-momentum equation and u_ξ term in the continuity equation are differenced based on a pressure based form of flux vector splitting [7] wherein the p_ξ term is represented by

$$p_{\xi} = \omega\,(p_i - p_{i-1})/\Delta\xi_i + (1-\omega)\,(p_{i+1} - p_i)/\Delta\xi_i$$

where $\omega = [\gamma M_{\xi}^2/(1 + (\gamma-1)M_{\xi}^2), 1]_{min}$

Here, M_{ξ} is the streamwise Mach number and γ is the ratio of specific heats. This reduces to a simple 'forward' difference for incompressible flow, and this introduces an elliptic acoustic interaction or upstream pressure influence through the $p_{\xi} = (p_{i+1} - p_i)/(\Delta\xi)_i$ contribution.

3) Grid Structure. In general, the N^{th} multigrid level consists of several subdomains. Each multigrid level has an equal or lesser extent than the coarser grids of the multigrid hierarchy. The first two grid levels cover the entire computational domain. The mesh size is initially quite coarse in all directions in which adaptivity is to be prescribed. Each of the multigrid levels comprise several subdomains, which derive part of its topology from the subdomaining pattern of the coarser predecessor. Within each subdomain, of a given multigrid level the refinement is specified independently. Thus, each subdomain of a multigrid level can act as a parent for a subdomain or subdomains at the next finer multigrid level. If at a given multigrid level, a particular subdomain is refined in only one direction, e.g., η, then on subsequent multigrid levels, further refinement within this subdomain is performed only in the η-direction. A similar strategy is adopted for the ξ-direction. Only subdomains that result from refinement of a parent subdomain in both the ξ and η directions require further decomposition according to the direction selective refinement specifications.

4) Refinement Strategy. In most adaptive gridding methods, on any grid level, an estimate of the truncation error of the discretized system of equations is used to identify those regions that require finer grid resolution (Ref [1], [9]). The overall truncation error estimates, however, do not provide information on the specific direction(s) that require refinement. Therefore for regions requiring higher resolution, the grid is refined in both directions, even though only one coordinate gradient may be significant. In order to achieve directional refinement adaptivity it is necessary to monitor the truncation error of selected gradients or derivatives. For the problems considered herein, the truncation error for the pressure and vorticity gradients, e.g., p_{ξ} and $u_{\eta\eta}$, are monitored in order to define the regions that require refinement in, ξ and η, respectively.

The truncation error estimate is obtained from the solution on two successive grids of the multigrid hierarchy. In order to determine the truncation error in a ξ (and/or η) derivative, a finer grid must be used in the ξ (and/or η) direction. Although the p_{ξ} and $u_{\eta\eta}$ terms are the key derivatives for the present analysis, the truncation error of these terms alone will not suffice to ensure that uniform accuracy is achieved throughout the flow domain. The global truncation error for the full discrete system of equations is monitored for this purpose.

Two types of adaptive calculations are performed for the geometries considered herein.

a) One dimensional adaptive calculation (semi-coarsening multigrid), with adaptivity in the ξ direction and with a preset stretched (adapted) η grid.

b) Two dimensional adaptive calculation, in which the refinement

is automated in both directions and uniform grids are used in each subdomain. Grid stretching is not applied, except as the grids change discretely from subdomain to subdomain.

The underlying procedure is identical for both methods. The solution is first obtained on a coarse grid, the coarseness being in the direction(s) in which adaptive multigrid refinement is to be considered. The grid is refined over the entire domain, and an improved solution is obtained. For the one-dimensional adaptive calculation, refinement is performed only in the ξ direction. From the two full grid solutions, the truncation error of the key derivatives and also of the global discrete system is estimated using Richardson extrapolation. Two types of refinement criteria are used. In one procedure, a tolerance is set for the truncation error and, in the other, the truncation error at all points is normalized with the maximum value; a tolerance was set relative to this normalized value. The results obtained with the two methods were quite similar. The regions that require refinement in the respective direction(s) are identified.

For the one dimensional (in ξ) adaptive calculation only one subdomain results. This decreases in extent as the grid level increases. For the problems considered herein, the significant flow gradients in ξ are centered around the small region $|\xi| < \xi_0$. For more complicated flows, it is possible that even with a one-dimensional adaptive calculation disjoint subdomains are necessary to significantly improve the efficiency over a non-adaptive calculation. For the two dimensional adaptive calculation, however, different regions will have different refinement requirements; therefore, it is necessary to define regions that have disparate grid requirements. Subdomains requiring refinement in the η direction, or the ξ direction, or in both (ξ,η) directions, are identified. Although different grid sizes are used in different regions, within each subdomain, uniform grids are specified. This procedure is applied on the third and higher levels of the multigrid hierarchy. The calculation proceeds with intergrid multigrid transfers. On convergence, the truncation error estimation process is repeated using the N^{th} multigrid and the stored $(N-1)^{th}$ multigrid level grid solutions.

5) Multigrid Implementation. For the RNS system of equations, a semicoarsening multigrid procedure has been presented in [4] to accelerate the convergence of the global pressure relaxation procedure (Ref [6]). A von Neumann analysis of the linearized form of the RNS system shows that the rate of convergence of the global procedure is dictated by the maximum eigenvalue as given by

$$\lambda \sim 1 - c_1 \pi^2 (\Delta\xi)^4 N_\xi^2 / \eta_M^4$$

where c_1 is a constant of O(1); N_ξ is the number of stations in the ξ direction; η_M is the normal boundary location, and $\Delta\xi$ is the axial step size. The convergence rate is significantly improved if the extent of the domain in the two directions is reduced. The current multigrid domain decomposition procedure, in effect, reduces η_M whenever a fine $\Delta\xi$ is specified and thereby achieves comparable convergence rates even on fine grids.

In the present application the multigrid method is implemented in a Full Approximation Storage (FAS) mode. The global pressure relaxation essentially reduces to a block SOR (ξ = constant) procedure for the

pressure in attached flows and for the pressure and velocities in reversed flow regions. At each station, an implicit, fully coupled tridiagonal system is inverted. When highly stretched grids are used to resolve the boundary layer, a semi-coarsening mode of the multigrid method has been shown to be more effective than the standard full coarsening mode. In this mode, the streamwise grid alone is coarsened when the calculation shifts to coarser grids. The same η grid is retained. Significant gains in the overall effort was achieved using this approach. It was found that a source term, Israeli source term (IST), first introduced in [5], is required in order to achieve satisfactory performance of the multigrid procedure. The IST acts as a form of under-relaxation of the pressure field. This leads to much smoother residual fields, which are essential for good representation on coarser grids. However the IST leads to a slower asymptotic convergence rate on a given grid. The domain decomposition procedure reduces this effect to a certain extent. Since the truncation error in the p_ξ term is used to determine regions needing refinement in the ξ direction, subdomains in which the grid is only refined in the η direction, should have a well converged pressure field. Thus it is possible to perform the calculation without the IST in these subdomains. In the present calculation, the one-dimensional adaptive calculation adds an element of sub-domaining to the semi-coarsening analysis presented in [4], so that only portions of the global domain need fine grid resolution in the ξ direction. For the two-dimensional adaptive calculation, the multigrid algorithm is implemented in either the standard full coarsening mode, or the semi-coarsening mode. One fine grid work-unit is comprised of one sweep in each subdomain belonging to that multigrid level. This includes the interdomain transfer processes. The decision to move the calculation back to a coarser grid is based on the rate of convergence on each subdomain. If the ratio of the residual norm between two successive global iterations, in any subdomain belonging to that multigrid level, falls below a certain value, then the calculation is restricted to the coarser level. The fine grid solution is not corrected until the residuals in the coarse grid subdomains are all driven to a value one order of magnitude lower than the maximum residual over all subdomains in the finer level. The multigrid components are summarized as follows,

a) Relaxation : $u_n^k = S^k u_{n-1}^k$, where S^k is the global pressure relaxation operator and u^k on convergence satisfies $L^k u^k = f^k$. Here k represents the present or finest multigrid level and n represents the iterate. $L^k u^k = f^k$ is the discrete approximation of the continuous problem Lu = f

b) Restriction to coarse grid where the following equations are solved,

$L^{k-1}u^{k-1} = I_k^{k-1} r_n^k + L^{k-1}\hat{I}_k^{k-1} u_n^k$ for points on the coarse grid which lie within the fine grid and

$L^{k-1}u^{k-1} = f^{k-1}$ for points on the coarse grid that lie outside the extent of the fine grid. Here $r_n^k = f^k - L^k u_n^k$. I_k^{k-1} and $\hat{I}_k^{k-1}$ are fine to coarse grid transfer operators. The full-weighting operator recommended in [2] was used to transfer the residuals and the solution was restricted by using a simple injection operator.

c) Prolongation or Correction where the fine grid solution is corrected

with the solution from the coarse grid modified problem.

$u^k_{n+1} = u^k_n + I^k_{k-1}(u^{k-1} - I^{k-1}_k u^k_n)$, where I^k_{k-1} is a coarse to fine interpolation operator.

It should be noted that in the present calculation, the multigrid transfer operations play a dual role. Apart from accelerating the convergence of the relaxation procedure, they also provide information from finer grids to the coarser grids, and thus improve the accuracy of the solution in regions of the coarser grids where refinement was not required. This is because the second term in the multigrid restriction process, acts as a truncation error injection term and improves the discrete approximation on the coarse grid. Thus on the coarser grids, instead of solving $L^{k-1}u^{k-1} = f^{k-1}$ everywhere, we solve

$L^{k-1}\hat{u} = \tau$ in part of the domain, where $\tau = L^{k-1}\hat{I}^{k-1}_k u^k$

This is closer to the continuous problem $Lu = f$. Here L is the continuous counterpart of the discrete operator L^{k-1} and u is the exact solution to the continuous problem ; u^{k-1} is the exact solution to the discrete problem and $\hat{u}$ is the improved solution due to the modified right hand side of the discrete approximation.

6) Interdomain transfer of boundary conditions and Conservation at grid interfaces. For a given subdomain, the following boundary conditions are to be prescribed.

$u = v = 0$ at $\eta = 0$; $u = 1$, $p = 0$ at $\eta = \eta_{max}$; $p_\xi = 0$ at $\xi = \xi_{max}$;

u, and v = free stream values at $\xi = 0$.

If a subdomain has its outflow at some $\xi < \xi_{max}$, then the boundary condition on pressure changes from Neumann to Dirichlet type. Also, if the lower boundary of a subdomain is at some $\eta > 0$, then non-zero velocities have to be prescribed. In time dependent, characteristic based, Navier Stokes computations, that use such locally embedded grids, boundary conditions are required for all variables i.e., u, v, and p. Special care has to be taken to ensure that mass conservation is not violated locally or globally.

In the RNS formulation, this difficulty does not occur as the normal velocity v in η, or u in ξ, is not prescribed at the upper or lower, or outflow boundaries. Only the tangential component u is prescribed at the upper interface or interdomain boundary. The RNS differencing allows for the calculation of the normal velocity at the outer boundaries and pressure at the body surfaces. The normal velocity is computed from the continuity equation and therefore mass conservation is automatically satisfied on all levels, for all subdomains. This eliminates the need for special interpolation formulae to ensure conservation of mass when the boundary conditions are prescribed from the coarse grid solution. Thus weak instabilities, that arise when such methods are applied to Navier stokes formulations without satisfying mass conservation, do not appear in the present method. Direct evaluation of the pressure at inflow or lower boundaries also eliminates the need for special pressure boundary conditions.

The calculation is performed sequentially rather than in parallel in the various subdomains. As such the boundary conditions at the inflow and outflow stations for each subdomain are updated with the latest available values. The overlap allowed in the subdomaining

process follows the following rules.
a) The last station of any subdomain, which is at some $\xi < \xi_{max}$ coincides with the first station of the subdomain to its right, (if one exists), where the pressure is computed.
b) Similarly, the inflow station of any subdomain, which is at some $\xi>0$ coincides with the last station on the subdomain to its left, (if one exists), where the velocities are computed.
c) If the inflow station or the outflow station of a given subdomain coincides with the physical boundaries of the global flow field then the original boundary conditions of the problem are used for these subdomains.
d) If there are no subdomains to the right for the cases in a), or if there are no subdomains to the left for the cases in b), then these boundaries are updated using coarse grid values during the multigrid prolongation process.

In the vertical direction no overlap is necessary. If a subdomain has only one of its horizontal boundaries in common with another subdomain, then updating the boundary conditions along this edge, after one sweep in all subdomains, leads to an iterative divergence on this subdomain; this influence gradually filters through to other subdomains. If these boundaries were updated through the multigrid transfer processes, then the calculation is convergent. This reflects the fact that an update of just one boundary after each sweep, with the other three updated only during the multigrid transfer process, leads to an inconsistency. This constrains the variables from adjusting to changes that occur dynamically, as the solution evolves in the various subdomains. It is also possible to perform the calculations in parallel, for a given multigrid level that comprises more than one subdomain by lagging the updating process by one sweep. This approach has not yet been implemented in the present calculations.

7) Results and Discussion. The first problem considered is the flow past the trailing edge of a finite flat plate. Fig.1 shows the grid obtained from the one-dimensional adaptive calculation. Note that the finer grids zoom in around the trailing edge located at x = 1.0 (The figure is scaled in the x direction by a factor of 2). The finer grids also reduce in extent in the η direction although the adaptivity is prescribed only in the ξ direction. Each multigrid level contains only one subdomain that is further refined in the finer levels. Fig.2 shows the composite grid obtained by prescribing two dimensional adaptivity. Note that within each subdomain uniform grids are prescribed. Fig.2 is an overlay of seven multigrid levels, each comprising several subdomains. In each level it was found that the subdomain, in which refinement in both directions was performed, was always centered around the trailing edge. This validates the refinement strategy. Both of these computations were compared with a full refinement calculation, which used a uniform fine grid in ξ and a stretched η grid. The grid stretch factor for the latter was chosen by specifying the minimum and maximum $\Delta\eta$ values and the location of η_{max} applied in the two dimensional adaptivity calculation. Multigrid acceleration in a semi-coarsening mode was used for this calculation. The same stretched η grid was used for the one dimensional adaptive calculation. Fig. 3 shows the comparison of the C_p variation obtained from the three calculations. A global truncation error tolerance of

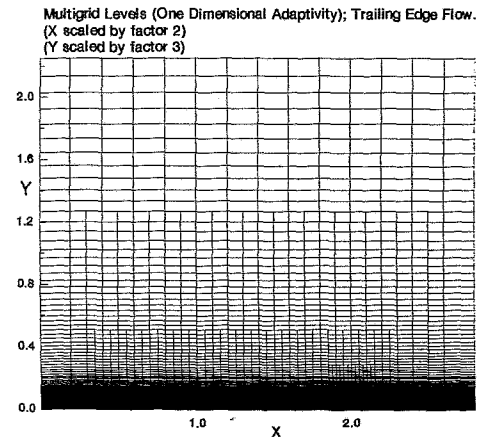

Figure 1.

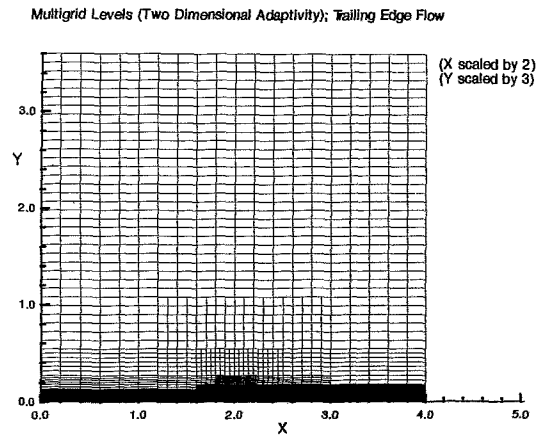

Figure 2.

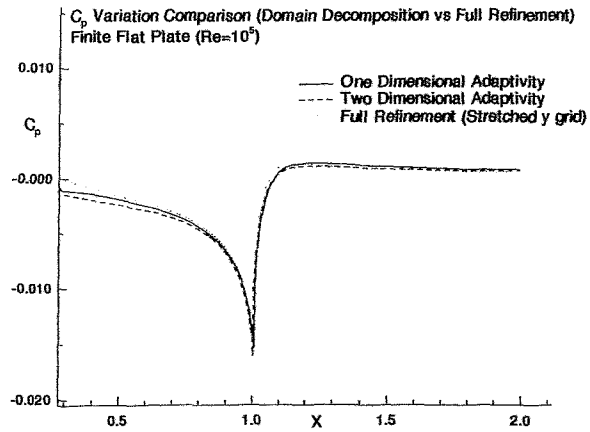

Figure 3.

0.05 was specified for this computation. Note that there is good agreement in the predicted peak pressures. Table 1 summarizes the computer memory and CPU requirements for the adaptive calculations. They are given as percentages of the full refinement calculation. Note that the memory requirement for the one and two dimensional calculations are very close. This signifiies that the η stretching used for the one dimensional calculation is nearly optimal. However it is not always possible to prescribe an optimal stretch factor a priori.

Table 1.

Aspect	Geometry	Two-D Adaptive	One-D Adaptive	Full refinement with stretched η grid
CPU	T.E.	18.03 %	15.10 %	100.0 %
	Trough (Sep)	39.80 %	---	100.0 %
	Trough (Unsep)	7.10 %	16.80 %	100.0 %
Memory	T.E.	12.90 %	13.22 %	100.0 %
	Trough (Sep)	16.32 %	---	100.0 %
	Trough (Unsep)	5.10 %	63.4 %	100.0 %

The grid of Fig.2 shows that the interaction from the trailing edge only affects a finite region around the trailing edge. From asymptotic triple deck theory, three layers with different length scales can be identified (Ref [8]), namely, a lower viscous rotational deck of $O(Re^{-5/8})$, a middle inviscid rotational deck of $O(Re^{-4/8})$ and an upper inviscid irrotational deck of $O(Re^{-3/8})$. Since the vorticity is zero in the upper deck, and since vorticity is the parameter monitored to prescribe refinement in the η direction, no η refinement should be necessary in the upper deck. The grid obtained from the two dimensional adaptive calculation displays, in each multigrid level, a region away from the body that is only refined in the ξ direction. This region, in the finest multigrid level, will represent the order of extent of the upper inviscid deck. Estimates for the extent of the other two decks can then be obtained. In more complicated flows, e.g., turbulent flow past the same geometry, for which analytical methods cannot be applied easily, such information can be very valuable in prescribing the necessary resolution in appropriate regions. In problems where the diffusion neglected in the RNS approximation becomes important, these effects can be added as an explicit deferred corrector. where they are necessary.

The second geometry to be considered is the flow past a trough. Both unseparated and separated flows were computed using the two refinement strategies. The trough geometry is specified as

$y_b = -D\ \mathrm{sech}[4(x-x_0)]$, where D represents the maximum depth at the location x_0 The values x_0=2.5 and Re = 80000 are used for the present calculations. The grid obtained from the two dimensional adaptive calculation, for separated (D=0.03) flow, is shown in Fig 4.

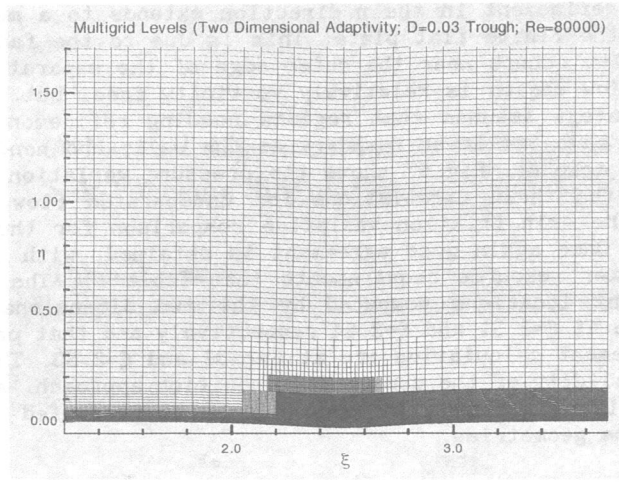

Figure 4.

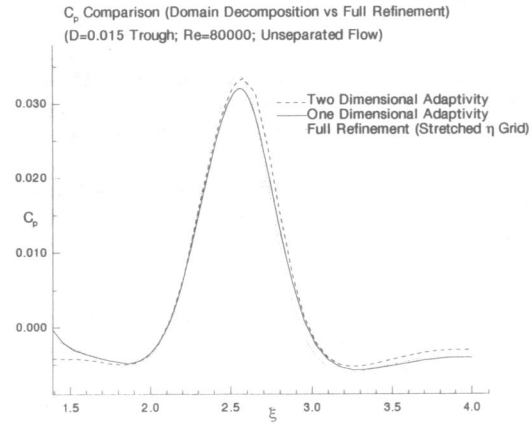

Figure 5.

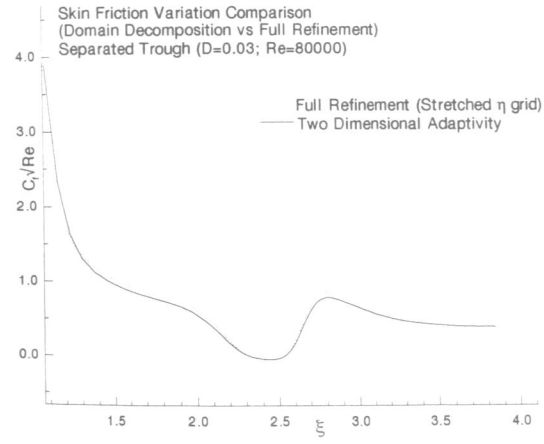

Figure 6.

Note that the refinement in the η direction extends to a much greater than that for the finite flat plate. This is due to the fact that the maximum vorticity occurs near the outer edge of the separation bubble. The reversed flow region is relatively vorticity free, but the current refinement strategy assumes that regions needing refinement in the η direction will have its lower boundary as the wall, and hence does not take this into account. Fig 5. shows the pressure variation comparison obtained from the three calculations for unseparated flow (D=0.015). Fig. 6 shows the skin friction variation comparison for the separated (D=0.03) flow. Once again good agreement is obtained, with significant gains in computer resource requirements (See Table 1). The separation and reattachment locations computed by the two dimensional adaptive calculation are at ξ=2.31 and ξ=2.54 respectively and that predicted by the full refinement calculation are at ξ=2.31 and ξ=2.53. This further confirms the validity of the domain decomposition approach. All results presented are in agreement with earlier results presented in [4] and [6] for the same geometries.

8) Summary. An adaptive multigrid domain decomposition method has been used to efficiently compute incompressible laminar flows with the RNS asymptotic system of equations and a pressure flux-split discretization. Significant gains in computer resources has been achieved. Good agreement is obtained between solutions computed using the present technique and standard non adaptive full refinement computations and earlier results presented. Investigation for internal geometries with multiple regions of flow reversal are in progress.

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